Housing Market Microstructure

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Abstract

In this article, we develop a model for the evolution of real estate prices with changes in the market structure over time. A wide range of inputs, including stochastic interest rates and demand for the asset, as well as random shocks to observe the sale process at the micro level, are considered. The holder of the asset makes optimal decisions in the face of changing market conditions by considering the level of interest rates and demand and by keeping the asset in the sale market for the optimal amount of time that maximizes expected utility. We analyze the patterns in the evolution of prices in our discussion of the simulation results and we use our results to explain the recent subprime lending crisis and meltdown in the housing market.

Keywords: Real estate market; Price evolution; Optimal waiting time

1 Introduction

The recent turmoil in financial markets triggered by defaults on subprime mortgages has revealed that market-related shocks affect risk appetite immediately and result in an expensive credit and liquidity environment. Market prices move mostly with the orders of the suppliers of liquidity (buyers) in these kinds of credit crunches compared to a competitive market where demanders of liquidity (sellers) have tantamount power. Similarly, with housing market's inherent illiquid nature, home prices are mostly driven by the demand from the buyers. In early 2000s, they rose to record levels due to high demand for mortgages with easy credit. The left-side graph in Figure 1 plots the Case-Shiller Composite-20 house-price index, which shows that the prices peaked 2006, fluctuated at that level for a while, and began to decrease lately with another upward shock in borrowing rates with the increase in the default rate of subprime borrowers. Similarly, Figure 1 also plots the level of mortgage rates and the existing home inventory in the same period. During the house price appreciation cycle (2000-2005), there is a sharp decrease in mortgage rates and a gradual or no increase in the existing home inventory level. On the other hand, the recent period with decreasing prices (2006-2007) corresponds to a sharp increases in mortgage rates and the existing home inventory. If we consider a lower level of home inventory a measure of higher demand for real estate properties, then these figures illustrate that price evolution in the housing market is a function of interest rates and demand. In this paper, we propose a mathematical model to explain this dependance from a market microstructure standpoint. We consider both market and personal shocks in this framework as an analogy to the subprime crisis and analyze how these shocks affect the individual owner's decision and the resulting sale process. Our model can also be employed to forecast the future price evolution of house prices under different interest rate and demand scenarios.

Although very closely linked in this paper, our contribution can be studied in two distinct parts: optimal waiting time (OWT) and sale price evolution. Individually, both of these topics have been studied extensively in the literature in various forms with nuances in definitions.

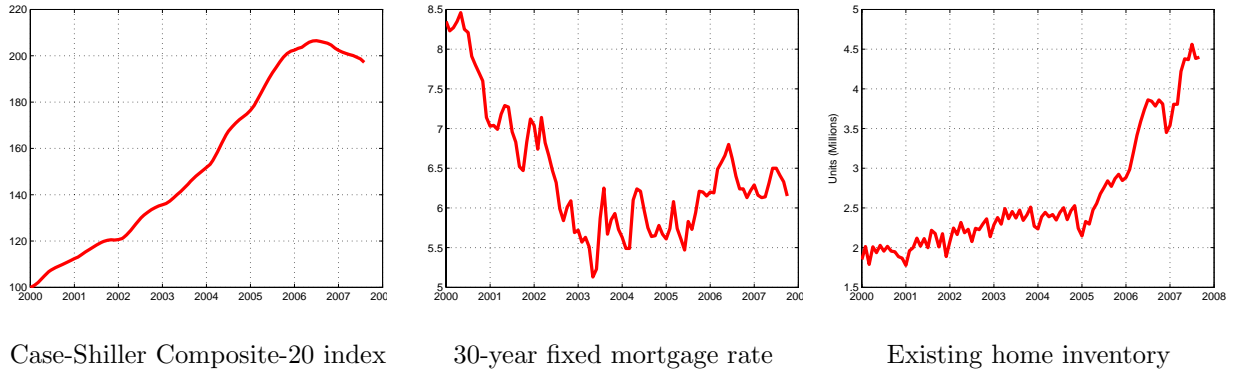| Case-Shiller Composite-20 index | 30-year fixed mortgage rate | Existing home inventory |

Figure 1: With the housing meltdown, the house-price index is falling, existing inventory is piling up and current mortgage rate is significantly higher than its trough.

Time-on-the-market (TOM), time-to-sale, optimal marketing time, and selling time are some of the frequently used terms that have similar connotations to OWT even though their exact economic definitions may differ [1]. The existing literature in this area mostly focuses on the explanation and the sign of the correlation between the length of TOM and the resulting sale price with empirical data. Earlier exemplary studies include Cubbin (1974) and Miller (1978). Cubbin builds an econometric model to explain the relationship between list price and selling time using sample data taken from Coventry housing market between 1968 and 1970. Cubbin finds that if the list price were higher, the selling time was quicker. Different from Cubbin, Miller cannot confirm his conclusion from 83 sample sales data from Columbus, Ohio and his empirical study does not prove the existence of optimal selling time. Kalra, Chan and Lai (1997) analyze 644 single-family houses sales records in the Fargo-Moorhead metropolitan area to conclude that TOM and sale price are positively related. Genesove and Mayer (1998) conclude from their empirical study from the Boston condominium market that an owner with a high loan-to-value ratio has a longer time on the market and sells his property for a higher price if he manages to sell. Taylor (1999) studies the relationship between TOM and the quality of the property in a theoretical framework. Knight (2002) examines how the changes in the listing price impact the resulting TOM and

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[1]In our model, OWT is an upper bound for TOM and is an algorithmic measure which is set at the beginning of the sale process.

sale price. Different from these articles, our paper does not explicitly test the relationship between expected TOM and the model parameters, but it rather investigates the relationship between OWT and list and reservation prices, and our findings partially match the results of Cubbin. In addition, we specifically examine the relationships between OWT and interest rates, order arrival and withdrawal intensities - which are not well-studied in the literature.

Many of the theoretical models studying TOM are based on information theory with search and matching models. With their corresponding optimal stopping rules, these models have been often used to explain the behaviors of the sellers and the buyers[2]. Buyers continue to search until the marginal benefit is equal to marginal cost of an additional search and similarly, sellers try to equate the marginal benefit to the cost of locating a bidder for his property. Haurin (1988) applies this theory of search model to investigate the relationship between the distribution of offers and the duration of marketing time. He concludes with his empirical study that as the variance of the distribution of the offers increases, the expected marketing time lengthens. Sarr (1988) examines the optimal list price adjustment under demand uncertainty. Wheaton (1990) investigates the role of vacancy rates in determining TOM, and reservation and sale prices with a search model. He finds that greater vacancy will increase selling time, lower the seller's reservation price, and will ultimately lead to lower market prices. Although we do not input vacancy rates in our model, we have similar findings with Wheaton (1990) for a seller with low reservation price. In this scenario, OWT increases, and the seller expects less from the transaction. Forgey, Rutherford and Springer incorporate a liquidity perspective to the search model. With the data collected from 3358 single-family housing transactions, they conclude that an optimal marketing period exists and properties with higher liquidity will sell at a higher selling price. Yavas (1992), Krainer and LeRoy (2002) and Williams (1995) also apply search and matching theory to analyze the sale prices of illiquid assets. Our paper is fundamentally different from these papers with its modeling approach and additional input parameters. We do not use a search and

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[2]Earlier search and matching models are used in labor market research and go back to Lucas and Prescott (1974) in which a worker departs from her job and searches for a new one. Housing market is an intuitive application of such models.

matching model, but instead employ a market microstructure approach by modeling offer values and their timing. We incorporate most of the parameters used individually in these papers (e.g., reservation price, list price, distribution of offers) in a single model with the additional inputs of withdrawal rates for the buyers' offers, deterministic and stochastic interest rates, and demand as a function of the interest rate and list price. We introduce a parameter for the seller's motivation that affects his utility from the sale [3]. We find the optimal selling time that maximizes the expected utility of the seller, which is a function expected profit discounted by the interest rate and the selling motivation parameter. OWT in our model is set at the beginning of the sale process and does not tell the exact optimal timing of the sale of the house. Optimal timing of investment has been studied in the case of known asset price dynamics in Grenadier and Wang (2005) and Evans, Henderson and Hobson (2007) but in these models, it is not possible to explain the price evolution for the asset as the dynamics are already assumed. In our paper, we also do not study the risk associated with the waiting period - which is well-documented by Lin and Vandell (2007).

Price evolution in the real estate market is the second part of our paper in which we use the OWT framework for multiple periods. Existing literature on estimating price evolution relies mostly on econometric models. Since the housing market is very heterogeneous in terms of the differences in the qualities of the properties in the market, most of the papers in the existing literature are devoted to developing statistical techniques to overcome this heterogeneity and forming a price index for a given geographical area. Earlier well-known models of this approach include Bailey, Muth and Nourse (1963), Case K. and Shiller (1987), Case K. and Shiller (1990), Case B. and Quigley (1991), and Poterba, Weill and Shiller (1991). Recent articles with similar econometric models are Goetzmann and Peng (2006) and McMillen and Thorsnes (2006). There is another stream of literature which uses equilibrium theory to estimate house price dynamics. Stein (1995) explains the large swings in prices by introducing an equilibrium model with the down-payment effects. Similarly, Ortalo-Magne and Rady (2006) present a recursive equilibrium model that accounts for income shocks

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[3]The existence of such a parameter is shown by Glower, Haurin and Hendershott (1998).

and credit constraints. Capozza, Hendershott and Mack (2004) estimate the parameters of mean reversion and serial correlation to explain the house price dynamics in equilibrium. Our model differs from these papers significantly as we apply the derived analytics in OWT framework and extend them into multiple selling periods to estimate the expected price evolution. Since there is an inherent economic linkage between the seller and the buyer in each sale (e.g., the transaction price becomes the reservation price of the buyer when he posts the property for sale), we can construct an expected price evolution for that property by tracking the expected sale price in each selling process. Since our OWT framework is very rich with the modeling of stochastic interest rates and demand, our price evolution model enables us to examine the price movements in the housing market in different rate and demand scenarios. We model the timing of the decision of sale for each owner with random income or personal shocks or with an optimality criterion that induces the seller to sell the asset for a profit opportunity. Simulation results of the model show that the fluctuation in prices are mostly driven by interest rates, demand for the asset, and reservation prices.

This paper is organized as follow. Section 2 introduces the model within the OWT framework. We start with an auxiliary model, and then extend it to more realistic cases and analyze the comparative statistics of OWT with respect to model parameters. Section 3 applies our model to price evolution in the real estate market. Section 4 provides the simulation of the price evolution of a property in the real estate market and analyzes the subprime lending crisis in the light of our simulation results. Finally, Section 5 concludes.

2   Model

Sellers of illiquid real assets often face a difficult decision regarding how long they should keep the asset in the market if they do not receive any offers matching the value of the list price. If the asset is highly desirable, the seller might remain undecided even in the case of having received an offer at the list price. He may want to wait an additional amount of time so that he allows the possibility of receiving an offer greater than the list price. If

he chooses to wait longer, he faces the risk of losing the current offer since the prospective buyer may withdraw his offer during the additional waiting time. As this scenario implies, the determination of waiting time becomes complicated when there is a mutual decision process with seller considering waiting for additional offers and the buyer withdrawing his offer as a response to the seller's waiting decision. In this section, we analyze the optimal amount of time that a seller should wait in this mutual decision process in order to maximize his expected payoff before making his final decision on the sale of the asset.

We consider two scenarios regarding our optimal waiting time (OWT) analysis. In the first case, the seller does not specify his final list price and accepts the highest available offer exceeding his reservation price, R, at the end of OWT. In the second case, he publicly announces the list price, L, and keeps R private. He sells the asset immediately if he receives an offer greater than L; otherwise, he waits until the end of OWT and chooses the best offer greater than R at that time[4].

We assume that buyers make their offers at random times with random magnitudes arising from their own valuations of the asset. Buyers may lose interest in the asset according to a known random process. The seller wants to set a waiting time such that he receives enough offers from which to choose. This waiting time should not be very long since he does lose too many offers with buyers' withdrawals or realize a large discount due to interest rates by waiting too long. Under these assumptions, the optimal waiting time turns out to be a function of the distribution of arrival and departure rate of offers as well as general market conditions such as interest rates.

We begin with the mathematical formulation of an auxiliary model, which will be handy during the analysis of our cases. We provide the details of the derivation of the OWT for this auxiliary model in the appendix. We then explain our models by first analyzing the case in which the reservation and list prices are private information, and then analyzing the case in which the seller announces a pre-determined list price.

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[4]The last case is a typical scenario in the real estate market, which we will specifically analyze in the next section.

2.1 Auxiliary model

In the auxiliary model, we assume that the seller's reservation price is $p_{\min}$ and that is public information (i.e., all of the offers that the seller receives are higher than $p_{\min}$). Furthermore, arrival times of the buyers' offers follow a one-dimensional Poisson point process with parameter λ, and the magnitudes of their offers are distributed uniformly with $U(p_{\min}, p_{\max})$ where $p_{\max}$ is finite. After making an offer, a buyer may withdraw his offer and the time to withdrawal follows the exponential distribution with parameter μ. Lastly, interest rates are constant and equal r. The seller wants to maximize expected payoff with respect to waiting time, T. With these assumptions, let the expected payoff function be $u(T, \lambda, \mu, p_{\min}, p_{\max}, r)$.

Lemma 2.1

$$u(T, \lambda, \mu, p_{\min}, p_{\max}, r) = \left(-\frac{g(T, \lambda, \mu, r)(p_{\max} - p_{\min})}{(1 - f(T, \mu))^2} \right) \times$$
$$\left(f(T, \mu)e^{\lambda T f(T, \mu)} - \frac{1}{\lambda T}(e^{\lambda T f(T, \mu)} - 1) - e^{\lambda T f(T, \mu)} + \frac{1}{\lambda T}(e^{\lambda T} - 1) \right)$$
$$+ \frac{p_{\max} g(T, \lambda, \mu, r)(e^{\lambda T} - e^{\lambda T f(T, \mu)})}{1 - f(T, \mu)} \tag{2.1}$$

where $f(T, \mu) = 1 - \frac{1}{\mu T}(1 - e^{-\mu T})$ and $g(T, \lambda, \mu, r) = (1 - f(T, \mu))e^{-rT}e^{-\lambda T}$.

PROOF.   See Appendix A.   ∎

2.2 Case without a List Price

In this scenario, we analyze the case in which the seller has a reservation price, R, which may be unknown to the public. The seller does not post a specific list price, but considers all offers up until the end of the waiting time. This scenario is very realistic in the sense that the seller has a minimum expectation from the sale, but does not limit the upside payoff.

All of the assumptions in the auxiliary model still hold with the additional reservation price information. Using the thinning principle for Poisson processes, Resnick (1992), the expected payoff, $v(T, \lambda, \mu, R, p_{\min}, p_{\max}, r)$, in this case can easily be found.
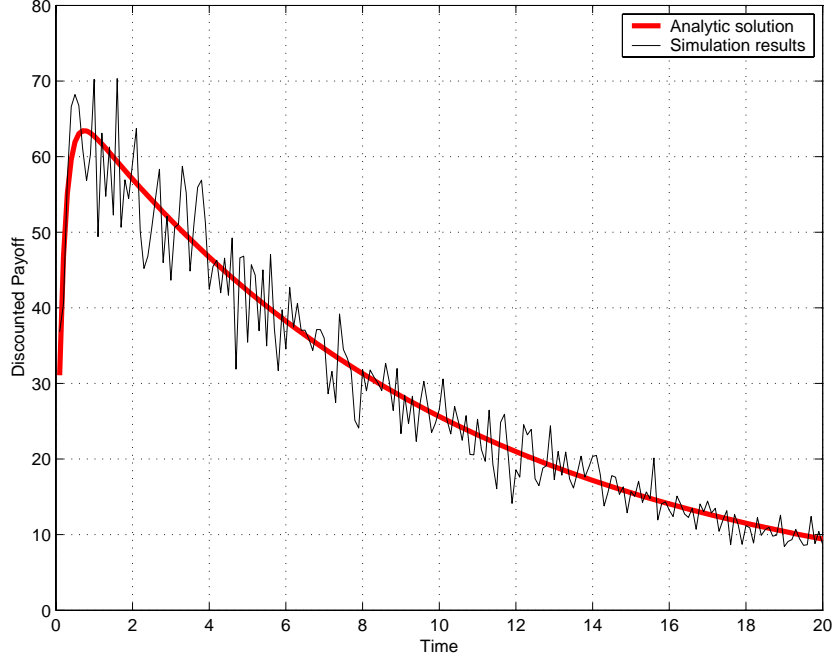
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Figure 2: Expected discounted payoff in the case without a list price.

Corollary 2.1 $v(T, \lambda, \mu, R, p_{\min}, p_{\max}, r) = u(T, \lambda_{thinned}, \mu, R, p_{\max}, r)$ *where* $\lambda_{thinned} = \lambda \frac{p_{\max} - R}{p_{\max} - p_{\min}}$.

Figure 2 plots the expected discounted payoff with respect to the waiting time in the case of no list price. Default values of the parameters used in the graph are presented in the appendix.

Our expected payoff function equals zero at $T = 0$, and starts to increase. It attains its maximum value and then starts to decrease with the effect of discounting term. This plot asserts that there is an optimal waiting time that would maximize the resulting payoff.

2.3 Case with an Announced List Price

In this case, the seller announces a public list price, L, and has a reservation price, R, which is private information. All of the assumptions in the auxiliary model still hold. This case can be simplified using our auxiliary model by dividing the payoff function into two parts. We know from Corollary 2.1 that, if the seller does not receive any offers higher than L,

the payoff becomes $u(T, \lambda \frac{L-R}{p_{\max}-p_{\min}}, \mu, R, L, r)$. Thus, we need to find the payoff function, $w(T, \lambda, \mu, R, L, p_{\min}, p_{\max}, r)$, by taking into account the possibility that there can be an offer greater than L.

Theorem 2.1

$$w(T, \lambda, \mu, R, L, p_{\min}, p_{\max}, r) = \left(1 - e^{-\lambda Ty}\right) \left(\frac{p_{\max} + L}{2}\right) \left(\frac{\lambda y}{\lambda y + r}\right)$$
$$+ e^{-\lambda Ty} u(T, \lambda x, \mu, R, L, r) \tag{2.2}$$

where $x = \frac{L-R}{p_{\max}-p_{\min}}$ *and* $y = \frac{p_{\max}-L}{p_{\max}-p_{\min}}$.

PROOF. If the seller does not receive any offer higher than L, the payoff equals $u(T, \lambda x, \mu, R, L, r)$. Given that there is an offer higher than L, our payoff equals $\frac{p_{\max}+L}{2} \mathbb{E}\left[e^{-r\beta}\right]$ where β is a random variable representing the arrival time of the first offer greater than L. If $y = \frac{p_{\max}-L}{p_{\max}-p_{\min}}$, then $\mathbb{E}\left[e^{-r\beta}\right]$ equals the moment-generating function of an exponential random variable with parameter λy. Thus, $\mathbb{E}\left[e^{-r\beta}\right] = \frac{\lambda y}{\lambda y + r}$. To find the resulting expected payoff, we only need the probability of receiving an offer greater than L, which equals $\left(1 - e^{-\lambda Ty}\right)$. As a result, our expected payoff function is the sum of these two parts multiplied by their corresponding probabilities. ∎

Figure 3 plots the expected discounted payoff with respect to the waiting time in the case of a pre-announced list price. Default values of the parameters used in the graph are presented in the appendix.

The resulting expected payoff function is a strictly increasing function with an asymptote:

$$\lim_{T \to \infty} w(T, \lambda, \mu, R, L, p_{\min}, p_{\max}, r) = \left(\frac{p_{\max} + L}{2}\right) \left(\frac{\lambda y}{\lambda y + r}\right) \tag{2.3}$$

When T increases, the payoff function is driven by the payoff of the case in which the seller receives an offer greater than L. Different from the first model, the payoff in the asymptotic case does not turn out to be a function of waiting time, T, and therefore, does not diminish with respect to T. By setting a longer waiting time, he can almost surely get an offer greater
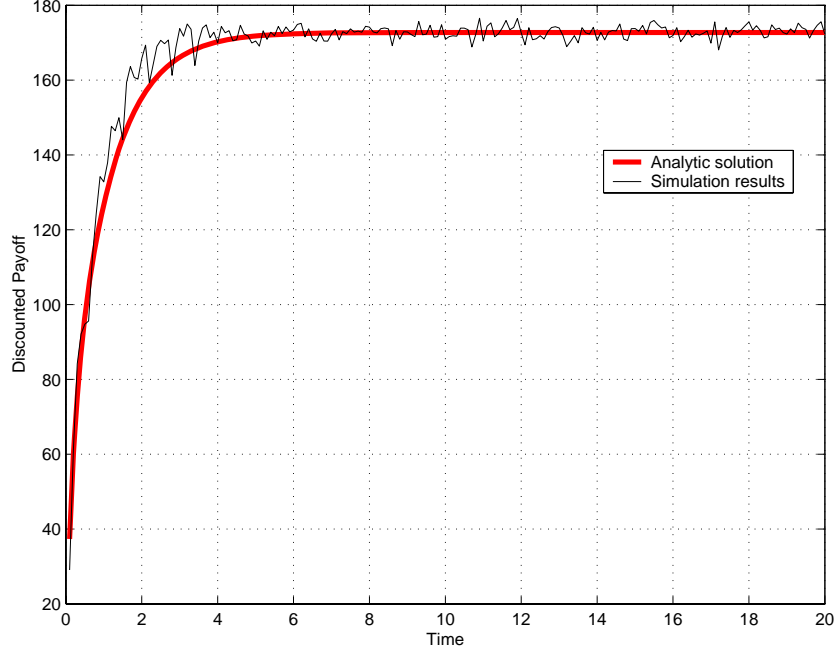
10

Figure 3: Expected discounted payoff in the case with an announced list price

than L because the discount factor, $\mathbb{E}\left[e^{-r\beta}\right]$, does not depend on T. This is due to the properties of exponential distribution which drives the first offer greater than L. This figure does not encourage the seller in any means to wait infinite amount of time to sell the asset. It only tells the seller to make a very conservative estimate of maximum waiting time before the sale process starts.

We can build the optimal waiting time framework on top of the maximum waiting time by introducing utility perspective. The reason why the discounted payoff function is increasing with respect to T is that there is no specific utility function associated with seller's motivation to sell the asset. In other words, if the seller is very motivated to sell the asset, his utility should be lower in the case of long waiting time. Glower, Haurin and Hendershott (1998) show that the seller's motivation is a significant parameter that affect selling time and sale price.

We need to incorporate the utility effect by maximizing the utility of the seller. Define the utility function, $\mathcal{U}(.)$, as follows.

11

Definition 2.1 $\mathcal{U}(D, T) = D e^{-\gamma T}$ *where D is the discounted payoff, T is the waiting time, and $\gamma \geq 0$ is the measure of selling motivation of the seller.*

As γ increases, the seller is more motivated to sell the asset and the seller has less utility from a sale event with significant waiting. Such a parameter does also exist in real world: If the seller needs solvency immediately for some reason, he will not have the same selling incentives as a seller who is trying to sell his asset for pure investment reasons. This parameter has functional properties comparable to *risk averseness*, which is a well-known term for risk appetite. Like risk averseness, *selling motivation* helps the model to be customized for different individuals, which would differentiate the sale process even in the same market conditions as long as the sellers are different.

With this assumption of utility, the expected utility function, $z(.)$, can be written as follows.

Corollary 2.2

$$z(T, \lambda, \mu, R, L, p_{\min}, p_{\max}, r, \gamma) = \mathbb{E}\Big[\mathcal{U}(.)\Big] \tag{2.4}$$

$$= e^{-\gamma T} w(T, \lambda, \mu, R, L, p_{\min}, p_{\max}, r) \tag{2.5}$$

Figure 4 plots the expected utility with respect to the waiting time in the case of an announced list price. Default values of the parameters used in the graph are presented in the appendix.

2.4   Analysis of OWT

In this section, we formally define OWT and analyze its comparative statistics with respect to model parameters. We also discuss the intuitive implications of the graphs. We use the case with the announced list price in Section 2.3 as our underlying model.
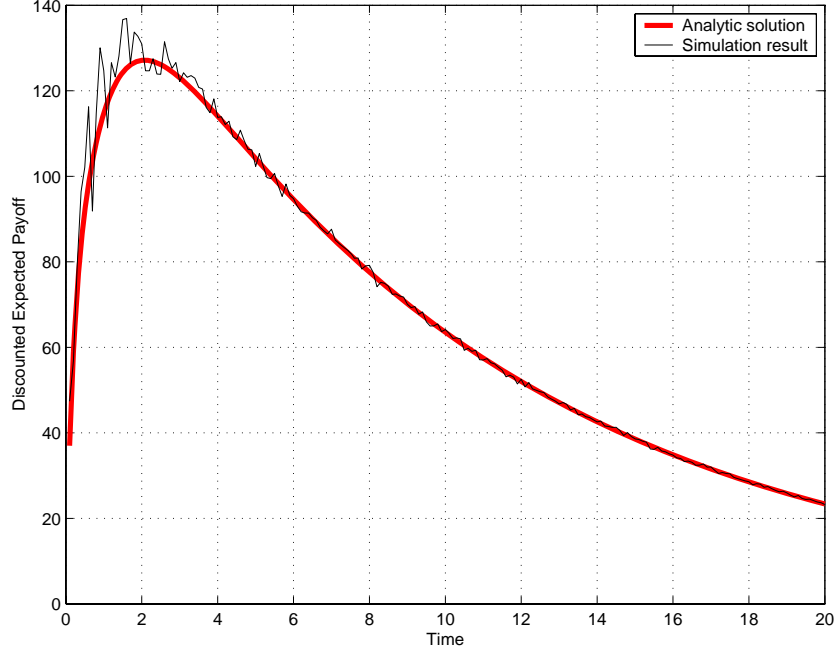
Figure 4: Expected utility in the case of an announced list price

Definition 2.2 *Let T^* denote OWT that maximizes expected utility. Then, it is equal to*

$$T^* = \arg\max\left\{T \geq 0 : z(T, \lambda, \mu, R, L, p_{\min}, p_{\max}, r, \gamma)\right\}. \tag{2.6}$$

Remark 1 *The expected utility function is continuous and its first derivative has a unique sign change from positive to negative. Since its derivative never attains positivity from thereon, the global maximum exists.*

We plotted T^* as a function of model parameters in Figures 5 through 7. Default values of the parameters used in the graphs are presented in the appendix.

The left-side graph in Figure 5 shows the change in T^* with respect to arrival intensity of the offers, λ, and interest rates. As λ increases, we expect the seller to declare a smaller waiting time because he expects that he will get sufficiently many offers from which to choose and, thus, he does not want to realize a large loss in utility by waiting longer. This plot confirms this expectation with a decreasing concave-up function when interest rates are low.
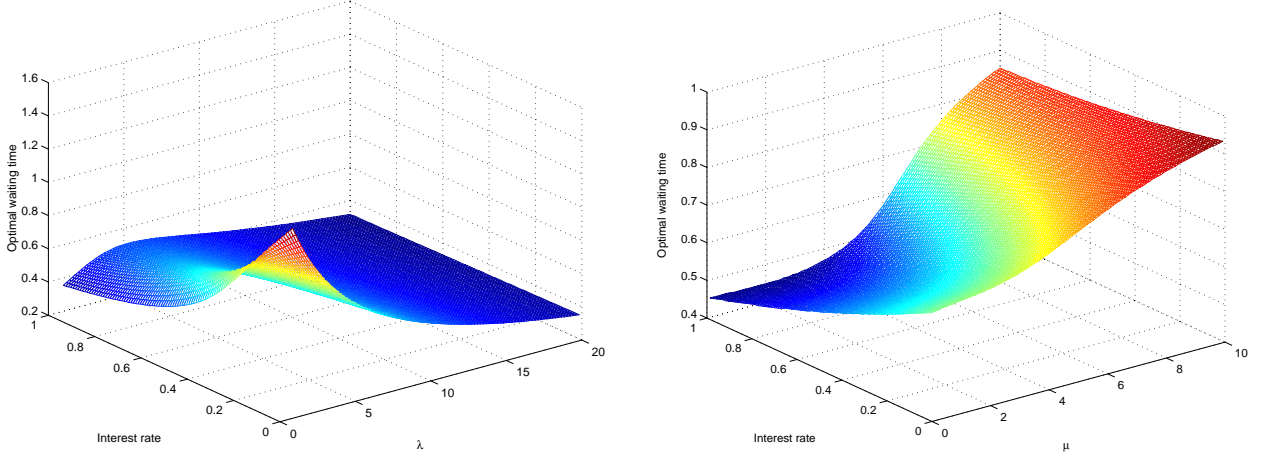
Figure 5: Optimal waiting time as a function of arrival intensity and interest rate (left) and optimal waiting time as a function of withdrawal intensity and interest rate (right)

However, when interest rates are very high, we see from the graph that T^* actually increases when λ is small and increasing. This implies that in that region, the amount of additional offers created with higher demand in the long-run is worth to wait even if the seller will face a higher discount.

The right-side graph in Figure 5 shows the change in T^* with respect to the withdrawal intensity of the buyers, μ, and interest rates. As μ increases, we expect the seller to declare a longer waiting time because he wants to increase the number of offers from which to choose, as in the case of increasing arrival intensity. Figure 5 shows that, in the cases of high interest rates, there is a period of constant optimal waiting time. Since the discount is very high, small changes in the withdrawal intensity do not affect T^*.

Both of these graphs show that interest rates are indirectly proportional with T^*. This is an intuitive result, as the seller does not want to lose potential payoff in the high interest environment by waiting longer.

The trade-off between λ and μ is also interesting to discuss from a theoretical perspective, noting that it is very difficult to measure these rates in the real world. Figure 6 illustrates that they do not always create the opposite impact on T^*. For small values of λ and μ, T^*
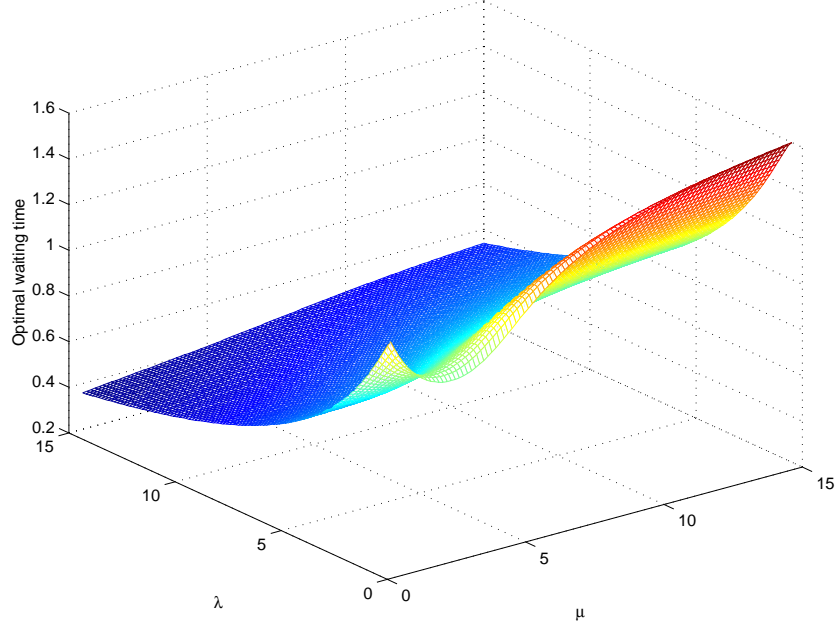
Figure 6: Optimal waiting time as a function of arrival intensity and withdrawal intensity

increases when they both rise to a certain threshold. Only after this point, they begin to create the opposite effect. This suggests that when the demand is low, T^* is highly affected by small positive changes in the demand and this does not change even if the demand is uncertain with the high probability of withdrawals.

The left-side graph in Figure 7 shows the change in T^* with respect to the list price and interest rates. When we increase the list price incrementally, we can also afford waiting a little longer as, in this case, we receive higher payoff with the sale. However, as Figure 7 illustrates, if our list price is already high, increasing the list price further diminishes the probability of receiving an offer greater than this new value and, thus, there is no incentive to wait longer.

The right-side graph in Figure 7 shows the change in T^* with respect to the reservation price and interest rate. As R increases, the seller wants to wait longer because, using the thinning principle in Poisson processes, this case theoretically implies decreasing λ. As we have seen earlier, decreasing the arrival intensity results in a larger T^*.
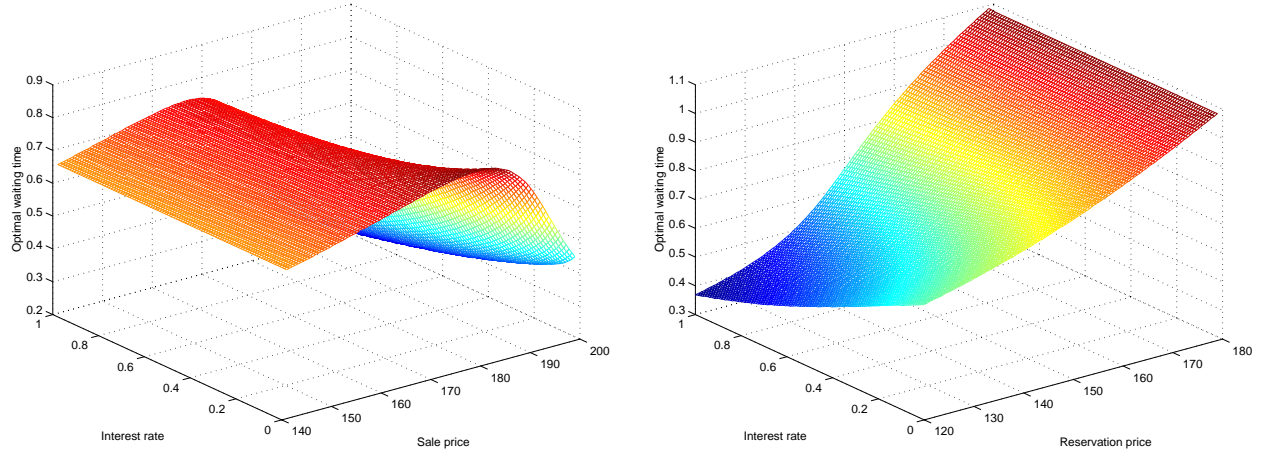
Figure 7: Optimal waiting time as a function of list price and interest rate (left) and optimal waiting time as a function of reservation price and interest rate (right)

3   Modeling Price Evolution in the Real Estate Market

In the previous sections of the paper, we characterized the optimal waiting time to set when selling an illiquid asset, and how it changes with various market parameters such as arrival and departure intensities of the buyers and interest rates. In the rest of the paper, we try to understand the time evolution of the sale prices of an asset in the real estate market when sellers maximize their payoff by considering optimal amount of waiting time. We first explain our market structure assumptions, and then propose our model governing the decision process of the holder of a real asset.

3.1   Stochastic demand and interest rates

The price evolution in the real estate market is strongly affected by changes in the broader economy such as recessionary or expansionary cycles and by shocks in the interest rates. In our model, we will track the recessionary or expansionary cycles with the stochastic demand function, $\lambda(t)$. We will also take stochastic interest rates, $r(t)$, as an input to the model. We then investigate the impact of the evolution of these inputs on the sale process of individual

assets.

In our model, the demand for the asset, $\lambda(t)$, will be a function of interest rates, $r(t)$, and the announced list prices, $L(t)$. $L(t)$ is non-random for the seller.

$$\lambda(t) = g\big(r(t), L(t)\big). \tag{3.7}$$

It is assumed that $L(t)$ is non-increasing and $L(t) \geq R > 0$ for all $t \geq 0$, where R is the seller's reservation price. Offers arrive according to a non-homogenous Poisson process with stochastic arrival intensity, $\lambda(t)$. Let $\mathcal{F}_t$ be the σ-algebra generated by $\{r(s)\}_{0 \leq s \leq t}$, and $N(t)$ be the number of arrivals in $[0, t]$. Then, the following are true:

$$
\begin{aligned}
\mathbb{P}\{N(t) = n\} &= \mathbb{E}\Big[\Lambda(t)^n \frac{\exp(-\Lambda(t))}{n!}\Big], \text{ and} \\
\mathbb{P}\{N(t) = n | \mathcal{F}_t\} &= \Lambda(t)^n \frac{\exp(-\Lambda(t))}{n!}
\end{aligned}
\tag{3.8}
$$

where $\Lambda(t) = \int_0^t \lambda(s) \mathrm{d}s$.

The value of the offers come from a known distribution, F_ξ, independently, where ξ represents the intensity of a generic offer. It is assumed that F_ξ has the density f_ξ. ξ_i and A_i, $i \geq 1$, represent the intensity and the arrival time of the ith offer respectively. Given $\mathcal{F}_t$ and the number of offers in $[0, t]$, then arrival times of all offers are independently distributed over $[0, t]$ with conditional density

$$f_{A | \{\mathcal{F}_t, N(t) = n\}}(a) = \frac{\lambda(a)}{\Lambda(t)}.$$

After an offer arrives, it is withdrawn after a random waiting time with distribution function F_τ, where τ is the waiting time of a generic offer. For the sake of analytical simplicity, F_τ is assumed to be continuous. However, it is not hard to extend the analysis below to more general offer waiting time distributions. τ_i, for $i \geq 1$, represents the waiting time of the ith offer. All τ_i's are assumed to be independent from one another.

We set $\xi_0 = A_0 = \tau_0 = 0$. Let β be the arrival time of the first offer greater than the list price. Then,

$$\beta = \inf\{A_i : \xi_i \geq L(A_i), i \geq 1\}. \tag{3.9}$$

We provide three theorems characterizing the expected payoff of the seller for three cases separately. In the first case, the seller announces a time-dependent list price for the asset. In this case, the asset is sold whenever there is an offer greater than the ask price of the asset. This is the most general case containing others as subcases. The second case is similar to the first one, but the list price of the asset does not change with time. Finally, in the third case, the seller does not limit his upside payoff by announcing a list price. He waits an optimal amount of time and chooses the best available offer greater than the reservation price to sell the asset. Let us start with the first case. In this case, the discounted payoff function, $X(t)$, at time t is written as follows:

$$
\begin{aligned}
X(t) = \quad & \exp\left(-\int_0^t r(s)ds\right) \cdot \left(\max_{0 \leq i \leq N(t)} \xi_i \cdot \mathbb{1}_{\{\xi_i \geq R\}} \cdot \mathbb{1}_{\{\tau_i \geq t - A_i\}}\right) \cdot \mathbb{1}_{\{\beta > t\}} \\
+ \quad & \exp\left(-\int_0^\beta r(s)ds\right) \xi_{i(\beta)} \mathbb{1}_{\{\beta \leq t\}}
\end{aligned} \tag{3.10}
$$

where $\xi_{i(\beta)}$ is the offer value at β. The first term in (3.10) accounts for the case that all offers until time t are smaller than the list price. The second term corresponds to the case that there is an offer greater than the list price before time t. We have the following theorem for the expected discounted payoff at time t.

Theorem 3.1 *Let $P(t) = \mathbb{E}[X(t)]$ and $P(t|\mathcal{F}_t) = \mathbb{E}[X(t)|\mathcal{F}_t]$. Then, $P(t) = \mathbb{E}[P(t|\mathcal{F}_t)]$, and*

$$
\begin{aligned}
P(t|\mathcal{F}_t) \quad = \quad & e^{-\int_0^t r(s)ds} e^{\Lambda(t)(\varphi(t)-1)} \left(L_0 - \int_0^{L_0} e^{-\Lambda(t)\psi(t,y)} dy\right) \\
+ \quad & \left(1 - e^{\Lambda(t)(\varphi(t)-1)}\right) \int_0^t \int_{L(a)}^\infty \frac{\lambda(a)\exp\left(-\int_0^a r(s)ds\right)}{\Lambda(t)\left(1 - F_\xi\left(L(a)\right)\right)} f_\xi(x) x \, dx \, da
\end{aligned} \tag{3.11}
$$

where $\psi(t,y) = \frac{1}{\Lambda(t)} \int_0^t \lambda(a)\left(1 - F_\tau(t-a)\right)\left(F_\xi\left(L(a) \vee y\right) - F_\xi\left(R \vee y\right)\right) da$ and $\varphi(t) =$

$\frac{1}{\Lambda(t)} \int_0^t \lambda(a) F_\xi\big(L(a)\big) \mathrm{d}a.$

PROOF.   See Appendix B.   ∎

The difficulty in proving Theorem 3.1 is that the changing list price introduces a coupling between the offer intensity and its arrival time. Theorem 3.1 can be further simplified if $L(t)$ is constant, which is what we analyze next.

In the second case, the seller's payoff can also be written exactly as in (3.10). Theorem 3.2 gives us the seller's expected payoff at time t for this case. We do not provide its proof as it is similar to that of Theorem 3.1.

Theorem 3.2 *Let* $P(t) = \mathbb{E}[X(t)]$ *and* $P(t|\mathcal{F}_t) = \mathbb{E}[X(t)|\mathcal{F}_t]$. *If* $L(t) = L \geq R > 0$ *is constant, then* $P(t) = \mathbb{E}\big[P(t|\mathcal{F}_t)\big]$ *and*

$$
\begin{aligned}
P(t|\mathcal{F}_t) &= \mathrm{e}^{-\int_0^t r(s)\mathrm{d}s} \mathrm{e}^{\Lambda(t)(F_\xi(L)-1)} \Big(L - \int_0^L \mathrm{e}^{-\Lambda(t)\psi(t,y)} \mathrm{d}y \Big) \\
&\quad + \big(1 - \mathrm{e}^{\Lambda(t)(F_\xi(L)-1)}\big) \int_0^t \int_{L(a)}^\infty \frac{\lambda(a)\exp\big(-\int_0^a r(s)\mathrm{d}s\big)}{\Lambda(t)\big(1 - F_\xi\big(L(a)\big)\big)} f_\xi(x) x \mathrm{d}x \mathrm{d}a \quad (3.12)
\end{aligned}
$$

where $\psi(t,y) = \big(F_\xi(L \vee y) - F_\xi(R \vee y)\big)\big(1 - \frac{1}{\Lambda(t)} \int_0^t \lambda(a) F_\tau(t-a) \mathrm{d}a\big)$.

Finally, in the third case, the seller does not restrict the upside payoff by announcing a list price. In this case, his payoff process at time t can be written as

$$
X(t) = \exp\Big(-\int_0^t r(s)ds\Big) \cdot \Big(\max_{0 \leq i \leq N(t)} \xi_i \cdot \mathbb{1}_{\{\xi_i \geq R\}} \cdot \mathbb{1}_{\{\tau_i \geq t - A_i\}} \Big). \quad (3.13)
$$

The simplicity of (3.13) compared to (3.10), results in a corresponding simplicity in the expected payoff formula at time t, which is given in Theorem 3.3 below. Again, we do not provide the proof for this theorem either since it is similar to the proof of Theorem 3.1.

Theorem 3.3 *Let* $P(t) = \mathbb{E}[X(t)]$ *and* $P(t|\mathcal{F}_t) = \mathbb{E}[X(t)|\mathcal{F}_t]$. *If the seller does not announce any list price (i.e.,* $L = \infty$)*, then* $P(t) = \mathbb{E}[P(t|\mathcal{F}_t)]$ *and*

$$
P(t|\mathcal{F}_t) = \exp\Big(-\int_0^t r(s)\mathrm{d}s\Big) \int_0^\infty \big(1 - \exp(-\Lambda(t)\psi(t,y))\big) \mathrm{d}y, \quad (3.14)
$$

19

where $\psi(t,y) = (1 - F_\xi(R \vee y)) \left(1 - \frac{1}{\Lambda(t)} \int_0^t \lambda(a) F_\tau(t-a) \mathrm{d}a\right)$.

Further reductions to Theorems 3.1, 3.2 and 3.3 are possible depending on the distribution of the offer intensities, the distribution of offer waiting times, and the distribution of the stochastic process governing the offer arrival times. In the next section, based on our analytical analysis in this part, we provide a simulation-based study to show how the list price of a real asset changes over time if the seller sets an optimal amount of time to sell it.

3.2 Microstructure of holding a real asset

In this section, we focus on a single asset in the real estate market and track its price evolution for a desired period of time. During this fixed time period, the asset may be sold a number of times and the resulting sequence of sale prices constitute the price evolution. For each holder of the asset, the model evolves similarly with common subperiods:

1. Occupation period

2. Exogenous shocks: personal crisis and profit opportunity

3. Sale process with optimal waiting time

4. Updating reservation and list prices.

3.2.1 Occupation period

Occupation starts after the sale of the asset. The buyer (now owner) knows how much he paid for the asset and this constitutes his reservation price, R. Thus, the owner will try to sell the asset for at least this amount when he posts the asset for sale. We assume that each owner needs the asset for at least a certain amount of time, O years, which may vary from individual to individual. Within this period, he does not want to sell the asset unless there is an inevitable shock (such as relocation necessity, unsuitability of the asset after a change in the size of his family, personal insolvency or bankruptcy, etc.). If the holder of the asset

does not face any of these unpredictable shocks, he will not try to sell the asset until O years has passed.

3.2.2 Exogenous shocks: Personal crisis and profit opportunity

There are two types of exogenous shocks to the holder of the asset in the model:

- *Personal crisis:* If the holder of the asset encounters an external shock that is not market-related (such as losing his job, family- or job-related relocation necessity, personal insolvency, etc.), he will try to sell the asset as soon as possible without considering how many years he has held the asset. In our model, the holder of the asset receives a random shock in this nature and he posts the asset for sale at the moment that this shock occurs whether or not he has owned the asset more than O years.

- *Market related profit opportunity:* After using the asset for X years, the holder of the asset begins to seek an optimal market environment to post the asset for sale. While waiting for the optimal market environment, he can still face a personal crisis after which he must post the asset for sale immediately. In our model, the holder posts the asset for sale whenever the interest rates fall below a certain threshold, ϕ. At this level, the relative demand to the asset compared to the high interest rate environment becomes higher and the holder of the asset expects to receive many offers exceeding his reservation price.

As a result, the asset will eventually be put on the sale market in our model, but it may be due to either a personal crisis or a possible profit opportunity that occurs after time O. If we denote time to personal crisis by ω and time to a possible profit opportunity by π, then time to posting the asset for sale, υ, becomes

$$\upsilon = \min(\omega, \pi) \text{ where } \pi = \inf(t \geq X : r(t) \leq \phi). \tag{3.15}$$

Consequently, υ becomes the total occupation period after which the holder decides to sell the asset. He now posts the asset for sale with an initial list price, L_0, which must satisfy

$L_0 \geq R$. He does not disclose his reservation price, R, to the market and may adjust the list price until the asset is sold.

3.2.3 Sale process with optimal waiting time analysis

After the asset is posted for sale, the seller sets an optimal waiting time, OWT, which would maximize his expected utility from the sale. During the OWT, he will collect offers from prospective buyers and will sell the asset immediately if he receives an offer greater than the current list price. Everyone in the market knows the list price, $L(T)$ after T waiting time, but they do not know the seller's reservation price, R. In our model, list price is a function of T, and gradually converges to R:[5]

Definition 3.1 $L(T) = R + (L_0 - R)e^{-\zeta T}$ where L_0 is the initial list price and ζ is a positive real number.

Let $p_1(T_1), ..., p_k(T_k)$ be the offers received during waiting time where $p_k(T_k)$ is the kth offer received at time T_k. The holder sells the asset immediately at time T_k where k satisfies:

$$p_k(T_k) \geq L(T_k) \text{ and } p_m(T_m) < L(T_m) \ \forall m < k \tag{3.16}$$

If the holder of the asset does not receive any offers greater than the list price but receives offers exceeding R, then the holder sells the asset to the buyer with the highest available offer, p, at the end of OWT.

Definition 3.2 If $\exists m$ such that $m \in \{1, 2, ..., n-1, n\}$, $p_m(T_m) \geq R$ and $p_m(T_m)$ is not withdrawn, then $p = \max\left(p_{k_1}(T_{k_1}), ..., p_{k_i}(T_{k_i})\right)$ where i is the number of the available offers that are not withdrawn until the end of OWT (Assume $i \geq 1$).

If there is no offer exceeding R at the end of OWT, then the holder must decrease his reservation and list price, and has to re-post the asset for sale with the updated list price.

---

[5]Note that T represents waiting time and t is actual time.

In our price evolution analysis, the framework introduced in Section 2.3 is used. We assume that the buyers make their offers at random times according to a non-homogeneous Poisson distribution whose intensities are defined by the demand function, $\lambda(t, r(t), L(T))$. At these random times, they offer a random price distributed uniformly arising from their own valuations of the asset. Buyers may also lose their interest in the asset according to a known random process.

Different from the model described in Section 2.3, the arrival intensity of the offers becomes a function of time and waiting time, $\lambda(t, T)$, which equals $\lambda(t, r(t), L(T))$. Interest rates in this analysis are no longer constant, but follow a stochastic process. If we denote the expected utility function by $\mathcal{Y}(T, \lambda(t, T), \mu, R, L(T), p_{\min}, p_{\max}, r(t), \gamma)$, then OWT is defined as follows.

Definition 3.3 $T^* = \arg\max\{T \geq 0 : \mathcal{Y}(T, \lambda(t, T), \mu, R, L(T), p_{\min}, p_{\max}, r(t), \gamma)\}$.

3.2.4 Updating reservation and list prices

There are two different scenarios for updating the list and reservation prices for the next period. If the seller achieves selling the asset before the end of the OWT, the new reservation price for the next period equals the agreed sale price in this period (as the buyer of the asset would not want to sell the asset for a price less than he paid). The initial post price in this case, L_0, becomes the maximum value that an offer could be arbitrarily close to with positive probability. If the distribution of the offers lies between $p_{\min}$ and $p_{\max}$, then L_0 equals $p_{\max}$ and $L(T)$ gradually decreases from L_0 to R during the sale process.

If the seller does not succeed in selling the asset, a new period starts for this sale process with lowered reservation and list prices. The seller chooses a new reservation price which is between R and $p_{\min}$ and sets L_0 to R. With these adjustments, he increases the probability of selling the asset in the next period.

4 Simulation

In this section, we first explain the parameters in our model and then discuss the simulation results of the price evolution process.

4.1 Model parameters

We assume that interest rates evolve according to a Cox-Ingersoll-Ross (1985) process:

$$dr(t) \;=\; \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t). \tag{4.17}$$

We define $\lambda(t, r(t), L(T))$ such that it satisfies the general relationship between demand, interest rates, and list price. Keeping list price constant, we assume that the demand for the asset decreases as the interest rate rises; if we keep the interest rate level, the demand for the asset decreases as the list price increases:

$$\lambda(t, r(t), L(T)) = \frac{K_1}{r(t)} + \frac{K_2}{L(T)}, \tag{4.18}$$

where K_1 and K_2 are constants.

We first simulate occupation periods, exogenous shocks and check whether there is market-related profit opportunity that the owner can take advantage of. The sale process starts either with a personal shock or a profit opportunity with low interest rates. For each sale process, we numerically calculate T^* and model the arrival times of the offers by using the demand function as a non-homogeneous Poisson process. At the arrival times, we produce independent values for the offers using a uniform distribution around $p_{\min}$ and $p_{\max}$. For the possible withdrawal of the offers, we use exponential distribution with parameter μ. At the end of the period, we update our reservation and list prices depending on the outcome of the sale process. If there is a sale, the new reservation price equals R and L_0 became $p_{\max}$. If there is no sale, then the new reservation price equals $\frac{R + p_{\min}}{2}$ and L_0 takes the value of R. A new period starts with these new parameters, current level of interest rates and demand.
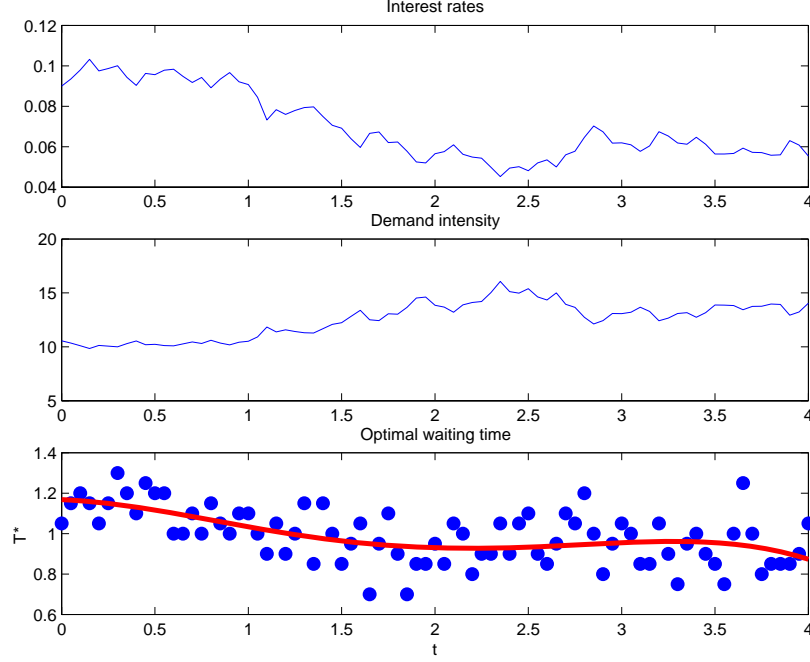
Figure 8: Evolution of optimal waiting time in a single realization of interest rates and a stochastic demand function

The appendix includes all the parameters used in the simulation.

4.2 Simulation results

Figure 8 shows the evolution of the OWT in a four-year period and illustrates its reaction against demand intensity and interest rates in a single realization of interest rates and demand function. This graph is fundamentally different from Figure 5 as it captures the evolution of all the parameters of the model. The horizontal axis no longer represents the waiting time, but instead shows the actual simulation time.

This figure illustrates that, as the interest rates decrease, demand intensity and OWT show opposite trends. With the increase in demand intensity and the decrease in the interest rates level, OWT decreases since, in these circumstances, the seller will get more offers and his payoff will be discounted with a lower factor. Therefore, the seller sets a smaller OWT because he knows that he will get enough offers to choose from in a smaller time period. This
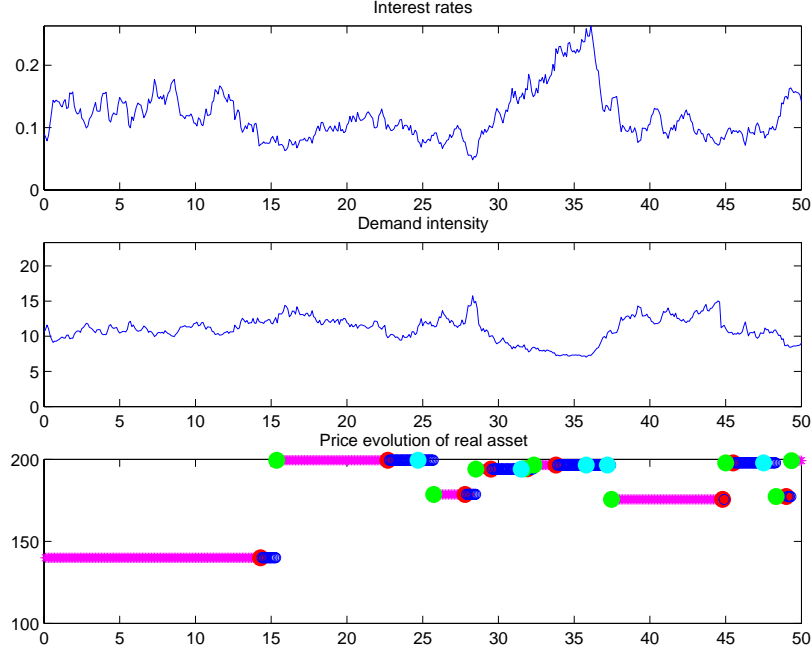
Figure 9: Single price evolution of a real asset in a realization of interest rates and a stochastic demand function

figure summarizes that changing market conditions has a broad impact on the sale process of the real asset, and they even affect the behavior of the seller. In a low-interest rate and high-demand environment, the seller keeps the asset in the sale market for a shorter amount of time than he would do in a high-interest and low-demand environment.

Figure 9 illustrates the agreed sale price of the real asset with the corresponding functions of interest rates and demand intensity in a fifty-year period. The graph of the price evolution in the third row specifically shows the random occupation periods, moments of exogenous shocks due to personal crises or convenient market conditions for the seller to make a probable profit. It also includes whether or not the price process results in success with the appropriate labels shown in the legend of Figure 10. This figure expands the plot of the price evolution shown in Figure 9.

Figure 10 specifically focuses on the decision process of the seller. All of the subperiods, occupation period, decision of sale, time on the market, and time of the sale are illustrated in the figure. Until the moment of sale, all of these different time periods are shown at the
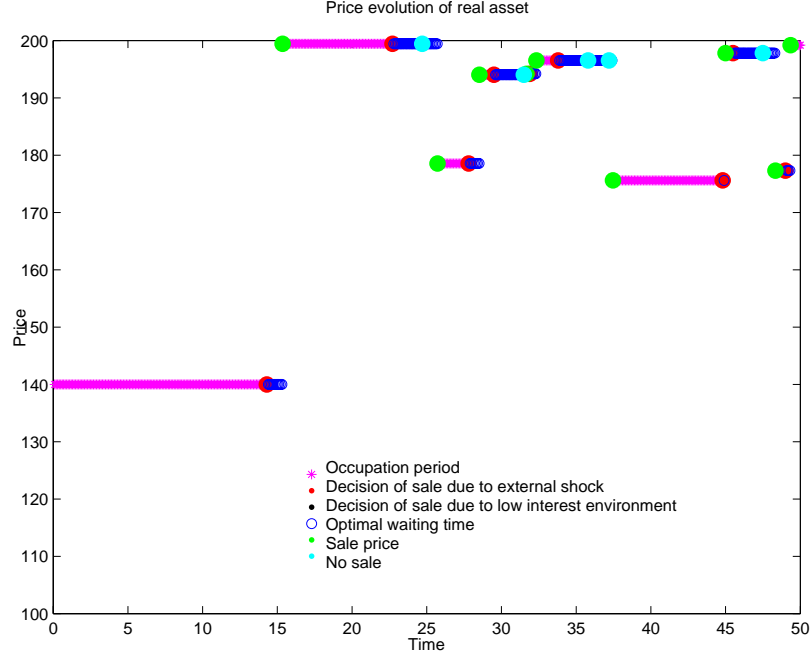
26

Figure 10: Single price evolution of a real asset with illustration of subperiods and sale or no sale outcomes

same price level which constitutes the price that the seller paid at his initial purchase of the asset.

In Figure 10, pink circles represent the occupation period in which the seller has not decided to sell the asset yet. Pink circles are always followed by a black or red dot which represents the decision of sale due to a either probable profit opportunity or personal crisis, respectively. At this time of sale decision, the seller sets an OWT by taking the market conditions as of that moment into account. The waiting period is shown by the blue circles. After this waiting time, a green or magenta dot follows, signifying the event of sale and no-sale respectively. If there is a sale, a new period starts with the occupation period of the new holder of the asset.

If the seller does not succeed in selling the asset, he lowers his reservation and list price and sets a new OWT by using the new parameters and considering market conditions. Blue circles follow the magenta dot in this case of no-sale. In this second attempt of sale, he may
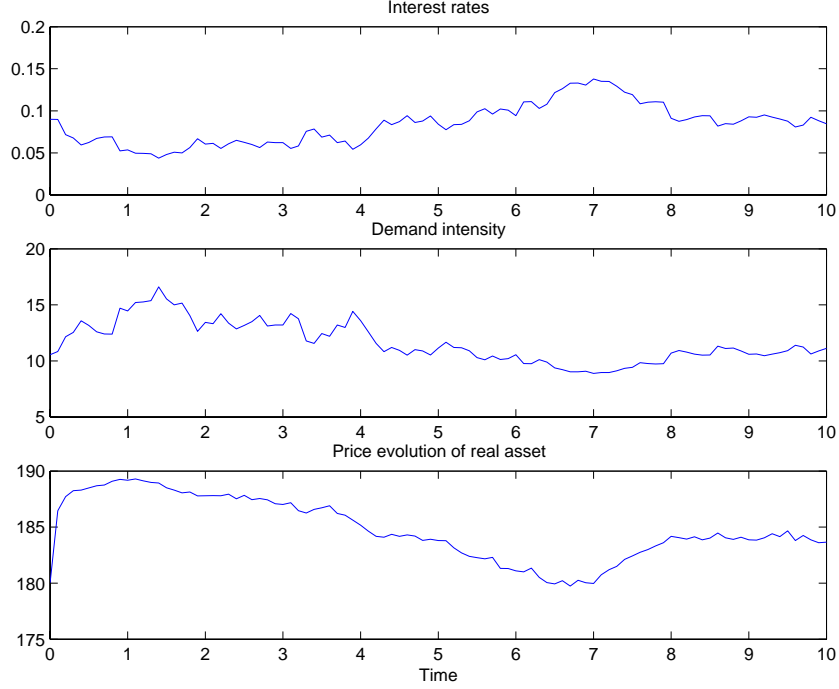
27

Figure 11: Expected price evolution of a real asset in a single realization of interest rates and a stochastic demand function

still sell the asset for a price higher than his initial reservation price, but this probability is less than the probability of such a result he had in the first attempt. Our results support this conclusion, as in the figure, four cases result in no-sales out of which only one is sold for a higher price compared to the initial reservation price. The seller may not achieve selling the asset in this attempt either. In this case, he will be further required to lower his reservation and list prices. However, this probability is also lower: Out of four cases of no-sale, only one seller encountered two successive failed attempts.

Figure 11 illustrates the expected price evolution of the real asset without considering occupation periods or exogenous shocks. This figure shows the mean sale price to expect if the seller posts the asset for sale at a given time. This figure still assumes that the seller sets an optimal waiting time for the sale process and has his own reservation price. In this example, the asset is sold in every small time period, and the model parameters are updated after the sale. This figure supports our conclusions from the results of the single simulation.

28

The expected sale price declines (increases) with the decrease (increase) in the demand and the rise (fall) in interest rates.

Simulation results show that the fluctuation in the prices are mostly driven by interest rates, demand, and time-on-the-market (TOM). In Figure 9, interest rates increase drastically approximately between the 25th and 40th year and this increase results in two unsuccessful attempts of sale in Figure 10. During this period, the reservation price of the real asset is very close to its maximum and, with the high-interest and low-demand environment, the seller could not succeed in selling the house in two trials. Eventually, the seller sells the asset in the third trial with a significant discount compared to his reservation price because the long time on the market adversely affected the final sale price.

4.3   Analysis of the recent Subprime Lending Crisis

Our simulation results coincide with what the housing market observed with the subprime lending crisis. The historically record low borrowing rates during 1999-2004 increased housing affordability with the corresponding up-trend in home prices as shown in Figure 1. When the borrowing rates increased and the rate of home buyers' mortgage contracts began to reset to these higher rates, low credit borrowers had a difficult time making their monthly payments back and are forced to put their homes for sale. This coincides exactly with our modeling of *personal crisis* scenario. Due to this personal shock, the home buyer had to put his asset for sale in a very unfavorable market condition: high interest rates and low demand. As shown in our simulation results, the expected price in such a market environment is lower compared to other scenarios and, as the home equity value of his asset decreased, the home owner was not be able to pay back his debt and defaulted.

Another catalyst for the defaults in the meltdown was the significantly higher reservation prices locked in during the bull period. Although demand for the real estate market should have been down during this period, borrowers still succeeded in financing - helped by the lower standards of mortgage originators. When borrowing rates began to increase, the low-credit environment was lost. Depressed by this shock, mortgage originators began to lower

their lending standards to keep their market share. As an indication of this, during 2002-2006, Combined Loan to Value (CLTV) ratio increased for the collateral. In the subprime category, it rose to 88% from 81% and in Alt-A category, it rose to 85% from 73%[6]. When the subprime borrowers had to put their homes for sale to pay back their mortgages, they could not get any offers matching their list price. As we discussed in our simulation results, in a high-interest and low demand environment, it is very difficult to sell your property if you have a high reservation price in the first place. Since most of the subprime borrowers paid their homes the historically record value, they had to lower their reservation prices during the sale process. This result is quite similar to our analysis of the successive no-sale events when the reservation of the owner was very close to the $p_{\max}$. Subprime borrowers ended up lowering their reservation prices to sell their asset and, since the mortgage debt exceeded the resulting sale price, they could not cover all their debt. On the other hand, if they could not sell their asset, then they defaulted and the property is sold for a discount in the foreclosure.

5   Conclusion

In this article, we propose a new model to describe how shifts in market conditions affect the evolution of real asset prices. We subdivide the seller's holding period into realistic components and enable exogenous shocks to the system that may be the result of a profit opportunity or an inevitable personal crisis that requires the seller to sell the asset immediately.

We extensively investigate the sale process by introducing a new model for analyzing time-on-the-market (TOM) with a new construct, optimal waiting time (OWT), which is generally larger than TOM but still has similar characteristics. When the asset is posted for sale, the seller sets a deterministic OWT that maximizes his expected utility. He may not wait the full OWT when there is an offer that matches the list price. While calculating OWT, he takes all market conditions into account which makes OWT time-dependent and

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[6]Source: UBS Mortgage Research

useful to apply to price evolution in the real estate market.

We study the comparative statistics of OWT with respect to different model parameters such as arrival intensity, withdrawal (cancelation) rates of the offers, and interest rates. We specifically look at the pairwise impact of these parameters and how they affect the resulting OWT in the different regions of the surface. We observe that arrival intensity and interest rates are indirectly proportional to OWT, but the withdrawal rate is directly proportional. The pairwise impact of arrival intensity and withdrawal rate against the interest rate shows that the sensitivity of the model is different in high-interest and low-interest environments.

We incorporate the derived theory of OWT into our price evolution framework that also includes occupation period, personal crisis or profit-taking opportunity, and the deterministic updating of model parameters with the occurrence of the sale. The seller first has a random occupation period, during which he does not consider selling the asset for investment purposes. However, he may still face a personal crisis due to which he has to post the asset for sale. After the occupation period, he may sell the asset to undertake a probable profit.

After the decision of sale, the seller sets OWT by considering interest rate, demand for the asset, reservation price, withdrawal intensity, selling averseness, and list price. This OWT maximizes his expected utility. Different from the OWT framework, our price evolution model uses stochastic interest rates and demand. Interest rates and demand are merely inputs for our model, thus, in this paper, we do not try to explain the time evolution of these functions.

Our model specifically considers the occurrence of sale and no sale conditions and how the seller responds to the no sale scenario. When the asset is not sold during the sale process, the seller decreases his reservation price and initial list price. With this dynamic update, the probability of sale increases whereas the expected payoff from the sale decreases. Our simulation results show us that it becomes more difficult to sell the real asset in the high-interest, low-demand environment and these conditions may require the seller to sell the asset below his initial reservation price.

We employ a deterministic method to update our parameters that are aligned with real-

world applications. The new seller has a different reservation price than the previous holder of the asset and faces different market conditions that shape his decision process. Current market conditions and the level of prices constitute the most effective factors in the determination of the evolution process. We combine these external factors with the seller's individual choices and, as a result, have a dynamic, time-dependent, and stochastic price process.

References

[1] Bailey, M., Muth, R. and Nourse, H. (1963), "A Regression Method for Real Estate Price Index Construction", *Journal of the American Statistical Association,* 58: 933-942

[2] Capozza, D.R., Hendershott, P.H. and Mack, C. (2004), "An Anatomy of Price Dynamics in Illiquid Markets: Analysis and Evidence from Local Housing Markets ", *Real Estate Economics*, 32(1), 1-32.

[3] Case, K.E. and Shiller, R. (1987), "Prices of Single-Family Homes Since 1970: New Indexes For Four Cities", *New England Economic Review*, Sept/Oct: 45-56.

[4] Case, K.E. and Shiller, R. (1990), "Forecasting Prices and Excess Returns in the Housing Market", *Journal of the American Real Estate and Urban Economics Association*, 18(3), 263-273.

[5] Case, B. and Quigley, J.M. (1991), "The Dynamics of Real Estate Prices", *The Review of Economics and Statistics*, 73(1), 50-58.

[6] Cubbin, J. (1974), "Price, Quality, and Selling Time in the Housing Market", *Applied Economics*, 6, 171-187.

[7] Cox, J.C., Ingersoll, J.E. and Ross, S.A. (1985), "A Theory of the Term Structure of Interest Rates", *Econometrica*, 53: 385-407.

[8] Evans, J., Henderson, V. and Hobson, D. (2007), "Optimal Timing For an Indivisible Asset Sale ", to appear in *Mathematical Finance.*

[9] Forgey, F.A., Rutherford, R.C. and Springer, T.A. (1996), "Search and Liquidity in Single-Family Housing", *Real Estate Economics*, 24(3), 273-292.

[10] Genesove, D. and Mayer, C. (1998), "Equity and Time to Sale in the Real Estate Market", *The American Economic Review*, 87(3), 255-269.

[11] Goetzmann, W. and Peng, L. (2006), "Estimating House Price Indexes in the Presence of Seller Reservation Prices", *The Review of Economics and Statistics*, 88(1): 100–112.

[12] Glower, M., Haurin, D.R., Hendershott, P.H. (1998), "Selling Time and Selling Price: The Influence of Seller Motivation", *Real Estate Economics*, 26(4), 719-740.

[13] Grenadier, S.R. and Wang, N. (2005), "Investment timing, agency, and information", *Journal of Financial Economics*, 75(3), 493-533.

[14] Haurin, D.R. (1988), "The Duration of Marketing Time of Residential Housing", *Journal of the American Real Estate and Urban Economics Association*, 16(4), 396-410.

[15] Kalra, R., Chan, C.C. and Lai, P. (1997), "Time on Market and Sales Price of Residential Housing: A Note", *Journal of Economics and Finance*, 21(2), 63-66.

[16] Knight, J.R. (2002), "Listing Price, Time on Market, and Ultimate Selling Price: Causes and Effects of Listing Price Changes", *Real Estate Economics*, 30(2), 213-237.

[17] Krainer, J. and LeRoy, S.F. (2002), "Equilibrium valuation of illiquid assets", *Economic Theory*, 19, 223-242.

[18] Lin, Z. and Vandell, K.D. (2007) "Illiquidity and Pricing Biases in the Real Estate Market", *Real Estate Economics*, 35(3), 291-330.

[19] Lucas, R.E. and Prescott, E.C. (1974), "Equilibrium Search and Unemployment", *Journal of Economic Theory*, 7, 188-209.

[20] McMillen, D.P. and Thorsnes, P. (2006) "Housing Renovations and the Quantile Repeat-Sales Price Index", *Real Estate Economics*, 34(4), 567-584.

[21] Miller, N.G. (1978), "Time on the Market and Selling Price", *Journal of the American Real Estate and Urban Economics Association*, 6(2), 164-174.

[22] Ortalo-Magné, F. and Rady S. (2006), "Housing Market Dynamics: On the Contribution of Income Shocks and Credit Constraints", *Review of Economic Studies*, 73, 459-485.

[23] Poterba, J.M. (1991), "House Price Dynamics: The Role of Tax Policy and Demography", *Brookings Papers on Economic Activity,* 143-203.

[24] Resnick, I.S. (1992), "Adventures in Stochastic Processes", Birkhauser Boston.

[25] Sarr, T. (1988), "A Note on Optimal Price Cutting Behavior under Demand Uncertainty", *The Review of Economics and Statistics*, 70(2), 336-339.

[26] Stein, J.C. (1995), "Prices and Trading Volume in the Housing Market: A Model with Down-Payment Effects", *Quarterly Journal of Economics,* 110(2), 379-406.

[27] Taylor, C. R. (1999), "Time-on-the-Market as a Sign of Quality", *Review of Economic Studies*, 66, 555-578.

[28] Wheaton, W.C. (1990), "Vacany, search, and prices in a housing market matching model", *Journal of Political Economy*, 61, 1270-1292.

[29] Williams, J.T. (1995), "Pricing real assets with costly search", *Review of Financial Studies*, 8, 55-90.

[30] Yavas, A. (1992), "A Simple Search and Barganing Model of Real Estate Markets", *Journal of the American Real Estate and Urban Economics Association*, 20(4), 533-548.

A Appendix: Auxiliary Model

We will derive the auxiliary model in this appendix. Buyers make offers with $\exp(\lambda)$ and their offers are distributed uniformly with $U(p_{\min}, p_{\max})$. The seller's reservation price is $p_{\min}$. After making an offer, a buyer may withdraw his offer with distribution $\exp(\mu)$. Interest rate is constant and equals r. The seller wants to maximize expected payoff with respect to waiting time, T. Let $N(T)$ be the number of offers received by time T and ξ_i be the offer from buyer i, B_i, at the arrival time, A_i, and $\xi_{(i)}$ be the ith minimum offer received by the seller. With these assumptions, the discounted expected payoff, $u(.)$, is a function of T, λ, μ, $p_{\min}$, $p_{\max}$, and r.

$$
\begin{aligned}
u(.) &= \mathbb{E}\left[\mathbb{E}\left[X | N(T) = n\right]\right] \\
&= \sum_{n=0}^{\infty} \mathbb{E}\Big[\xi_{(n)} e^{-rT} \mathbb{1}_{[B_n \text{ is still interested}]} + \xi_{(n-1)} e^{-rT} \mathbb{1}_{[B_{n-1} \text{ is still interested and } \xi_{(n)} \text{ is withdrawn}]} \\
&\qquad \ldots + \xi_{(1)} e^{-rT} \mathbb{1}_{[B_1 \text{ is still interested and } \xi_{(n)}...\xi_{(2)} \text{ are withdrawn}]} \Big| N(T) = n\Big] \mathbb{P}\left(N(T) = n\right) \\
&= \sum_{n=0}^{\infty} \sum_{i=1}^{n} \mathbb{E}\left[\xi_{(n-i+1)} e^{-rT} \mathbb{1}_{[B_{n-i+1} \text{ is still interested and } \xi_{(n)}...\xi_{(n-i+2)} \text{ are withdrawn}]} \Big| N(T) = n\right] \times \\
&\qquad \mathbb{P}\left(N(T) = n\right) \\
&= \sum_{n=0}^{\infty} \sum_{i=1}^{n} \mathbb{E}\left[\xi_{(n-i+1)} | N(T) = n\right] \times \mathbb{E}\left[e^{-rT} \mathbb{1}_{[B_{n-i+1} \text{ is still interested}]} \Big| N(T) = n\right] \times \\
&\qquad \mathbb{E}\left[\mathbb{1}_{[\xi_{(n)}...\xi_{(n-i+2)} \text{ are withdrawn}]} \Big| N(T) = n\right] \mathbb{P}\left(N(T) = n\right).
\end{aligned}
$$

Each of the components of the sum is as follows

$$
\mathbb{E}\left[\xi_{(i)}|N(T)=n\right] = p_{\min} + \frac{(p_{\max}-p_{\min})i}{n+1},
$$

$$
\mathbb{E}\left[\mathbb{1}_{[B_i \text{ is still interested}]}|N(T)=n\right] = \int_0^T \frac{1}{T}e^{-\mu x}dx = \frac{1}{\mu T}(1-e^{-\mu T}) := 1 - f(T),
$$

$$
\mathbb{E}\left[\mathbb{1}_{[\xi_{(n)}\cdots\xi_{(n-i+2)} \text{ are withdrawn}]}|N(T)=n\right] = [f(T)]^{i-1},
$$

$$
\mathbb{P}\left(N(T)=n\right) = \frac{(\lambda T)^n e^{-\lambda T}}{n!}.
$$

Using these components, expected payoff becomes

$$
\begin{aligned}
u(.) &= \sum_{n=0}^{\infty}\sum_{i=1}^{n}\left(p_{\min}+\frac{(p_{\max}-p_{\min})(n-i+1)}{n+1}\right)e^{-rT}(1-f(T))(f(T))^{i-1}\left(\frac{(\lambda T)^n e^{-\lambda T}}{n!}\right)\\
&= g(T)\sum_{n=0}^{\infty}\frac{(\lambda T)^n}{n!}\sum_{i=1}^{n}\left(p_{\max}(f(T))^{i-1}-\frac{(p_{\max}-p_{\min})}{n+1}i(f(T))^{i-1}\right),
\end{aligned}
$$

where $g(T)=(1-f(T))e^{-rT}e^{-\lambda T}$. We use the following facts in the final computation.

$$
\sum_{i=1}^{n}ix^{i-1} = \frac{d(\sum_{i=1}^{n}x^i)}{dx} = \frac{d(\frac{1-x^{n+1}}{1-x})}{dx} = \frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}
$$

and

$$
\sum_{i=1}^{n}x^{i-1} = \frac{1-x^n}{1-x}.
$$

Finally, the discounted expected payoff equals

$$
\begin{aligned}
u(T,\lambda,\mu,p_{\min},p_{\max},r) &= g(T)\sum_{n=0}^{\infty}\frac{(\lambda T)^n}{n!}\Bigg[\frac{p_{\max}-p_{\min}}{(n+1)(1-f(T))^2}\left(nf(T)^{n+1}-(n+1)f(T)^n+1\right)\\
&\qquad\qquad + p_{\max}\frac{1-f(T)^n}{1-f(T)}\Bigg]\\
&= \frac{-g(T)(p_{\max}-p_{\min})}{(1-f(T))^2}\left(f(T)e^{\lambda Tf(T)}-\frac{1}{\lambda T}(e^{\lambda Tf(T)}-1)-e^{\lambda Tf(T)}\right.\\
&\qquad\qquad \left. +\frac{1}{\lambda T}(e^{\lambda T}-1)\right)+\frac{p_{\max}g(T)(e^{\lambda T}-e^{\lambda Tf(T)})}{1-f(T)},
\end{aligned}
$$

where $f(T) = 1 - \frac{1}{\mu T}(1 - e^{-\mu T})$ and $g(T) = (1 - f(T))e^{-rT}e^{-\lambda T}$.

B   Appendix: Theorem 3.1

We first establish some auxiliary results that will be used while proving Theorem 3.1. Lemma B.1 gives us the formula for the conditional probability, which is conditioned on $\mathcal{F}_t$ and $N(t) = n$, that an offer value is smaller than the announced list price at its arrival time.

Lemma B.1 *Let* $\varphi(t) = \mathbb{P}\{\xi_1 < L(A_1)|\mathcal{F}_t, N(t) = n\}$ *for* $n \geq 1$. *Then,*

$$\varphi(t) = \frac{1}{\Lambda(t)} \int_0^t \lambda(a) F_\xi\big(L(a)\big) \mathrm{d}a.$$

PROOF.   Given $A_1 = a$, the event $\{\xi_1 < L(A_1)\}$ is independent of $\mathcal{F}_t$ and $N(t)$. Thus,

$$
\begin{aligned}
\varphi(t) &= \int_0^t f_{A|\{\mathcal{F}_t, N(t)=n\}}(a) \mathbb{P}\{\xi_1 < L(a)\} \mathrm{d}a \\
&= \frac{1}{\Lambda(t)} \int_0^t \lambda(a) F_\xi\big(L(a)\big) \mathrm{d}a.
\end{aligned}
$$

∎

The following lemma provides the formula for the conditional probability, which is conditioned on $\mathcal{F}_t$ and $N(t) = n$, that none of the offers arrived in the time interval $[0, t]$ is greater than the announced sale at their arrival times.

Lemma B.2 $\mathbb{P}\{\beta > t|\mathcal{F}_t, N(t) = n\} = \varphi(t)^n$ *for* $n \geq 1$.

PROOF.

$$
\begin{aligned}
\mathbb{P}\{\beta > t|\mathcal{F}_t, N(t) = n\} &= \mathbb{P}\Big(\bigcap_{i=0}^n \{\xi_i < L(A_i)\}|\mathcal{F}_t, N(t) = n\Big) \\
&= \mathbb{P}\{\xi_1 < L(A_1)|\mathcal{F}_t, N(t) = n\}^n = \varphi(t)^n.
\end{aligned}
$$

∎

Now, we calculate the conditional density of β, $f_{\beta|\{\mathcal{F}_t, N(t)=n, \beta \leq t\}}(s)$, conditioned on $\mathcal{F}_t$, $N(t) = n$, and $\beta \leq t$.

Lemma B.3 $f_{\beta|\{\mathcal{F}_t, N(t)=n, \beta \leq t\}}(s) = \frac{\lambda(s)}{\Lambda(t)}$.

PROOF.   Let $i(\beta)$ be the index of the offer at time β. Then,

$$\mathbb{P}\{\beta \leq s|\mathcal{F}_t, N(t) = n, \beta \leq t\} = \mathbb{P}\{A_{i(\beta)} \leq s|\mathcal{F}_t, N(t) = n, \beta \leq t\}.$$

Since $\beta \leq t$, we know that $i(\beta) \leq n$. Given $N(t) = n$ and $\mathcal{F}_t$, all offer arrival times are independently distributed over $[0, t]$ according to density $\frac{\lambda(s)}{\Lambda(t)}$. Thus,

$$\mathbb{P}\{\beta \leq s|\mathcal{F}_t, N(t) = n, \beta \leq t\} = \frac{\Lambda(s)}{\Lambda(t)}.$$

Thus, $f_{\beta|\{\mathcal{F}_t, N(t)=n, \beta \leq t\}}(s) = \frac{\lambda(s)}{\Lambda(t)}$. ■

In Lemma B.4, we derive the conditional probability, conditioned on $\mathcal{F}_t$ and $N(t) = n$, of an offer, which is not withdrawn up to time t and greater than the reservation price but not exceeding the list price at its arrival time, to be greater than a positive real number y.

Lemma B.4 Let $\psi(t, y) = \mathbb{P}\{\xi_1 \mathbb{1}_{\{R \leq \xi_1 < L(A_1)\}} \mathbb{1}_{\{\tau_1 \geq t - A_1\}} > y|\mathcal{F}_t, N(t) = n\}$ for $n \geq 1$. Then,

$$\psi(t, y) = \frac{1}{\Lambda(t)} \int_0^t \lambda(a)\Big(1 - F_\tau(t - a)\Big)\Big(F_\xi\big(L(a) \vee y\big) - F_\xi\big(R \vee y\big)\Big)\mathrm{d}a. \qquad (2.19)$$

PROOF.   Conditioned on $A_1 = a$, events $\{\tau_1 \geq t - A_1\}$ and $\{R \leq \xi_1 < L(A_1)\}$ are independent of each other as well as being independent of $\mathcal{F}_t$ and $N(t)$. Thus,

$$\begin{aligned}
\psi(t, y) &= \frac{1}{\Lambda(t)} \int_0^a \lambda(a)\mathbb{P}\{\tau_1 \geq t - a\}\mathbb{P}\{\xi_1 \mathbb{1}_{\{R \leq \xi_1 < L(a)\}} > y\} \\
&= \frac{1}{\Lambda(t)} \int_0^t \lambda(a)\Big(1 - F_\tau(t - a)\Big)\Big(F_\xi\big(L(a) \vee y\big) - F_\xi\big(R \vee y\big)\Big)\mathrm{d}a.
\end{aligned}$$

■

Let $p_n(t) = \mathbb{P}\{N(t) = n|\mathcal{F}_t\}$. The following summation formula will also be used in the proof of Theorem 3.1, and gives us the conditional moment generating function of a Poisson process with stochastic intensity.

Lemma B.5 *For $q > 0$, the following holds.*

$$\sum_{n=0}^{\infty} p_n(t) q^n = \exp\big(\Lambda(t)(q-1)\big).$$

We now start proving Theorem 3.1. Let $X_1(t)$ be the first term in (3.10), and $X_2(t)$ be the second term in (3.10). Let us define $P_1(t|\mathcal{F}_t) = \mathbb{E}[X_1(t)|\mathcal{F}_t]$, and $P_2(t|\mathcal{F}_t) = \mathbb{E}[X_2(t)|\mathcal{F}_t]$. We first calculate $P_1(t|\mathcal{F}_t)$.

$$P_1(t|\mathcal{F}_t) = \sum_{n=0}^{\infty} p_n(t) \mathbb{E}\big[X_1(t)|\mathcal{F}_t, N(t) = n\big].$$

Put $M(t) = \max_{0 \le i \le N(t)} \xi_i \mathbb{1}_{\{\xi_i \ge R\}} \mathbb{1}_{\{\tau_i \ge t - A_i\}}$. Then,

$$\mathbb{E}[X_1(t)|\mathcal{F}_t, N(t) = n] = \mathrm{e}^{-\int_0^t r(s)\mathrm{d}s} \mathbb{E}\big[M(t)\mathbb{1}_{\{\beta > t\}}|\mathcal{F}_t, N(t) = n\big]. \tag{2.20}$$

The expectation in (2.20) can be calculated by conditioning on the event $\{\beta > t\}$.

$$\begin{aligned}
\mathbb{E}\big[M(t)\mathbb{1}_{\{\beta > t\}}|\mathcal{F}_t, N(t) = n\big] &= \mathbb{P}\{\beta > t|\mathcal{F}_t, N(t) = n\}\mathbb{E}\big[M(t)|\mathcal{F}_t, N(t) = n, \beta > t\big] \\
&= \varphi(t)^n \mathbb{E}\big[M(t)|\mathcal{F}_t, N(t) = n, \beta > t\big].
\end{aligned}$$

Since $M(t)$ is positive, its conditional expectation can be calculated by integrating $\mathbb{P}\{M(t) > y|\mathcal{F}_t, N(t) = n, \beta > t\}$ with respect to y over $[0, \infty]$. Let us calculate $\mathbb{P}\{M(t) > y|\mathcal{F}_t, N(t) = n, \beta > t\}$.

$$\begin{aligned}
\mathbb{P}&\{M(t) > y|\mathcal{F}_t, N(t) = n, \beta > t\} \\
&= 1 - \Big(1 - \mathbb{P}\{\xi_1 \mathbb{1}_{\{\xi_1 \ge R\}}\mathbb{1}_{\{\tau_1 \ge t - A_1\}} > y|\mathcal{F}_t, N(t) = n, \xi_1 < L(A_1)\}\Big)^n \\
&= 1 - \Big(1 - \frac{\mathbb{P}\{\xi_1 \mathbb{1}_{\{R \le \xi_1 \le L(A_1)\}}\mathbb{1}_{\{\tau_1 \ge t - A_1\}} > y|\mathcal{F}_t, N(t) = n\}}{\mathbb{P}\{\xi_1 < L(A_1)|\mathcal{F}_t, N(t) = n\}}\Big)^n \\
&= 1 - \Big(1 - \frac{\psi(t, y)}{\varphi(t)}\Big)^n
\end{aligned}$$

Integrating $\mathbb{P}\{M(t) > y | \mathcal{F}_t, N(t) = n, \beta > t\}$ over y, we obtain $\mathbb{E}[M(t) | \mathcal{F}_t, N(t) = n, \beta > t]$.

$$
\begin{aligned}
\mathbb{E}[M(t) | \mathcal{F}_t, N(t) = n, \beta > t] &= \int_0^\infty \mathbb{P}\{M(t) > y | \mathcal{F}_t, N(t) = n, \beta > t\} \mathrm{d}y \\
&= \int_0^\infty \Big(1 - \big(1 - \frac{\psi(t, y)}{\varphi(t)}\big)^n \Big) \mathrm{d}y.
\end{aligned}
\tag{2.21}
$$

Noting that $\psi(t, y) = 0$ a.s. when $y > L_0$, we can further simplify (2.21) to

$$
\mathbb{E}[M(t) | \mathcal{F}_t, N(t) = n, \beta > t] = L_0 - \int_0^{L_0} \big(1 - \frac{\psi(t, y)}{\varphi(t)}\big)^n \mathrm{d}y.
$$

As a result, $\mathbb{E}[X_1(t) | \mathcal{F}_t, N(t) = n]$ is equal to

$$
\mathbb{E}[X_1(t) | \mathcal{F}_t, N(t) = n] = \mathrm{e}^{-\int_0^t r(s)\mathrm{d}s} \Big(L_0 \varphi(t)^n - \int_0^{L_0} \big(\varphi(t) - \psi(t, y)\big)^n \mathrm{d}y \Big).
$$

Now, we average $\mathbb{E}[X_1(t) | \mathcal{F}_t, N(t) = n]$ over $N(t)$. By using Fubini's theorem and Lemma B.5, we obtain

$$
P_1(t | \mathcal{F}_t) = \mathrm{e}^{-\int_0^t r(s)\mathrm{d}s} \mathrm{e}^{\Lambda(t)(\varphi(t) - 1)} \Big(L_0 - \int_0^{L_0} \mathrm{e}^{-\Lambda(t)\psi(t,y)} \mathrm{d}y \Big).
$$

Let us now calculate $P_2(t | \mathcal{F}_t)$. It is equal to

$$
P_2(t | \mathcal{F}_t) = \sum_{n=0}^\infty p_n(t) \mathbb{E}[X_2(t) | \mathcal{F}_t, N(t) = n].
$$

where $\mathbb{E}[X_2(t) | \mathcal{F}_t, N(t) = n]$ is calculated as

$$
\begin{aligned}
\mathbb{E}[X_2(t) | \mathcal{F}_t, N(t) = n] &= \mathbb{E}\big[\mathrm{e}^{-\int_0^\beta r(s)\mathrm{d}s} \xi_{i(\beta)} \mathbb{1}_{\{\beta \le t\}} | \mathcal{F}_t, N(t) = n\big] \\
&= \mathbb{P}\{\beta \le t | \mathcal{F}_t, N(t) = n\} \mathbb{E}\big[\mathrm{e}^{-\int_0^\beta r(s)\mathrm{d}s} \xi_{i(\beta)} | \mathcal{F}_t, N(t) = n, \beta \le t\big] \\
&= \big(1 - \varphi(t)^n\big) \int_0^t f_{\beta | \{\mathcal{F}_t, N(t) = n, \beta \le t\}}(a) \mathrm{e}^{-\int_0^a r(s)\mathrm{d}s} \mathbb{E}\big[\xi_{i(a)}\big].
\end{aligned}
$$

42

The last equality follows from the fact that the magnitude of $\xi_{i(\beta)}$ is independent of $\mathcal{F}_t$ and $N(t)$, and only depends on a given the event $\{\beta = a\}$. Note also that

$$
\begin{aligned}
\mathbb{E}[\xi_{i(a)}] &= \mathbb{E}[\xi_1 | \xi_1 \geq L(a)] \\
&= \frac{\int_{L(a)}^{\infty} x f_\xi(x) \mathrm{d}x}{1 - F_\xi\big(L(a)\big)}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathbb{E}[X_2(t)|\mathcal{F}_t, N(t) = n] &= \big(1 - \varphi(t)^n\big) \int_0^t \int_{L(a)}^{\infty} f_{\beta|\{\mathcal{F}_t, N(t)=n, \beta \leq t\}}(a) \frac{\exp(-\int_0^a r(s)\mathrm{d}s)}{1 - F_\xi\big(L(a)\big)} x f_\xi(x) \mathrm{d}x \mathrm{d}a \\
&= \big(1 - \varphi(t)^n\big) \int_0^t \int_{L(a)}^{\infty} \frac{\lambda(a) \exp\big(-\int_0^a r(s)\mathrm{d}s\big)}{\Lambda(t)\big(1 - F_\xi\big(L(a)\big)\big)} f_\xi(x) x \mathrm{d}x \mathrm{d}a.
\end{aligned}
$$

$P_2(t|\mathcal{F}_t)$ is obtained by averaging $\mathbb{E}[X_2(t)|\mathcal{F}_t, N(t) = n]$ over $N(t)$. By using Lemma B.5, we obtain

$$
P_2(t|\mathcal{F}_t) = \big(1 - e^{\Lambda(t)(\varphi(t)-1)}\big) \int_0^t \int_{L(a)}^{\infty} \frac{\lambda(a) \exp\big(-\int_0^a r(s)\mathrm{d}s\big)}{\Lambda(t)\big(1 - F_\xi\big(L(a)\big)\big)} f_\xi(x) x \mathrm{d}x \mathrm{d}a.
$$

This completes the proof since $P(t|\mathcal{F}_t) = P_1(t|\mathcal{F}_t) + P_2(t|\mathcal{F}_t)$.

C Appendix: Parameter Assumptions

Parameter	Value
Arrival intensity (λ)	5
Withdrawal intensity (μ)	5
Interest rate (r)	0.1
Reservation price (R)	140
List price (L)	180
Selling motivation (γ)	0.1
$p_{\min}$	100
$p_{\max}$	200

Table 1: **Default parameter values in OWT analysis in Section 2**

Parameter	Value
Withdrawal intensity (μ)	10
Occupation period (O)	$U(4,6)$
Rate of personal crisis	$Exp(10)$
Initial reservation price(R_0)	140
Initial list price (L_0)	200
Selling motivation (γ)	0.8
Interest rate threshold (ϕ)	0.06
$p_{\min}$	100
$p_{\max}$	200
K_1	0.5
K_2	1000
θ	0.1
σ	0.08
κ	0.25
r_0	0.09

Table 2: **Default parameter values in the simulation**