Transform Analysis for Pricing American Options under Low-Dimensional Stochastic Volatility Models

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ABSTRACT

This paper presents a new transform-based approach for *path-independent* lattice construction for pricing American options under low-dimensional stochastic volatility models. We derive multidimensional transforms which allow us to construct efficient path-independent lattices for virtually *all* low-dimensional stochastic volatility models given in the literature including one volatility factor-based stochastic volatility (SV) models, two volatility factors-based SV models, stochastic volatility jump (SVJ) models with one and two jump factors in the asset returns, and SVJ models with jumps in both asset returns and volatility. The lattice-based approximations of the prices of European options converge rapidly to their true prices obtained using quasi-analytical solutions.

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The breakthroughs in European option pricing have outpaced those in American option pricing with the discovery of Heston [1993] stochastic volatility model, followed by the stochastic volatility jump (SVJ) models of Bakshi, Cao, and Chen [1997], Bakshi and Madan [2000], Bates [1996, 2000, 2006], Duffie, Pan, and Singleton [2000], and Pan [2002]). The hugely popular book, *The Volatility Surface*, by Gatheral [2007] reveals immense practitioner interest in the SVJ models for pricing equity options. To get more insight on the challenges in pricing American equity options, consider the following classification of the option pricing models, based upon the type of stochastic processes followed by the state variables:

i) low-dimensional models with non-stochastic volatility,

ii) high-dimensional models with non-stochastic volatility,

iii) low-dimensional models with stochastic volatility,

iv) high-dimensional models with stochastic volatility.

Examples of the first type of models include the Black and Scholes [1973] model with constant volatility, the Cox and Ross [1976] CEV model with state variable-dependent volatility, and the jump-diffusion models of Merton [1976] and Kou [2002] with constant volatility and constant jump distribution parameters. At least three methods, given as the lattice method, the finite difference method, and the analytical approximation method, have been successful at pricing American options under these models. The lattice method was proposed by Cox, Ross, and Rubinstein [1979] for the Black and Scholes model. Neslon and Ramaswamy [1990] extend this method to build efficient recombining trees for the Cox and Ross's CEV model with state variable-dependent volatility. Amin [1993] extends this method to build efficient multinomial recombining trees for the Merton's jump diffusion model. The finite difference method was initially proposed by Brennan and Schwartz [1978] for the Black and Scholes model, and has been extended by Zhang [1997] and Carr and Hirsa [2003] for the Merton's jump diffusion model. Finally, analytical approximations for pricing American options have been derived by Barone-Adesi and Whaley [1987], Kim [1990] and Carr, Jarrow, and Myneni [1992] for the Black and Scholes model, and have been extended by Gukhal [2001] and Chiarella and Ziogas [2004] for the Merton's jump diffusion model, and by Kou and Wang [2004] for the double exponential jump diffusion model of Kou [2002].

The absence of analytical approximations and practical limitations on computational time prevent the three methods given above for pricing American options under the second type of models with many state variables. The significant breakthrough for pricing American options under the high-dimensional models with non-stochastic volatility is provided by Longstaff and Schwartz (LS) [2001], who building on the initial work by Carriere [1996] and Tsitsiklis and Van Roy [1999], develop an innovative least squares Monte Carlo (LSM) approach that uses least square regressions to estimate the conditional expected payoff to the option holder from continuation. LS demonstrate the effectiveness of this approach for high-dimensional models using the examples of pricing an American swaption using a "twenty-factor" string model, and an American option on the maximum of five risky asset prices. The solutions, which would normally take hours or days to compute using the traditional methods, are obtained within minutes using the LSM approach.

Though the LSM approach is powerful for the second type of models, it is not generally recommended for the first type of models.[1] For example, the traditional binomial method, the Neslon and Ramaswamy [1990] extension of the binomial method, and the Amin [1993] multinomial extension of the binomial method are significantly more efficient than the LSM approach for pricing American options under the Black and Scholes model, the Cox and Ross CEV models, and the Merton jump diffusion model, respectively. Further, as mentioned earlier, finite differencing methods and analytical approximations also work well under the first type of models. The traditional methods can generate the price of an American option within a fraction of a second or a few seconds, while the LSM approach requires a significant fraction of a minute, or even longer to run the least squares regressions using the simulated data with tens of thousands of paths.

Clearly, the traditional methods (i.e., lattice methods, finite difference methods, and analytical approximation methods) are well suited for the first type of models, and the LSM approach is advantageous for the second type of models. However, it is not immediately obvious if the LSM approach is useful for the third type of models characterized by low-dimensional stochastic processes with *stochastic volatility.* The main difficulty here is that the ordinary least squares regressions can no longer be used for estimating the conditional expected payoff to the option holder from continuation, due to the hetroscedastic error terms when the underlying processes follow stochastic volatility, as in the models of Heston [1993], Hull and White [1987], Bates [2000, 2006],

[1]The only exception is when the underlying state variables follow non-Markovian processes, making the lattice methods difficult to use for pricing American options. However, as shown by Amin and Morton [1994], for low-dimensional, non-Markovian HJM models, non-recombining trees are quite accurate even with as few as 8 to 10 steps. This finding may not hold over other classes of non-Markovian models, however.

Jiang and Oomen [2006], Bakshi, Cao, and Chen [1997, 2000], Pan [2002], Duffie, Pan, and Singleton [2000], and others. Though, in principle, some type of non-linear regression such as GMM, etc., could be used, this would further slow down the computational speed of the LSM procedure, as doing GMM would require time-consuming non-linear optimizations to be performed at every time step of the simulation.

As an alternative to the LSM approach, this paper derives a multidimensional transform that allows efficient path-independent lattice construction for virtually all low-dimensional stochastic volatility models (with and without jumps) given in the literature, using which American options can be priced accurately within seconds. Hence, even if some non-linear regression methods can be used to make the LSM procedure applicable to the third type of models, doing this would be relatively inefficient since the lattice methods derived in this paper can achieve the same objective with significantly higher computational efficiency. Using the multidimensional transform derived in this paper, we show how to construct efficient path-independent lattices for the following types of low-dimensional stochastic volatility models:

i) one volatility factor-based stochastic volatility (SV) models (e.g., Chesney and Scott [1989], Heston [1993], Hull and White [1987], Stein and Stein [1991], and Wiggins [1987]);

ii) two volatility factors-based stochastic volatility (SV2) models (e.g., Bates [2000], and Jiang and Oomen [2006]);

iii) stochastic volatility jump models with one volatility factor and one jump factor (SVJ) and one volatility factor and two jump factors (SVJ2) (e.g., Andersen, Benzoni, and Lund [2002], Bates [1996, 2000, 2006], Bakshi, Cao, and Chen [1997, 2000], Chacko and Viceira [2003], and Pan [2002]);

iv) two volatility factor-based models with one jump factor (SV2J) (e.g., Jiang and Oomen [2006]).

Much econometric research has already investigated the empirical properties of the above models (for example, see Bates [2000, 2006], Chacko and Viceira [2003], Chernov and Ghysels [2000], Eraker [2004], Eraker, Johannes, and Polson [2003], Jiang and Oomen [2007], and Pan [2002], among others). These studies finds jump to be an important feature in addition to stochastic volatility for explaining the features of equity index returns and option prices. Most notably, Eraker [2004] and Eraker, Johannnes, and Polson [2003] find that jumps in volatility are also required - in addition to jumps in stock returns - for explaining the index option prices, and returns data on the S&P 500 and Nasdaq 100. The existence of a rich econometric literature on the low-dimensional stochastic volatility models makes the results of our paper even more useful for future empirical investigations into American option pricing under these models.

Before outlining our approach to American option pricing for the above models, we would like to highlight a related approach that uses GARCH models for pricing equity options in the presence of stochastic volatility. This approach developed by Duan [1995] for European option pricing using simulation methods, has been extended to American option pricing by Ritchken and Trevor [1999] using a lattice approximation method. Duan [1996, 1997] also shows that virtually all of the bivariate diffusion stochastic volatility models (e.g., Heston [1993], Hull and White [1987], and others) can be obtained as special cases of a family of generalized GARCH models. This allows the lattice algorithm of Ritchken and Trevor (RV) to be applicable to all of the bivariate diffusion stochastic volatility models, as well.

Since the variance process under the GARCH method is a *path-dependent* process - implying an explosive number of variances at any given node of the asset price tree - RV use the approximation methods of Hull and White [1993] and Ritchken, Sankarasubramanian, and Vijh [1993], which keep track of only the minimum and the maximum values of variances at each node, and then use linear interpolation to generate K variances at that node. With suitable parameterizations, RV find that a value of $K = 20$ is generally sufficient for accurate pricing of options using the GARCH model. Due to the volatility interpolations and other approximations involved in the modeling of the path-dependent variance process, the RV lattice algorithm is relatively slower and less accurate than a typical path-independent binomial or trinomial model. The RV lattice algorithm also requires a forward dynamic program to determine the values of the minimum and maximum variance at each node.

In contrast to the RV algorithm, this paper derives a multidimensional transform in order to model both the asset price process and the volatility process as *path-independent* trees. The efficiency and accuracy of our approach is of similar order as a two-factor Cox, Ross, and Rubinstein [1979] model, which represents a significant improvement over the RV algorithm for the special case of the bivariate diffusion stochastic volatility models. Our approach also generalizes parsimoniously to other low-dimensional stochastic volatility models, such as those with two volatility state variables, or jump factors in the asset returns. In contrast, the GARCH option pricing models cannot allow more than one volatility state variable, and it is virtually impossible to extend the RV lattice approximation method to the GARCH-jump models (e.g., Duan, Richken, and Sun [2006]), which allow jumps in the asset return and/or the volatility process.

Though a few other researchers have provided different approaches for pricing American options under low-dimensional stochastic volatility models, none are as general in terms of applicability to a diverse set of models with stochastic volatility and jumps, or numerically as efficient as our approach (e.g., see Clarke and Parrott [1999], Finucane and Tomas [1996], Guan and Xiaoqiang [2001], Hilliard and Schwartz [1996], Leisen [2000], Tzavalis and Wang [2003], and Chiarella and Ziogas [2005]). For example, Leisen [2000] shows how to construct recombining lattices for the class of bivariate diffusion stochastic volatility models. Leisen's approach requires both forward induction and backward induction with a complex tree structure of the order $O(N^4)$ for an N-step tree. The requirement of forward induction puts huge burden on the computer memory and the high order (i.e., $O(N^4)$) for the tree structure slows down the Leisen approach significantly. Though Hilliard and Schwartz [1996] generate a path independent lattice for the special case of Hull and White [1987] model using a two-dimensional transform (along the lines of Nelson and Ramaswamy [1990]), their approach cannot be generalized to any other stochastic volatility models. Chiarella and Ziogas [2005] extend McKean's [1965] incomplete Fourier transform method to solve the two-factor partial differential equation for the price and early exercise surface using numerical methods. However, Chiarella and Ziogas approach is limited only to the case of Heston model. Clarke and Parrott [1999] consider a multigrid implicit finite difference scheme for pricing American options under bivariate diffusion stochastic volatility models. This approach requires analytic strike-price related transformation of asset prices, and adaptive time-stepping for efficient computation of option prices, and cannot be generalized easily to models with jumps in asset prices and/or volatility. Guan and Xiaoqiang [2001] use an interpolation-based approach to construct recombining trees for pricing American options under stochastic volatility models. However, their interpolation-based approximations lead to large pricing errors for long-dated options.

This paper is organized as follows. Section 1 demonstrates the main difficulty in the construction of a path-independent lattice for low-dimensional stochastic volatility models, using Heston's model as an example. The traditional lattice technique leads to negative transition probabilities in the tree for stock return when the volatility of the stock return becomes small relative to the absolute size of the correlation between the stock return and its volatility. In order to remove the negative transition probabilities, Section 2 proposes a multi-dimensional transform that is *uncorrelated* with the volatility state variable(s). A zero correlation between the transform process and the volatility processes allows path-independent lattice construction without the occurrence of negative transition probabilities for virtually all stochastic volatility models given in the literature with one and two volatility factors. Section 3 extends the multi-dimensional transform to stochastic volatility models with jumps. The results are derived for models with one and two volatility factors combined with one and two jumps (i.e., SVJ, SV2J, and SVJ2 models outlined earlier). The details of the transforms used and the lattice-based simulation results for pricing European and American options are given in each of the sections 2 and 3 for the models considered in these sections.

I. The Problem of Negative Transition Probabilities using a *Simple* Path-Independent Lattice for Stochastic Volatility Models

This section outlines the main difficulty underlying the construction of a simple path-independent lattice for low-dimensional stochastic volatility models using the specific example of Heston [1993] model. The *simple* path-independent lattice has computational simplicity similar to that of the two-factor binomial tree of CRR [1979], and hence, is simpler than the lattices considered by Ritchken and Trevor [1999] and Leisen [2000]. We demonstrate why such a simple path-independent lattice leads to negative transition probabilities when the volatility of the stock return becomes small. This problem applies to all stochastic volatility models that allow non-zero correlation between the stock return and its volatility, and remains the main impediment in the construction of simple path-independent lattices for these models. This demonstration motivates the development of the multi-dimensional transform in the next section, which allows the construction of the simple path-independent lattice of the transform, while keeping the transition probabilities positive.

To illustrate this problem, consider the risk-neutral stochastic processes for the log of the stock price and the volatility process under the Heston model, given as follows:

$$dy(t) = \left(r - v(t)/2\right)dt + \sqrt{v(t)}\,dZ(t) \tag{1}$$

and,

$$dv(t) = a\,(m - v(t))dt + \sqrt{v(t)}\,dW(t) \tag{2}$$

where $y(t) = \ln S(t)$, $S(t)$ is the stock price, $v(t)$ is the instantaneous variance of the stock return, and $dZ(t)$ and $dW(t)$ are the associated Wiener processes, respectively. Let the conditional correlation between the changes in $y(t)$ and $v(t)$ be given as ρ, where by definition $dZ(t)dW(t) = \rho dt$.

A necessary (but not sufficient) condition for constructing a simple two-dimensional path-independent lattice for the log stock price process and the variance process is that separate path-independent trees must exist for each of these processes. Using the method of Nelson and Ramaswamy (NR) [1990], a discrete-time path-independent trinomial tree is constructed for the variance process, by using the NR transform.

Let $v_h(t)$ represent the discrete-time trinomial-tree approximation of the $v(t)$ process, and let the up, middle, and down node values of the variance after a discrete time-interval h, obtained using the NR method, be given as, V_h^u, V_h^m, and V_h^d. The intuition behind the NR transform is that it shifts the node values on a *non-recombining tree* by the order of O(h), which shifts the up, middle, and down probabilities, slightly. But this shift is done in a manner that allows the nodes in the second, third, and higher steps to recombine, while matching the mean and the variance of the underlying process at each time step in the order of O(h).

Let $y_h(t)$ be the discrete-time trinomial-tree approximation of the $y(t)$ process. Construction of a path-independent trinomial tree for $y(t)$ process requires that the span between the up and down nodes increases as the variance $v(t)$ increases (see Ritchken and Trevor [1999]). Let $\sqrt{\bar{v}}$ represent the value used for spacing the adjacent nodes on the grid for the trinomial tree, such that $\sqrt{\bar{v}}\sqrt{h}$ is the vertical distance between any two adjacent nodes on the entire grid. Then a one-step trinomial tree for the $y(t)$ process will lead to the following three nodes at time $t + h$:

$$
\begin{aligned}
y_h^u(t+h) &= y_h(t) + J_h(t)\sqrt{\bar{v}}\sqrt{h} \\
y_h^m(t+h) &= y_h(t) \\
y_h^d(t+h) &= y_h(t) - J_h(t)\sqrt{\bar{v}}\sqrt{h}
\end{aligned}
\tag{3}
$$

where the variable $J_h(t)$ is a function of $v_h(t)$ process, and takes integer values greater or equal to 1. The parameter $\sqrt{\bar{v}}$ used for defining node spacing for the grid structure can be set equal to the initial volatility $\sqrt{v(0)}$. The integer variable $J_h(t)$ is used to increase the span between the up node and the down node, as the variance increases, which prevents the probability of the middle node from becoming negative. The value of $J_h(t)$ is computed as follows. For all $t \geq 0$,

$$
J_h(t) = \begin{cases} CEILING\left(\dfrac{\sqrt{v_h(t)}}{\sqrt{\bar{v}}}\right), \text{ if } v_h(t) > 0 \\ 1, \text{ if } v_h(t) = 0. \end{cases}
\tag{4}
$$

where $CEILING(z)$, such that $z > 0$, defines the first integer value that is greater or equal to z. Restricting $J_h(t)$ to be an integer value ensures that the tree *recombines* at the future nodes, while the specific definition in equation (4) ensures that the probability associated with the middle node value $y_h^m(t+h)$ does not become negative as the variance increases.

By matching the mean and variance of the discrete-time $y_h(t)$ process over the time interval t to $t + h$, such that these moments converge to instantaneous moments of the continuous-time $y(t)$ process defined in equation (1), in the limit as $h \to 0$, we obtain the three probabilities associated with the three nodes in equation (3), as follows:

$$p^{u}_{h,y} = \frac{1}{2}\frac{v_h(t)}{\left(J_h(t)\sqrt{\bar{v}}\right)^2} + \frac{1}{2}\frac{\left(r - v_h(t)/2\right)}{J_h(t)\sqrt{\bar{v}}}\sqrt{h}$$

$$p^{d}_{h,y} = \frac{1}{2}\frac{v_h(t)}{\left(J_h(t)\sqrt{\bar{v}}\right)^2} - \frac{1}{2}\frac{\left(r - v_h(t)/2\right)}{J_h(t)\sqrt{\bar{v}}}\sqrt{h} \qquad (5)$$

$$p^{m}_{h,y} = 1 - \frac{v_h(t)}{\left(J_h(t)\sqrt{\bar{v}}\right)^2}$$

The joint lattice of the $y_h(t)$ process and the $v_h(t)$ process requires the computation of joint probabilities. Using standard approaches such as Hull and White [1994], the joint probabilities take the following form:

$$p(Y^i_h, V^j_h) = p^i_{h,y}\ p^j_{h,v} + c_{ij}\,r \qquad (6)$$

for $i, j = u, m$, and d, where c_{ij} are constants, and r is the correlation coefficient defined earlier. Now consider following two cases.

Case 1. The process $v_h(t)$ is strictly greater than 0.

It is well known that under specific parameter choice, the square root process remains strictly greater than zero. Under this case, the three probabilities in equation (5) remain positive as $h \to 0$. However, as $v_h(t)$ becomes small, the probabilities $p^{u}_{h,y}$ and $p^{d}_{h,y}$ in equation (5) become small, and hence some of the joint probabilities in equation (6) can become negative, depending on the size and magnitudes of the correlation coefficient r and the coefficients c_{ij}.

Case 2. The process $v_h(t)$ converges to 0.

Under this case, as $v_h(t)$ converges to zero, the probability of the down node, $p^d_{h,y}$, in equation (5) becomes negative in the order O($\sqrt{h}$), because $r > 0$. It is easy to show that this problem cannot be solved by either allowing the trinomial lattice of $y_h(t)$ to move in a different direction with multiple node jumps, or by imposing artificial bound on $p^d_{h,y}$, such that it remains greater than zero. Further, both probabilities $p^u_{h,y}$ and $p^d_{h,y}$ become of the order O($\sqrt{h}$), and hence, similar to case 1, the joint probabilities in equation (6) can become negative, depending on the size and magnitudes of the correlation coefficient r and the coefficients c_{ij}.

Though the problem of negative transition probabilities under the above two cases has been demonstrated using the Heston model, similar demonstration can also made when constructing simple path-independent lattices for other stochastic volatility models. It can be shown that the problem outlined in the first case applies to all stochastic volatility models in which the stock return and the volatility process have a *non-zero* conditional correlation. The problem outlined in the second case applies to all stochastic volatility models in which the instantaneous variance of the $y(t)$ process converges to zero because of the singularity reached by the $v(t)$ process.

The next section derives a multidimensional transform that allows building a simple path independent lattice – i.e., with the similar simple structure as for the lattice developed earlier in this section for the Heston model - that disallows negative transition probabilities for all for stochastic volatility models given in the finance literature. The multidimensional transform is derived as a function of the stock price, and the volatility variables such that the stochastic process of the transform remains conditionally *uncorrelated* with the stochastic processes of the volatility variables. Applying this transform allows the construction of a joint lattice of the transform and the volatility variables, without requiring the correlation term (as in equation (6)), which eliminates the first reason for causing negative transition probabilities. The transform also uses a time-dependent deterministic term, which forces the instantaneous mean of the transform process to *converge to zero*, when the instantaneous variance of the transform process converges to zero. This eliminates the second reason for causing negative transition probabilities as outlined in case 2 above. The lattice of the stock price process is obtained by using the inverse transform.

A MULTI-DIMENSIONAL TRANSFORM FOR STOCHASTIC VOLATILITY MODELS

Assume that the stock price $s(t)$, follows an N-dimensional stochastic process under the risk-neutral measure, given as:

$$\frac{ds(t)}{s(t)} = r\,dt + \sum_{i=1}^{N} \psi_i(v_i)\,dZ_i(t) \tag{7}$$

where the Wiener processes $Z_i(t)$ are mutually independent, and $\psi_i(v_i)$ are continuous differentiable functions of the state variables v_i. The risk-neutral stochastic processes followed by the state variables are given as:

$$dv_i(t) = \alpha_i(\mu_i - v_i(t))\,dt + \varphi_i(v_i)\,dW_i(t) \tag{8}$$

where the Wiener processes $W_i(t)$ are mutually independent, and $j_i(v_i)$ are continuous differentiable functions of the state variables v_i. The correlations between the diffusion factors related to stock returns and the corresponding volatility factors are given as follows:

$$
\begin{aligned}
dZ_i(t)dW_i(t) &= r_i dt, \text{ for all } i = 1,2,..., N, \text{ and} \\
dZ_i(t)dW_j(t) &= 0, \text{ for all } i \neq j
\end{aligned}
\tag{9}
$$

Equations (7) through (9), nest virtually all continuous-time stochastic volatility (SV) models in the equity option pricing literature. Specifically, the SV models of Chesney and Scott [1989], Heston [1993], Hull and White [1987], Stein and Stein [1991], and Wiggins [1987] can be derived with appropriate specifications of the functions $y(v)$ and $j(v)$ with one diffusion factor and one volatility factor. Also, the SV2 models (i.e., based upon two diffusion factors and two volatility factors) considered in Alizadeh, Brandt, and Diebold [2002], Bates [2000], Chacko and Viceira [2003], and Jiang and Oomen [2007] can be derived with $N = 2$, and $y_i(v_i)$ and $j_i(v_i)$ given as square-root functions.

The above model uses $2N$ factors, given as N diffusion factors and N volatility factors, of which N pairs are correlated as shown in equation (9). Generating efficient recombining lattices for such a model is difficult even with $N = 1$ or 2. The following proposition outlines the main result of this paper, given as the multidimensional transform which reformulates the Markovian system given by equations (7) through (9) with $N + 1$ conditionally correlated state variables into an equivalent Markovian system with $N + 1$ *conditionally independent* state variables. This transform allows construction of parsimonious recombining lattices for virtually all of continuous-time stochastic volatility models given in the literature.

Proposition 1. *For all stochastic processes nested in equations (7) through (9),the multidimensional transform given as:*

$$y(t) = \ln\left(s(t)\right) - \sum_{i=1}^{N} r_i \int^{v_i(t)} \left(\frac{y_i(u)}{j_i(u)} \right) du - h(t) \tag{10}$$

is conditionally independent of each of the volatility processes, $v_i(t)$ *(i.e.,* $dy(t)dv_i(t)=0$*), where* $h(t)$ *is a deterministic function of time.*

Proof: Using Ito's lemma, the stochastic process for $y(t)$ is given as:

$$dy(t) = m_y(t)dt + s_y(t)dZ_y(t) \tag{11}$$

where,

$$\begin{aligned} m_y(t) = {} & r - \frac{\partial h(t)}{\partial t} - \frac{1}{2}\sum_{i=1}^{N} y_i^2(v_i) - \sum_{i=1}^{N} r_i \frac{y_i(v_i)}{j_i(v_i)} a_i (m_i - v_i(t)) \\ & + \frac{1}{2}\sum_{i=1}^{N} r_i \left(\frac{\partial j_i(v_i)}{\partial v_i} y_i(v_i) - \frac{\partial y_i(v_i)}{\partial v_i} j_i(v_i) \right) \end{aligned} \tag{12}$$

$$s_y(t) = \sqrt{\sum_{i=1}^{N} y_i^2(v_i)\left(1 - r_i^2\right)} \tag{13}$$

and the Wiener process $dZ_y(t)$ is defined as:

$$dZ_y(t) = \frac{\sum_{i=1}^{N} y_i(v_i)\left(dZ_i(t) - r_i dW_i(t)\right)}{\sqrt{\sum_{i=1}^{N} y_i^2(v_i)\left(1 - r_i^2\right)}} \tag{14}$$

Using the definition of the correlation terms in equation (9), and the definition of $dZ_y(t)$ in equation (14), it follows that $dZ_y(t)dW_i(t) = 0$, for all $i = 1, 2, \ldots, N$. Hence, the multidimensional transform $y(t)$ is conditionally independent of each of the volatility processes, $v_i(t)$.

Equation (11) together with equation (8) provides an equivalent Markovian system, which requires a joint evolution of $N + 1$ conditionally independent state variables, instead of $N + 1$ conditionally correlated state variables given by equations (7) through (9). In order to build a recombining lattice for pricing American options under stochastic volatility models, we build a separate recombining tree for each of the N volatility processes $v_i(t)$, and the $y(t)$ process. Then using the *conditional independence* between these $N + 1$ processes, and the inverse transform given in the next section, we obtain a recombining lattice for $s(t)$ by joining the $N + 1$ recombining trees, with joint probabilities computed as simple products of the marginal probabilities at each node of the recombining lattice.

The definition of the transform $y(t)$ in equation (10) contains three separate terms. The first term is the standard log transformation of the stock price, which allows the construction of a recombining tree for the $y(t)$ process and is needed even in the absence of volatility factors (see Cox, Ross, and Rubinstein [1977] and Nelson and Ramaswamy [1990]), as in the case of binomial tree construction for the standard Black and Scholes [1973] model. The second term eliminates the correlation between the $y(t)$ process and the N volatility processes, thereby allowing the Markovian system given in equations (7) through (9) to be represented with $N + 1$ conditionally independent processes. The third term $h(t)$, is a deterministic function of time. The next section demonstrates that negative probabilities arise for some nodes of the tree of the $y(t)$ process, *if* $m_y(t)$ does not equal zero, when $s_y(t)$ becomes zero. Proposition 2 given in the next section assumes that an appropriate specification of term $h(t)$ exists, which allows the term $m_y(t)$ to go to zero when $s_y(t)$ goes to zero.

LATTICE CONSTRUCTION FOR STOCHASTIC VOLATILITY MODELS

For the purpose of lattice construction, let $S(t)$, $V_i(t)$, and $Y(t)$, denote the discrete-time variables, that correspond to the continuous-time variables, $s(t)$, $v_i(t)$, and $y(t)$, respectively, in equations (7) through (10). To get some insight on the multidimensional lattice construction for stochastic volatility models, first consider the one-dimensional trees for the volatility variables $V_i(t)$, for the one volatility factor-based SV1 models, as well as the two volatility factors-based SV2 model given in Table I.

Tree Construction for the Volatility Variables

The first row of Table I specifies the functions $j_i(.)$ (see equation (8)) for various stochastic volatility models with both one volatility factor and two volatility factors. Specifically, we consider the SV1 models (i.e., based on a single volatility factor) of Hull and White [1987], Chesney and Scott [1989], Stein and Stein [1991], and Heston [1993], and the SV2 model (i.e., based on two volatility factors) given in Alizadeh, Brandt, and Diebold [2002], Bates [2000], Chacko and Viceira [2003], and Jiang and Oomen [2007]. For the Hull and White model, the volatility variable follows a lognormal distribution, and hence, the standard binomial method can be used for tree construction of this variable. For other SV1 models given by Chesney and Scott, and Stein and Stein, the volatility variable follows the mean-reverting Vasicek [1977] process, and the binomial Vasicek trees can be constructed.

For the case of Heston model, the volatility variable follows the well-known square root process. For this process, many authors including Li, Ritchken, and Sankarasubramanian [1995], and Acharya and Carpenter [2002] have demonstrated the application of the Nelson and Ramaswamy (NR) [1990] transform for generating a recombining tree. We use a slightly modified version of this approach given as the truncated trinomial tree by Nawalkha, Beliaeva, and Soto (NBS) [2007], which corrects a small error in the original NR transform when the short rate hits the zero boundary, and truncates the tree at the zero value of the transform, reducing the number of nodes significantly. Finally, since the two volatility variables under the SV2 model are uncorrelated, the NBS trinomial method is applied separately for modeling the one-dimensional trees for both these variables.

Tree Construction for the Multidimensional Transform

The main contribution of this paper is the derivation of the multidimensional transform in proposition 1, which can be used to build a parsimonious recombining lattice for the stock price that captures the correlations between the stock price and the volatility variables. Doing this, however, requires the construction of a tree for the multidimensional transform, in addition to the trees of the volatility variables discussed above. The second row of Table I specifies the functions $y_i(.)$ (see equation (7)), and the third row of this table derives the specific multidimensional transforms for the various SV1 and SV2 models using equation (10). The fourth row of Table I defines the deterministic term $h(t)$ that is used for avoiding negative probabilities for some of the stochastic volatility models when the volatility of the transform become zero.

The discrete time processes $Y(t)$ for the multidimensional transform is represented as follows:

$$\Delta Y(t) = m_Y(t)\Delta t + s_Y(t)\Delta Z_Y(t) \tag{15}$$

where $m_Y(t)$ and $s_Y(t)$ are obtained by replacing $v_i(t)$ with $V_i(t)$, in the definitions of $m_y(t)$ and $s_y(t)$ given in equations (12) and (13), respectively. The terms $m_Y(t)$ and $s_Y(t)$ for the various stochastic volatility models are specified in the fifth row and the sixth row of Table I, respectively. The term $DZ_Y(t)$ is the discrete-time approximation of the continuous change in the Wiener process, $dZ_y(t)$ defined in equation (14). The procedure to build the tree for $Y(t)$ transform is the same for all models nested in equations (7) through (9), and is outlined as follows. The discrete-time $Y(t)$ process represented in equation (15) is modeled as a trinomial tree, with the up and down nodes determined by a *changing node span*. However, before modeling the changing node span, first consider the normal node span, as shown in Figure 1.

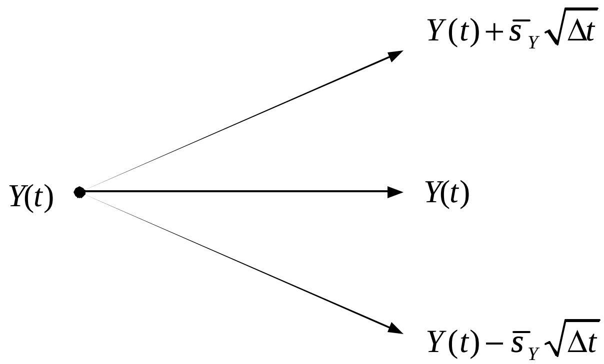

Figure 1. Normal node span of the trinomial tree for the $Y(t)$ process

The parameter $\bar{s}_Y$ is used to define the *normal* node span for building the grid for the tree for the $Y(t)$ process. The normal node span is defined as the distance between the up node and the down node in Figure 1, or:

$$\text{Normal node span} = 2\bar{s}_Y\sqrt{\Delta t} \tag{16}$$

The parameter $\bar{s}_Y$ used in defining the normal node span can be set equal to the initial volatility of the $Y(t)$ process, given as $s_Y(0)$ (see equation (13)), or in case the initial volatility is much lower or much higher than the expected volatility of $Y(t)$ process in the time-window over which the option is being priced, then $\bar{s}_Y$ can be set close to the expected volatility of $Y(t)$ process. Since the stochastic process of $Y(t)$ does not follow constant volatility, the up node and the down nodes are chosen to match the volatility $s_Y(t)$, at any given time t. This is done by changing the node span as shown in Figure 2 using the integer function $k(t)$, computed as the first positive integer that is greater or equal to the ratio $s_Y(t)/\ \bar{s}_Y$, or:

$$k(t) = \begin{cases} CEILING\left(\dfrac{s_Y(t)}{\bar{s}_Y}\right) = CEILING\left(\dfrac{\sqrt{\sum_{i=1}^{N} y_i^2(V_i)\left(1 - r_i^2\right)}}{\bar{s}_Y}\right), & \text{if } s_Y(t) > 0 \\ 1, & \text{o.w.} \end{cases} \quad (17)$$

where as mentioned earlier, $V_i(t)$ denotes the ith discrete-time volatility process, which approximates the continuous-time $v_i(t)$ process for all i = 1,2,…N.

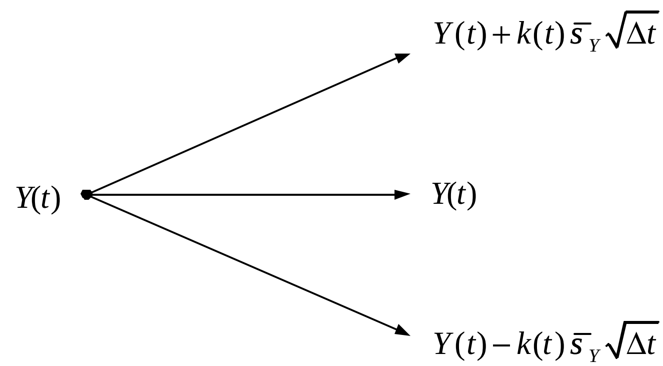

Figure 2. Changing node span of the trinomial tree for the $Y(t)$ process

The changing node span in Figure 2 is the distance between the up node and the down node for the trinomial tree for $Y(t)$, at time t, and is defined as follows:

$$\text{Changing Node Span} = 2k(t)\bar{s}_Y\sqrt{\Delta t} \quad (18)$$

It can be seen that the changing node span is always an integer multiple of the normal node span given in equation (16). If $k(t)$ equals 1, then we obtain the normal node span of the trinomial tree. When the volatility $s_Y(t)$ increases, then node span must increase to avoid getting a negative middle probability for the $Y(t)$ process in Figure 2. The changing nodes span can be twice, thrice, or even a higher multiple of the normal node span. Restricting $k(t)$ to be an integer value ensures that the tree *recombines* at the future nodes.

The computation of the changing node span in equation (18) involves N volatility terms $V_i(t)$ (for $i = 1,2,\ldots,N$), which are available using the discrete-time trees for the N volatility processes. The combined effect of these volatilities determines the size of the changing node span. For illustration, Figure 3A displays a normal node span with $k(t)$ =1, in all states, over two time intervals. In contrast, Figure 3B displays a changing node span with $k(t) = 1$ in all states in the first time interval, and with $k(t) = 2$ in the up state and the middle state in the second time interval. Note that even though the node span when $k(t) = 2$ is twice the node span when $k(t) = 1$, the total number of nodes goes up only to seven from five, at the end of the second time interval, due to the recombining nature of the tree. The non-explosive nature of the tree for $Y(t)$ process allows the building of an *efficient* lattice that combines the tree for the $Y(t)$ process with the trees of the volatility processes, $V_i(t)$.

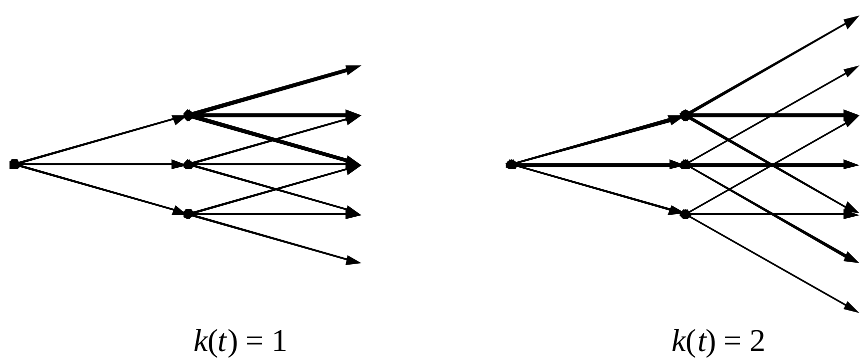

Figure 3. The 2-period $Y(t)$ tree with a changing node span

The marginal probabilities of the $Y(t)$ process are obtained by matching the conditional mean and the conditional variance, in the order of O(Δt), and are given as follows:

$$p_Y^u = \frac{1}{2}\frac{s_Y^2(t)}{k(t)^2 \bar{s}_Y^2} + \frac{1}{2}\frac{m_Y(t)}{k(t)\bar{s}_Y}\sqrt{\Delta t}$$

$$p_Y^d = \frac{1}{2}\frac{s_Y^2(t)}{k(t)^2 \bar{s}_Y^2} - \frac{1}{2}\frac{m_Y(t)}{k(t)\bar{s}_Y}\sqrt{\Delta t} \qquad (19)$$

$$p_Y^m = 1 - p_Y^u - p_Y^d$$

The roles played by the functions $k(t)$ and $h(t)$, in ensuring positive probabilities can be seen from the definition of these probabilities in equation (19). Whenever the current volatility $s_Y(t)$ exceeds $\bar{s}_Y$, the integer value of $k(t)$ becomes greater than 1 (see equation (17)), which ensures that both the up probability and the down probability remain less than 0.5, and the middle probability remains non-negative in the limit as $\Delta t \to 0$.

By assumption, the function $h(t)$ is defined to make $m_Y(t)$ go to zero, when $s_Y(t)$ goes to zero. Note that if $m_Y(t)$ did not converge to zero, whenever $s_Y(t)$ goes to zero, then either the up probability or the down probability is negative in equation (19). However, the function $h(t)$ is not required for the models of Hull and White, and Chesney and Scott, given in Table I. In the case of Hull and White model, the volatility process $V(t)$ follows a lognormal distribution, and hence $s_Y(t)$ remains strictly above zero. In the case of Chesney and Scott model, the volatility process $V(t)$ follows a mean-reverting Gaussian distribution, and hence, can become zero. But, since $s_Y(t)$ is defined as an exponential function of $V(t)$ (see Table I), it remains strictly above zero. However, the function $h(t)$ is required under the models of Stein and Stein, Heston, and the SV2 model. Under these models, the specific definitions of $h(t)$ given in Table I, ensure that whenever all of the volatility variables (which follow square root processes) become zero, making $s_Y(t)$ go to zero, then $m_Y(t)$ goes to zero, as well.

We now state proposition 2, according to which the discrete-time process $Y(t)$ converges to the continuous-time process $y(t)$ in the limit as Δt ® 0.

Proposition 2. *The discrete-time process Y(t) given by equation (15), and approximated using the trinomial tree specified in equations (16) through (19),converges to the continuous-time process y(t), in the limit as* Δt ® 0, *for each of the stochastic volatility models given in Table I.*

A recombining lattice for stock price $S(t)$ is obtained by joining the recombining trees of the transform $Y(t)$ and the volatility processes $V_i(t)$, using the *inverse transform* obtained by rearranging the terms in equation (10), and expressing it using discrete-time variables as follows:

$$S(t) = \exp\left(Y(t) + \sum_{i=1}^{N} r_i \int^{V_i(t)} \left(\frac{y_i(u)}{j_i(u)} \right) du + h(t) \right) \tag{20}$$

The last row of Table I gives the inverse transform using the above equation, for the various stochastic volatility models with one and two volatility factors. The recombining lattice for $S(t)$ is easy to construct since by construction the tree for the $Y(t)$ process is conditionally independent of the trees of the volatility processes $V_i(t)$. Hence, the probabilities of nodes of the multidimensional lattice for $S(t)$ are obtained simply by multiplying the marginal probabilities of the respective nodes of the trees for the $Y(t)$ process and the $V_i(t)$ processes. Since the tree for $Y(t)$ process converges to $y(t)$, and the trees for $V_i(t)$ processes converge to $v_i(t)$, in the continuous-time limit, the multidimensional lattice for $S(t)$ defined using the inverse transform in equation (20) also converges to $s(t)$ in the limit, using the transform relationship given in equation (10) in proposition 1, for all SV1 and SV2 models given in Table I.

SIMULATIONS FOR PRICING AMERICAN OPTIONS UNDER STOCHASTIC VOLATILITY MODELS

Table II documents the performance of SV1 model. It shows the European and American put option prices and Heston [1993] closed form solution for the case of European option. The computations have been performed for the options with strikes X = \$90, \$100 and \$110 maturing in 1, 3 and 6 months for the starting volatility value of $\sqrt{v(0)}$ = 0.1, 0.2 and 0.3 and starting stock value of $S(0)$ = \$100. The other parameters are based on Jun Pan [2002] and given as follows: r = 0.03; s = 0.38; a = 5.3; m = 0.0242; and r = -0.57. The number of steps in a tree is N = 100 and 300. The American put prices were obtained using Control Variate (CV) technique. CV technique computes the value of the put option as follows:

American Put = Tree American Put + (Closed Form Euro Put – Tree Euro Put)

CV technique is particularly useful for options with longer maturity when a regular tree needs a significant number of steps to provide accurate answers.

Table IV documents the performance of SV2 model. It shows the European and American put option prices and closed form solution for the case of European option. The computations have been performed for the options with strikes X = \$90, \$100 and \$110 maturing in 1, 3 and 6 months for the starting volatility value of $\sqrt{v(0)}$ = 0.1, 0.2 and 0.3 and starting stock value of $S(0)$ = \$100. The other parameters were adopted from Christoffersen, Heston, and Jacobs [2007] and given as follows: r = 0.03; s_1 = 1.072; a_1 = 0.2563; m_1 = 0.004; r_1 = -0.8084; s_2 = 0.2875; a_2 = 1.8817; m_2 = 0.0233; and r_2 = -0.6997. The number of steps in a tree is N = 50 and 150.

EXTENSIONS TO STOCHASTIC VOLATILITY JUMP MODELS

Under stochastic volatility models, the volatility is modeled as a diffusion process, therefore, it does not have enough variation in the short run to generate high short-term kurtosis and therefore prices of short-term options might be biased. To deal with this problem, many papers extend stochastic volatility models with jumps in asset returns (for example, see Bates [1996, 2000], Bakshi, Cao and Chen [1997], Pan [2002], Andersen, Benzoni and Lund [2002], Chernov, Ghysels, Gallant and Tauchen [1999]). It has been suggested that jumps could account for the skewness and high kurtosis in option prices. In the second part of this paper we further extend stochastic volatility model of Heston [1993] by allowing jumps in asset returns.

There are not many papers that use or develop numerical procedures for pricing

American options under stochastic volatility and jumps in asset returns. For example, Bates [1996] in his paper uses a bivariate explicit finite-difference method influenced by Omberg [1988] to compute American option prices. He subsequently adjusts the obtained prices for observed biases in European prices using the control variate technique. Another approach using Monte Carlo simulation that works with multiple factors was developed by Longstaff and Schwartz [2001]. The key to their approach is to use a cross-sectional least squares regression to estimate the conditional expected payoff to the option holder from continuation. This way they are able to obtain a complete specification of the optimal exercise strategy along each sample path. With this specification, American options can then be valued by simulation.

In this section we consider the construction of parsimonious recombining lattices for *affine* stochastic volatility jump models. Due to their analytical tractability, these models have been studied extensively, and much empirical estimation work supports the basic features of these models. Assume that the stock price $s(t)$, follows a risk-neutral stochastic process with N volatility factors and M jump factors (selecting N = 1 or 2, and M = 1 or 2, leads to different affine stochastic volatility jump models studied in the literature), given as:

$$\frac{ds(t)}{s(t)} = \left(r - \sum_{i=1}^{M} h_i l_i(t) \right) dt + \sum_{i=1}^{N} \sqrt{v_i(t)}\, dZ_i(t) + \sum_{i=1}^{M} (e^{J_i(t)} - 1) dN_i(t) \qquad (21)$$

where the Wiener processes $Z_i(t)$ are mutually independent. The jump variables $J_i(t)$ follow Gaussian distributions, under the risk-neutral measure, specified as follows:

$$J_i(t) \sim \mathcal{N}\left(m_{Ji},\, s_{Ji}^2\right), \text{ for } i = 1,\ldots,M \tag{22}$$

where m_{Ji} and s_{Ji}^2 are the first two central moments of $J_i(t)$, and the compensator terms are defined under the risk-neutral measure as follows:

$$h_i = E(e^{J_i(t)} - 1) = e^{m_{Ji} + 0.5 s_{Ji}^2} - 1, \text{ for } i = 1,\ldots,M \tag{23}$$

The risk-neutral intensity of the ith Poisson variable $N_i(t)$, is given as $l_i(t)$, where $l_i(t)$ is defined as a linear function of the ith volatility, or:

$$l_i(t) = a_i + b_i v_i(t), \text{ for } i = 1,\ldots,M \tag{24}$$

where a_i and b_i are positive constants. The Wiener processes, jump variables, and Poisson processes given in equation (21), are all independently distributed.

The volatility variables follow square-root processes under the risk-neutral measure, given as follows:

$$dv_i(t) = a_i(m_i - v_i(t))dt + s_i\sqrt{v_i(t)}\,dW_i(t) \tag{25}$$

for $i = 1,\ldots,N$, where the Wiener processes $dW_i(t)$ are mutually independent. The correlations between the diffusion factors related to stock returns and the corresponding volatility factors are given as follows:

$$dZ_i(t)dW_i(t) = \rho_i dt, \text{ for all } i = 1,..., N, \text{ and} \\ dZ_i(t)dW_j(t) = 0, \text{ for all } i \neq j \tag{26}$$

The model specified in equations (21) through (26) nest a variety of affine models with one or two volatility factors, and one or two jumps. When $N = 1$ and $M = 1$, we get the SVJ model given by Andersen, Benzoni, and Lund [2002], and Pan [2002]. When $N = 1$ and $M = 1$, and $b_1 = 0$ in equation (24), we get the SVJ model given by Bakshi, Cao, and Chen [1997, 2000] and Bates [1996, 2000]. When $N = 1$ and $M = 2$, we get the SVJ2 model given by Bates [2006]. When $N = 2$ and $M = 1$, and $b_1 = 0$ in equation (24), we get the SV2J model given by Jiang and Oomen [2007]. We limit our simulations to models with a maximum of two volatility factors and two jumps (i.e., $N = 2$ and $M = 2$), which nest virtually all affine stochastic volatility jump models given in the literature.

The following proposition gives a multidimensional transform that reformulates the Markovian system given by equations (21) through (26), with $N + 1$ conditionally correlated state variables into an equivalent Markovian system with $N + 1$ conditionally independent state variables.

Proposition 3. *For the stochastic processes nested in equations (21) through (26), the multidimensional transform given as:*

$$y(t) = \ln\left(s(t)\right) - \sum_{i=1}^{N} \frac{\rho_i}{\sigma_i} v_i(t) - h(t) \tag{27}$$

is conditionally independent of each of the volatility processes, $v_i(t)$ (i.e., $dy(t)dv_i(t)=0$) where $h(t)$ is a deterministic function of time.

Proof: Using Ito's lemma, the stochastic process for $y(t)$ is given as:

$$dy(t) = m_y(t)dt + s_y(t)dZ_y(t) + \sum_{i=1}^{M} J_i(t)dN_i(t) \qquad (28)$$

where,

$$m_y(t) = \sum_{i=1}^{N} v_i(t)\left(\frac{r_i a_i}{s_i} - \frac{1}{2}\right) - \sum_{i=1}^{M} v_i(t)\left(h_i b_i\right)$$
$$-\frac{\partial h(t)}{\partial t} + \left(r - \sum_{i=1}^{N} \frac{r_i a_i m_i}{s_i} - \sum_{i=1}^{M} h_i a_i\right) \qquad (29)$$

$$s_y(t) = \sqrt{\sum_{i=1}^{N} v_i(t)\left(1 - r_i^2\right)} \qquad (30)$$

and the Wiener process $dZ_y(t)$ is defined as:

$$dZ_y(t) = \frac{\sum_{i=1}^{N} \sqrt{v_i(t)}\left(dZ_i(t) - r_i dW_i(t)\right)}{\sqrt{\sum_{i=1}^{N} v_i(t)\left(1 - r_i^2\right)}} \qquad (31)$$

The proof follows using the same logic as in proposition 1 (i.e., $dZ_y(t)dW_i(t) = 0$, for all $i = 1,\ldots,N$), and noting that jump variables $J_i(t)$, and Poisson processes $N_i(t)$, for all $i = 1,\ldots,M$, are distributed independently of the Wiener process $dZ_y(t)$ defined in equation (31).

As in the case of stochastic volatility models in the previous section, by appropriately choosing $h(t)$ in equation (29), we can ensure that when all of volatility processes $v_i(t)$ become zero, and make $s_y(t)$ go to zero in equation (30), then $m_y(t)$ also goes to zero. Similar to the case of stochastic volatility models in the previous section, doing this prevents negative diffusion probabilities for the tree of the $y(t)$ process, when $s_y(t)$ goes to zero. An inspection of equation (29) reveals that setting $h(t)$ to:

$$h(t) = \left(r - \sum_{i=1}^{N} \frac{r_i a_i m_i}{s_i} - \sum_{i=1}^{M} h_i a_i \right) t \tag{32}$$

makes $m_y(t)$ equal to only the first two summation terms in equation (29), which are linear in $v_i(t)$, and hence, achieves the desired objective.

We now consider lattice construction for three models introduced earlier as the SVJ model, the SVJ2 model, and the SV2J model. The lattice construction is similar to the case of stochastic volatility models considered in the previous section, except that jump lattices are superimposed on the diffusion lattice of the $y(t)$ process, using the explicit finite difference method outlined by Amin [1993], with slight adjustments made for the changing node span of the diffusion lattice.

Let the discrete-time process $Y(t)$ corresponding to $y(t)$ process be represented as follows:

$$\Delta Y(t) = m_Y(t)\Delta t + s_Y(t)\Delta Z_Y(t) + \sum_{i=1}^{M} J_i(t)\Delta N_i(t) \tag{33}$$

where, the term $DZ_Y(t)$ represents the discrete-time approximation of the continuous change in the Wiener process $dZ_y(t)$, and the variables $DN_i(t)$ represent the discrete-time approximations of the continuous changes in the Poisson processes, $dN_i(t)$, for all i = 1,…,M. The variable $DN_i(t)$ is assumed to equal 1 with probability $\overline{\lambda}_i(t)\Delta t$, and 0 with probability 1 - $\overline{\lambda}_i(t)\Delta t$, for all i = 1,…,M. The discrete-time variables $\mu_Y(t)$, $\sigma_Y(t)$, and $\overline{\lambda}_i(t)$ are obtained by replacing $v_i(t)$ with $V_i(t)$, for all i = 1,…N, in the definitions of $\mu_y(t)$, $\sigma_y(t)$, and $\lambda_i(t)$ given in equations (29), (30), and (24), respectively, and $V_i(t)$ represents the discrete-time process corresponding to the $v_i(t)$ process, for all i = 1,…,N.

The procedure to build trees for the discrete-time volatility variables $V_i(t)$ is identical to that outlined in the previous section for the cases of Heston model and the SV2 model. To illustrate the construction of the tree for the $Y(t)$ process, consider Table V which gives the definitions of the transform $y(t)$; the function $h(t)$; the discrete-time functions $\mu_Y(t)$ and $\sigma_Y(t)$, which are needed to model the diffusion part of the multinomial tree for $Y(t)$; and the discrete-time inverse transform $S(t)$ for the SVJ, SVJ2, SV2J, and SV2J2 models.

In order to approximate the $Y(t)$ process in equation (33), using a multinomial jump diffusion tree, consider the case of SV2J2 model which nests all other models in Table V. For this model equation (33) can be approximated as follows:

$$\Delta Y(t) \approx \begin{cases} \mu_Y(t)\Delta t + \sigma_Y(t)\Delta Z_Y(t), \text{ with probability } 1 - \overline{\lambda}_1(t)\Delta t - \overline{\lambda}_2(t)\Delta t, \\ J_1(t), \text{with probability } \overline{\lambda}_1(t)\Delta t, \\ J_2(t), \text{with probability } \overline{\lambda}_2(t)\Delta t \end{cases} \quad (34)$$

In the limit as Δt ® 0, equation (34) is different from equation (33) only by terms of the order of $o(0)$. Amin [1993] using a similar approximation shows to build a jump-diffusion multinomial tree for the Merton's [1976] jump option pricing model. We first show how the diffusion component is modeled on the grid for the jump-diffusion tree. Specifically, from any given node $Y(t)$ at time t, the three diffusion nodes are defined as follows:

$$\begin{aligned} Y^{u}(t+\Delta t) &= Y(t)+k(t)(J+1)\bar{s}_{Y}\sqrt{\Delta t} \\ Y^{m}(t+\Delta t) &= Y(t)+k(t)J\bar{s}_{Y}\sqrt{\Delta t} \\ Y^{d}(t+\Delta t) &= Y(t)+k(t)(J-1)\bar{s}_{Y}\sqrt{\Delta t} \end{aligned} \tag{35}$$

where J is an integer closest in absolute distance to the following expression:

$$\frac{m_Y(t)}{k(t)\bar{s}_Y}\sqrt{\Delta t} \tag{36}$$

By comparing equations (35) and (36), it can be seen that the definition of J ensures that the middle node $Y^{m}(t+\Delta t)$ is as close as possible to the expression $Y(t) + m_Y(t)\Delta t$, allowing the three probabilities to remain positive, even when the drift $m_Y(t)\Delta t$ has a high magnitude. In general, since $m_Y(t)$ (defined for the stochastic volatility jump models in row 3 of Table V) depends upon jump intensity parameter b_i (see equation (24)), and jump size parameters m_{Ji} and s_{Ji} (see equation (23)), the magnitude of $m_Y(t)$ can be high, for some of the models given in Table V.

The three diffusion node probabilities *conditional* on no-jump occurrence sum up to 1 and are obtained by matching both the drift and the variance of $dY(t)$ *exactly* (using the extra degree of freedom available using a trinomial tree), as follows:

$$p_Y^{u*} = \frac{1}{2}\frac{s_Y^{2}(t)\Delta t + e_m e_d}{k(t)^2 \bar{s}_Y^2 \Delta t}$$

$$p_Y^{m*} = \frac{-\left(s_Y^{2}(t)\Delta t + e_u e_d\right)}{k(t)^2 \bar{s}_Y^2 \Delta t} \qquad (37)$$

$$p_Y^{d*} = \frac{1}{2}\frac{s_Y^{2}(t)\Delta t + e_u e_m}{k(t)^2 \bar{s}_Y^2 \Delta t}$$

where,

$$\begin{aligned} e_u &= Y^u(t+\Delta t) - \left(Y(t) + m_Y(t)\Delta t\right) \\ e_m &= Y^m(t+\Delta t) - \left(Y(t) + m_Y(t)\Delta t\right) \\ e_d &= Y^d(t+\Delta t) - \left(Y(t) + m_Y(t)\Delta t\right) \end{aligned} \qquad (38)$$

Since the two Poisson jumps occur with intensities $\bar{l}_1(t)$ and $\bar{l}_2(t)$, the unconditional probabilities of the three diffusion nodes are given as:

$$\begin{aligned} p_Y^u &= p_Y^{u*}\left(1 - \bar{l}_1(t)\Delta t - \bar{l}_2(t)\Delta t\right) \\ p_Y^m &= p_Y^{m*}\left(1 - \bar{l}_1(t)\Delta t - \bar{l}_2(t)\Delta t\right) \\ p_Y^d &= p_Y^{d*}\left(1 - \bar{l}_1(t)\Delta t - \bar{l}_2(t)\Delta t\right) \end{aligned} \qquad (39)$$

To see how Amin's method can be applied to build a multinomial tree for the $Y(t)$ process, first consider the case of only one jump by assuming that $\bar{l}_2(t) = 0$, in equation (34), consistent with the SV2J model in Table V. Figure 4 compares the multinomial jump tree for the Merton model as modeled by Amin [1993], versus the multinomial jump tree for the SV2J model. The normal node span for the grid for Merton tree is based upon stock return volatility σ, while the normal node span for the grid for the SV2J model is based upon the constant $\bar{s}_Y$. While the diffusion nodes for Merton tree are always the first two nodes surrounding the central node, the diffusion nodes for SV2J tree are the three nodes that include the central node, and the up and down nodes with the distance between them determined by the function $k(t)$.

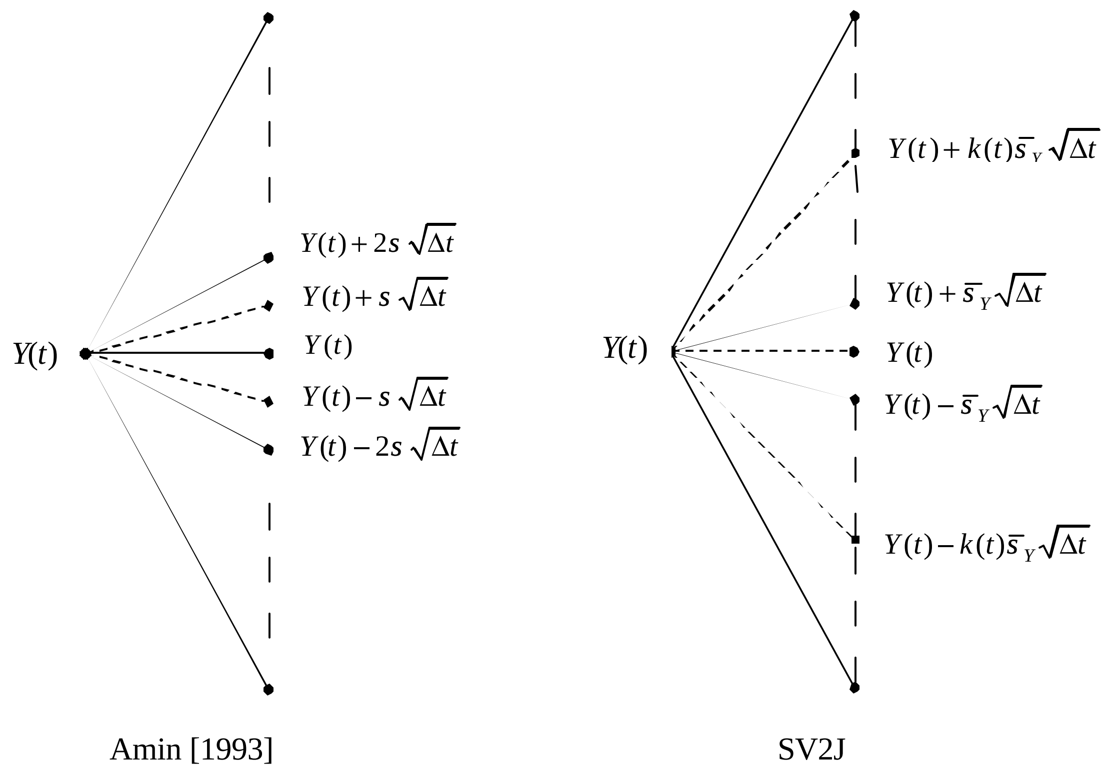

Figure 4. Jump Distribution of Amin [1993] versus Multinomial SV2J Tree.

As the time t volatility increases, the integer value of the function $k(t)$ increases, and the span between the up and down diffusion nodes increases for the SV2J tree. Due to the slightly different structure of the diffusion nodes under these two models, the jump probabilities are allocated slightly differently, using the algorithm given in Amin [1993].

SIMULATIONS FOR PRICING AMERICAN OPTIONS UNDER STOCHASTIC VOLATILITY JUMP MODELS

Table VI and Table VII document the performance of SVJ model using parameters obtained by Pan [2002] and Bates [2006] respectively. Table VIII documents the performance of SVJ2 model using parameters obtained by Bates [2006]. Table IX documents the performance of SVJ2 model using parameters obtained by Jiang and Oomen [2006]. The tables show the European and American put option prices and closed form solutions for the case of European options. The computations have been performed for the options with strikes X = \$90, \$100 and \$110 maturing in 1, 3 and 6 months for the starting volatility values of $\sqrt{v(0)}$ = 0.1, 0.2 and 0.3 and starting stock price value of $S(0)$ = \$100. The number of steps in a tree used for SVJ and SVJ2 models was N = 100 and 300. The number of steps in a tree for SV2J model was N = 50 and 100. The American put prices were obtained using Control Variate technique. All tables show good conversion for the European options. The convergence is worse for options with longer maturity and higher level of starting volatility. The convergence becomes better with the increase in the number of steps in a tree.

CONCLUSIONS

This paper presented a new transform-based approach for *path-independent* lattice construction for pricing American options under low-dimensional stochastic volatility models. We derive multidimensional transforms which allow us to construct efficient path-independent lattices for virtually *all* low-dimensional stochastic volatility models given in the literature including:

i) one volatility factor-based stochastic volatility (SV) models (e.g., Chesney and Scott [1989], Heston [1993], Hull and White [1987], Stein and Stein [1991], and Wiggins [1987]);

ii) two volatility factors-based stochastic volatility (SV2) models (e.g., Bates [2000], and Jiang and Oomen [2006]);

iii) stochastic volatility jump models with one volatility factor and one jump factor (SVJ) and one volatility factor and two jump factors (SVJ2) (e.g. Andersen, Benzoni, and Lund [2002], Bates [1996, 2000, 2006], Bakshi, Cao, and Chen [1997, 2000], Chacko and Viceira [2003], and Pan [2002]);

iv) two volatility factor-based models with one jump factor (SV2J) (e.g., Jiang and Oomen [2006]).

The related lattice approach of Ritchken and Trevor (RV) uses GARCH models for pricing equity options in the presence of stochastic volatility. Since the variance process under the GARCH method is a *path-dependent* (which implies an explosive number of variances at any given node of the asset price tree), RV method has to use approximation procedures to model volatility. Due to the volatility interpolations and other approximations involved in the modeling of the path-dependent variance process, the RV lattice algorithm is relatively slower and less accurate than a typical path-independent binomial or trinomial model. In contrast to the RV algorithm, this paper derives a multidimensional transform in order to model both the asset price process and the volatility process as *path-independent* trees. The efficiency and accuracy of our approach is of similar order as a two-factor Cox, Ross, and Rubinstein [1979] model, which represents a significant improvement over the RV algorithm for the special case of the bivariate diffusion stochastic volatility models. Our approach also generalizes parsimoniously to other low-dimensional stochastic volatility models, such as those with two volatility state variables, or jump factors in the asset returns. In contrast, the GARCH option pricing models cannot allow more than one volatility state variable, and it is virtually impossible to extend the RV lattice approximation method to the GARCH-jump models. Though a few other researchers have provided different approaches for pricing American options under low-dimensional stochastic volatility models, none are as general in terms of applicability to a diverse set of models with stochastic volatility and jumps, or numerically as efficient as our approach. Our lattice-based approximations of the prices of European options converge rapidly to their true prices obtained using quasi-analytical solutions.

Table I. Variable Definitions for Alternative SV1 and SV2 models

	Hull-White	**Chesney-Scott**	**Stein-Stein**	**Heston**	**SV2**
$j(v)$	$s\,v(t)$	s	s	$s\sqrt{v(t)}$	$j_i(v_i)=s_i\sqrt{v_i(t)}$, for $i=1,2$
$y(v)$	$\sqrt{v(t)}$	$\exp(v(t))$	$v(t)$	$\sqrt{v(t)}$	$y_i(v_i)=\sqrt{v_i(t)}$, for $i=1,2$
$y(t)$	$\ln(s(t))-\frac{2r}{s}\sqrt{v(t)}$	$\ln(s(t))-\frac{r}{s}\exp(v(t))$	$\ln(s(t))-\frac{r}{2s}v^2(t)$ $-h(t)$	$\ln(s(t))-\frac{r}{s}v(t)$ $-h(t)$	$\ln(s(t))-\sum_{i=1}^{2}\frac{r_i}{s_i}v_i(t)$ $-h(t)$
$h(t)$	-	-	$\left(r-\frac{1}{2}rs\right)t$	$\left(r-\frac{ram}{s}\right)t$	$\left(r-\sum_{i=1}^{2}\frac{r_i a_i m_i}{s_i}\right)t$
$m_Y(t)$	$r-\frac{ra(m-V(t))}{s\sqrt{V(t)}}$ $-\frac{1}{2}V(t)+\frac{1}{4}sr\sqrt{V(t)}$	$r-\frac{re^{V(t)}}{s}a(m-V(t))$ $-\frac{1}{2}\left(e^{2V(t)}+e^{V(t)}rs\right)$	$-\frac{rV(t)}{s}a(m-V(t))$ $-\frac{1}{2}V^2(t)$	$V(t)\left(\frac{ra}{s}-\frac{1}{2}\right)$	$\sum_{i=1}^{2}V_i(t)\left(\frac{r_i a_i}{s_i}-\frac{1}{2}\right)$
$s_Y(t)$	$\sqrt{V(t)(1-r^2)}$	$\sqrt{e^{2V(t)}(1-r^2)}$	$\sqrt{V^2(t)(1-r^2)}$	$\sqrt{V(t)(1-r^2)}$	$\sqrt{\sum_{i=1}^{2}V_i(t)(1-r_i^2)}$
$S(t)$	$\exp\left(Y(t)+\frac{2r}{s}\sqrt{V(t)}\right)$	$\exp\left(Y(t)+\frac{r}{s}e^{V(t)}\right)$	$\exp\left(Y(t)+\frac{r}{2s}V^2(t)+h(t)\right)$	$\exp\left(Y(t)+\frac{r}{s}V(t)+h(t)\right)$	$\exp\left(Y(t)+\sum_{i=1}^{2}\frac{r_i}{s_i}V_i(t)+h(t)\right)$

Table II. *SV1 Model.* European and American put prices computed assuming r = 0.03; s = 0.38; a = 5.3; m = 0.0242; and r = -0.57 (Jun Pan [2002]).

$\sqrt{v(t)}$	*X*	*T*	European Tree		Heston	American Tree	
			N = 100	*N* = 300		*N* = 100	*N* = 300
0.1	90	0.0833	0.0083	0.0086	0.0086	0.0087	0.0087
0.1	100	0.0833	1.1518	1.1523	1.1524	1.1669	1.1671
0.1	110	0.0833	9.7257	9.7257	9.7257	10.0000	10.0000
0.1	90	0.25	0.2201	0.2222	0.2232	0.2254	0.2254
0.1	100	0.25	2.1202	2.1260	2.1267	2.1746	2.1745
0.1	110	0.25	9.2840	9.2953	9.2954	10.0114	10.0001
0.1	90	0.5	0.7022	0.7089	0.7141	0.7319	0.7308
0.1	100	0.5	3.0987	3.1044	3.1122	3.2268	3.2274
0.1	110	0.5	9.1849	9.1875	9.1994	10.0144	10.0122
0.2	90	0.0833	0.1031	0.1034	0.1036	0.1039	0.1040
0.2	100	0.0833	2.0698	2.0700	2.0700	2.0865	2.0867
0.2	110	0.0833	9.7835	9.7831	9.7830	9.9994	9.9998
0.2	90	0.25	0.6196	0.6219	0.6232	0.6298	0.6300
0.2	100	0.25	3.1827	3.1843	3.1850	3.2503	3.2511
0.2	110	0.25	9.7605	9.7588	9.7577	10.1471	10.1485
0.2	90	0.5	1.2076	1.2145	1.2179	1.2481	1.2489
0.2	100	0.5	4.0260	4.0318	4.0344	4.1913	4.1933
0.2	110	0.5	9.9098	9.9116	9.9110	10.5276	10.5290
0.3	90	0.0833	0.3945	0.3963	0.3972	0.3984	0.3984
0.3	100	0.0833	3.0567	3.0582	3.0589	3.0756	3.0758
0.3	110	0.0833	10.1382	10.1369	10.1362	10.2401	10.2409
0.3	90	0.25	1.3036	1.3123	1.3168	1.3306	1.3310
0.3	100	0.25	4.4596	4.4651	4.4678	4.5414	4.5427
0.3	110	0.25	10.6921	10.6893	10.6877	10.9511	10.9533
0.3	90	0.5	1.9832	2.0022	2.0117	2.0655	2.0670
0.3	100	0.5	5.2543	5.2680	5.2746	5.4699	5.4726
0.3	110	0.5	11.0172	11.0174	11.0169	11.5545	11.5582

Table III *Hull and White model.* European put prices computed assuming r = 0.0; s = 1.0; a = 0.0; m = 0.0; and $v(0)$= 0.1 (Leisen [2000]).

r	***X***	**European Tree**		**Leisen**	**Monte Carlo**
		***N* = 40**	***N* = 100**		
0.5	95	0.7904	0.7881	0.793	0.788
0.5	96	1.0632	1.0612	1.068	1.061
0.5	97	1.3948	1.3930	1.403	1.393
0.5	98	1.7877	1.7874	1.800	1.787
0.5	99	2.2439	2.2440	2.259	2.243
0.5	100	2.7613	2.7614	2.775	2.761
0.5	101	3.3382	3.3389	3.359	3.337
0.5	102	3.9698	3.9709	3.993	3.969
0.5	103	4.6528	4.6535	4.678	4.652
0.5	104	5.3813	5.3829	5.408	5.381
0.5	105	6.1512	6.1523	6.179	6.151
0	95	0.9252	0.9227	0.922	0.922
0	96	1.1835	1.1808	1.181	1.180
0	97	1.4936	1.4912	1.491	1.490
0	98	1.8575	1.8558	1.858	1.855
0	99	2.2802	2.2815	2.283	2.279
0	100	2.7595	2.7616	2.762	2.763
0	101	3.3078	3.3089	3.310	3.307
0	102	3.9112	3.9093	3.910	3.908
0	103	4.5699	4.5656	4.566	4.565
0	104	5.2778	5.2749	5.273	5.273
0	105	6.0309	6.0277	6.027	6.028
-0.5	95	1.0280	1.0289	1.022	1.029
-0.5	96	1.2714	1.2726	1.265	1.272
-0.5	97	1.5607	1.5612	1.553	1.562
-0.5	98	1.8995	1.9008	1.891	1.901
-0.5	99	2.2930	2.2941	2.282	2.294
-0.5	100	2.7434	2.7439	2.725	2.745
-0.5	101	3.2537	3.2542	3.237	3.255
-0.5	102	3.8235	3.8235	3.804	3.825
-0.5	103	4.4528	4.4524	4.429	4.454
-0.5	104	5.1386	5.1385	5.112	5.140
-0.5	105	5.8779	5.8771	5.849	5.879

Table IV. *SV2 Model.* European and American put prices computed assuming r = 0.03; s_1 = 1.072; a_1 = 0.2563; m_1 = 0.004; r_1 = -0.8084; s_2 = 0.2875; a_2 = 1.8817; m_2 = 0.0233; and r_2 = -0.6997 (Christoffersen, Heston, and Jacobs [2007]).

$\sqrt{v_1(t)}$	$\sqrt{v_2(t)}$	***X***	***T***	**European Tree**		**Closed Form**	**American Tree**	
				***N* = 50**	***N* = 150**		***N* = 50**	***N* = 150**
0.1	0.1	90	0.0833	0.0858	0.0869	0.0875	0.0877	0.0877
0.1	0.1	100	0.0833	1.4246	1.4249	1.4248	1.4389	1.4392
0.1	0.1	110	0.0833	9.7267	9.7266	9.7266	10.0000	10.0000
0.1	0.1	90	0.25	0.4710	0.4764	0.4788	0.4832	0.4826
0.1	0.1	100	0.25	2.2875	2.2897	2.2913	2.3396	2.3401
0.1	0.1	110	0.25	9.2724	9.2753	9.2767	10.0044	10.0015
0.1	0.1	90	0.5	0.9750	0.9913	1.0000	1.0210	1.0191
0.1	0.1	100	0.5	3.0953	3.1020	3.1112	3.2214	3.2211
0.1	0.1	110	0.5	8.9607	8.9598	8.9703	10.0096	10.0105
0.1	0.3	90	0.0833	0.5609	0.5649	0.5669	0.5682	0.5683
0.1	0.3	100	0.0833	3.3828	3.3874	3.3898	3.4043	3.4047
0.1	0.3	110	0.0833	10.2955	10.2963	10.2965	10.3803	10.3812
0.1	0.3	90	0.25	1.9292	1.9505	1.9608	1.9733	1.9742
0.1	0.3	100	0.25	5.3075	5.3296	5.3405	5.3953	5.3978
0.1	0.3	110	0.25	11.3447	11.3528	11.3564	11.5391	11.5443
0.1	0.3	90	0.5	3.1210	3.1745	3.1989	3.2426	3.2472
0.1	0.3	100	0.5	6.6380	6.6840	6.7072	6.8443	6.8534
0.1	0.3	110	0.5	12.1942	12.2156	12.2273	12.5890	12.6036
0.3	0.1	90	0.0833	0.7953	0.7973	0.7983	0.7997	0.79998
0.3	0.1	100	0.0833	3.3766	3.3768	3.3769	3.3896	3.3899
0.3	0.1	110	0.0833	10.0468	10.0442	10.0429	10.1526	10.1539
0.3	0.1	90	0.25	2.4661	2.4705	2.4737	2.4851	2.4855
0.3	0.1	100	0.25	5.2339	5.2346	5.2355	5.2820	5.2831
0.3	0.1	110	0.25	10.6953	10.6887	10.6849	10.9113	10.9129
0.3	0.1	90	0.5	3.7903	3.7995	3.8053	3.8430	3.8441
0.3	0.1	100	0.5	6.5255	6.5280	6.5305	6.6435	6.6463
0.3	0.1	110	0.5	11.3837	11.3744	11.3710	11.7701	11.7711
0.3	0.3	90	0.0833	1.3394	1.3442	1.3466	1.3489	1.3490
0.3	0.3	100	0.0833	4.5789	4.5832	4.5852	4.5985	4.5990
0.3	0.3	110	0.0833	11.0074	11.0071	11.0070	11.0617	11.0628
0.3	0.3	90	0.25	3.5924	3.6098	3.6192	3.6357	3.6369
0.3	0.3	100	0.25	7.2201	7.2365	7.2451	7.2950	7.2978
0.3	0.3	110	0.25	12.8930	12.8995	12.9029	13.0357	13.0410
0.3	0.3	90	0.5	5.3733	5.4128	5.4331	5.4854	5.4905
0.3	0.3	100	0.5	9.0509	9.0886	9.1067	9.2333	9.2424
0.3	0.3	110	0.5	14.3213	14.3429	14.3526	14.6334	14.6477

Table V. Variable Definitions for Alternative Stochastic Volatility Jumps Models

	SVJ **(N = 1, M = 1)**	**SV2J** **(N = 2, M =1)**	**SVJ2** **(N = 1, M = 2)**
$y(t)$	$\ln\left(s(t)\right)-\frac{r_1}{s_1}v_1(t)-h(t)$	$\ln\left(s(t)\right)-\sum_{i=1}^{2}\frac{r_i}{s_i}v_i(t)-h(t)$	$\ln\left(s(t)\right)-\frac{r_1}{s_1}v_1(t)-h(t)$
$h(t)$	$\left(r-\frac{r_1 a_1 m_1}{s_1}-h_1 a_1\right)t$	$\left(r-\sum_{i=1}^{2}\frac{r_i a_i m_i}{s_i}-h_1 a_1\right)t$	$\left(r-\frac{r_1 a_1 m_1}{s_1}-\sum_{i=1}^{2}h_i a_i\right)t$
$m_Y(t)$	$V_1(t)\left(\frac{r_1 a_1}{s_1}-\frac{1}{2}-h_1 b_1\right)$	$\sum_{i=1}^{2}V_i(t)\left(\frac{r_i a_i}{s_i}-\frac{1}{2}\right)$ $-V_1(t)h_1 b_1$	$V_1(t)\left(\frac{r_1 a_1}{s_1}-\frac{1}{2}\right)$ $-\sum_{i=1}^{2}V_i(t)h_i b_i$
$s_Y(t)$	$\sqrt{V_1(t)\left(1-r_1^2\right)}$	$\sqrt{\sum_{i=1}^{2}V_i(t)\left(1-r_i^2\right)}$	$\sqrt{V_1(t)\left(1-r_1^2\right)}$
$S(t)$	$\exp\begin{pmatrix}Y(t)+\frac{r_1}{s_1}V_1(t)\\+h(t)\end{pmatrix}$	$\exp\begin{pmatrix}Y(t)+\sum_{i=1}^{2}\frac{r_i}{s_i}V_i(t)\\+h(t)\end{pmatrix}$	$\exp\begin{pmatrix}Y(t)+\frac{r_1}{s_1}V_1(t)\\+h(t)\end{pmatrix}$

Table VI *SVJ Model.* European and American put prices computed assuming *r* = 0.03; *s* = 0.28; *a* = 7.1; *m* = 0.0134; *r* = -0.52; *a* = 0; *b* = 27; μ_J = -0.199; and s_J = 0.033 (Jun Pan [2002]).

$\sqrt{v(t)}$	***X***	***T***	**European Tree**		**Closed Form**	**American Tree**	
			***N* = 100**	***N* = 300**		***N* = 100**	***N* = 300**
0.1	90	0.0833	0.1784	0.1933	0.1938	0.2061	0.1967
0.1	100	0.0833	1.2518	1.2664	1.2668	1.2935	1.2824
0.1	110	0.0833	9.7107	9.7253	9.7256	10.0150	10.0003
0.1	90	0.25	0.6142	0.6257	0.6268	0.6563	0.6465
0.1	100	0.25	2.3780	2.3887	2.3888	2.4584	2.4477
0.1	110	0.25	9.2724	9.2926	9.2914	10.0190	9.9988
0.1	90	0.5	1.2017	1.2128	1.2147	1.2796	1.2727
0.1	100	0.5	3.5350	3.5384	3.5372	3.7011	3.6962
0.1	110	0.5	9.2442	9.2339	9.2279	10.0039	10.0147
0.2	90	0.0833	0.5937	0.5960	0.5963	0.6042	0.6026
0.2	100	0.0833	2.5360	2.5367	2.5362	2.5580	2.5565
0.2	110	0.0833	9.8225	9.8230	9.8223	10.0052	10.0051
0.2	90	0.25	1.3385	1.3419	1.3435	1.3769	1.3772
0.2	100	0.25	3.9282	3.9279	3.9274	4.0240	4.0246
0.2	110	0.25	9.9897	9.9849	9.9818	10.3305	10.3326
0.2	90	0.5	1.9443	1.9529	1.9566	2.0466	2.0469
0.2	100	0.5	4.8411	4.8470	4.8478	5.0943	5.0922
0.2	110	0.5	10.3005	10.3016	10.2967	10.9100	10.9040
0.3	90	0.0833	1.2530	1.2564	1.2579	1.2663	1.2660
0.3	100	0.0833	3.9725	3.9727	3.9726	3.9992	3.9990
0.3	110	0.0833	10.4261	10.4234	10.4217	10.5088	10.5092
0.3	90	0.25	2.4105	2.4200	2.4248	2.4687	2.4698
0.3	100	0.25	5.7121	5.7161	5.7179	5.8362	5.8383
0.3	110	0.25	11.4498	11.4432	11.4396	11.6997	11.7064
0.3	90	0.5	3.0469	3.0685	3.0782	3.2063	3.2068
0.3	100	0.5	6.4807	6.4991	6.5054	6.8067	6.8053
0.3	110	0.5	11.9215	11.9304	11.9288	12.5218	12.5141

Table VII *SVJ Model.* European and American put prices computed assuming r = 0.03; s = 0.237; a = 4.25; m = 0.0142; r = -0.611; a = 0; b = 93.4; μ_J = -0.002; and s_J = 0.039 (Bates [2006]).

$\sqrt{v(t)}$	***X***	***T***	**European Tree**		**Closed**	**American Tree**	
			***N* = 100**	***N* = 300**	**Form**	***N* = 100**	***N* = 300**
0.1	90	0.0833	0.0064	0.0065	0.0066	0.0066	0.0066
0.1	100	0.0833	1.1201	1.1205	1.1207	1.1345	1.1346
0.1	110	0.0833	9.7279	9.7285	9.7288	10.0010	10.0003
0.1	90	0.25	0.1367	0.1385	0.1394	0.1409	0.1409
0.1	100	0.25	1.8887	1.8928	1.8949	1.9488	1.9483
0.1	110	0.25	9.2492	9.2562	9.2587	10.0095	10.0025
0.1	90	0.5	0.4403	0.4489	0.4534	0.4652	0.4646
0.1	100	0.5	2.5808	2.5985	2.6074	2.7378	2.7332
0.1	110	0.5	8.8622	8.8892	8.9003	10.0381	10.0111
0.2	90	0.0833	0.1134	0.1141	0.1145	0.1148	0.1148
0.2	100	0.0833	2.1810	2.1839	2.1854	2.2005	2.001
0.2	110	0.0833	9.8223	9.8271	9.8296	10.0100	10.0063
0.2	90	0.25	0.6055	0.6132	0.6176	0.6254	0.6248
0.2	100	0.25	3.2547	3.2734	3.2839	3.3574	3.3534
0.2	110	0.25	9.8232	9.8529	9.8682	10.2559	10.2370
0.2	90	0.5	1.0826	1.1073	1.1218	1.1588	1.1557
0.2	100	0.5	3.8798	3.9313	3.9607	4.1579	4.1439
0.2	110	0.5	9.7737	9.8562	9.8992	10.6292	10.5787
0.3	90	0.0833	0.4659	0.4705	0.4731	0.4744	0.4742
0.3	100	0.0833	3.2711	3.2840	3.2912	3.3087	3.3067
0.3	110	0.0833	10.2645	10.2858	10.2968	10.3960	10.3861
0.3	90	0.25	1.4414	1.4713	1.4901	1.5100	1.5067
0.3	100	0.25	4.7324	4.7966	4.8345	4.9271	4.9132
0.3	110	0.25	10.8958	10.9997	11.0564	11.3532	11.3100
0.3	90	0.5	2.0661	2.1368	2.1838	2.2608	2.2501
0.3	100	0.5	5.3801	5.5229	5.6097	5.8681	5.8328
0.3	110	0.5	11.0333	11.2651	11.3944	12.0596	11.9641

Table VIII *SVJ2 Model.* European and American put prices computed assuming r = 0.03; s = 0.24; a = 4.12; m = 0.0163; r = -0.627; a_1 = 0; b_1 = 121.3; μ_{J1} = -0.002; s_{J1} = 0.03; a_2 = 0; b_2 = 3.7; μ_{J2} = -0.216; and s_{J2} = 0.003 (Bates [2006]).

$\sqrt{v(t)}$	*X*	*T*	**European Tree**		**Closed Form**	**American Tree**	
			***N* = 100**	***N* = 300**		***N* = 100**	***N* = 300**
0.1	90	0.0833	0.0373	0.0374	0.0375	0.0379	0.0379
0.1	100	0.0833	1.1575	1.1576	1.1580	1.1722	1.1722
0.1	110	0.0833	9.7256	9.7264	9.7268	10.0012	10.0004
0.1	90	0.25	0.2383	0.2403	0.2415	0.2464	0.2463
0.1	100	0.25	2.0245	2.0280	2.0309	2.0845	2.0834
0.1	110	0.25	9.2517	9.2612	9.2659	10.0143	10.0047
0.1	90	0.5	0.6427	0.6535	0.6601	0.6835	0.6820
0.1	100	0.5	2.8554	2.8768	2.8895	3.0232	3.0153
0.1	110	0.5	8.9422	8.9900	9.0102	10.0680	10.0203
0.2	90	0.0833	0.2049	0.2050	0.2054	0.2068	0.2068
0.2	100	0.0833	2.2565	2.2576	2.2593	2.2760	2.2753
0.2	110	0.0833	9.8172	9.8236	9.8271	10.0127	10.0075
0.2	90	0.25	0.8064	0.8086	0.8125	0.8268	0.8255
0.2	100	0.25	3.4642	3.4743	3.4853	3.5649	3.5580
0.2	110	0.25	9.8768	9.9118	9.9321	10.3146	10.2878
0.2	90	0.5	1.3970	1.4069	1.4208	1.4760	1.4691
0.2	100	0.5	4.2439	4.2764	4.3077	4.5232	4.4978
0.2	110	0.5	9.9594	10.0457	10.1001	10.8083	10.7376
0.3	90	0.0833	0.6278	0.6281	0.6319	0.6353	0.6353
0.3	100	0.0833	3.4058	3.4150	3.4251	3.4463	3.4439
0.3	110	0.0833	10.2833	10.3081	10.3239	10.4287	10.4162
0.3	90	0.25	1.7582	1.7569	1.7801	1.8100	1.8040
0.3	100	0.25	5.0523	5.0820	5.1291	5.2362	5.2172
0.3	110	0.25	11.0645	11.1502	11.2226	11.5270	11.4770
0.3	90	0.5	2.5267	2.5401	2.5856	2.6944	2.6637
0.3	100	0.5	5.8959	5.9618	6.0497	6.3537	6.2880
0.3	110	0.5	11.4023	11.5798	11.7202	12.4223	12.2782

Table IX *SV2J Model.* European and American put prices computed assuming r = 0.03; s_1 = 0.99; a_1 = 0.897; m_1 = 0.0094; r_1 = 0.028; s_2 = 2.711; a_2 = 6.23; m_2 = 0.0125; r_2 = -0.478; a = 0.751; b = 0; μ_J = -0.067; and s_J = 0.051 (Jiang and Oomen [2006]).

$\sqrt{v_1(t)}$	$\sqrt{v_2(t)}$	X	T	European Tree		Closed Form	American Tree	
				N = 50	N = 100		N = 50	N = 100
0.1	0.1	90	0.0833	0.1385	0.1448	0.1466	0.1525	0.1485
0.1	0.1	100	0.0833	1.3396	1.3357	1.3315	1.3664	1.3595
0.1	0.1	110	0.0833	9.7812	9.7858	9.7870	10.0058	10.0035
0.1	0.1	90	0.25	N/A	0.5160	0.5221	N/A	0.5381
0.1	0.1	100	0.25	N/A	2.1844	2.1829	N/A	2.2752
0.1	0.1	110	0.25	N/A	9.5725	9.5761	N/A	10.1415
0.1	0.1	90	0.5	N/A	0.9677	0.9803	N/A	1.0256
0.1	0.1	100	0.5	N/A	2.9903	2.9955	N/A	3.1961
0.1	0.1	110	0.5	N/A	9.2724	9.2743	N/A	10.3170
0.1	0.3	90	0.0833	0.6967	0.7010	0.7040	0.7067	0.7063
0.1	0.3	100	0.0833	2.7822	2.7780	2.7710	2.7993	2.7988
0.1	0.3	110	0.0833	10.0773	10.0747	10.0726	10.2133	10.2140
0.1	0.3	90	0.25	1.3942	1.3999	1.4032	1.4285	1.4248
0.1	0.3	100	0.25	3.7138	3.7007	3.6788	3.7961	3.7908
0.1	0.3	110	0.25	10.2339	10.2099	10.1935	10.6190	10.6340
0.1	0.3	90	0.5	1.8723	1.8681	1.8634	1.9329	1.9288
0.1	0.3	100	0.5	4.4177	4.3441	4.3025	4.5253	4.5372
0.1	0.3	110	0.5	10.2921	10.1837	10.1008	10.8546	10.9047
0.3	0.1	90	0.0833	0.5822	0.5844	0.5853	0.5872	0.5871
0.3	0.1	100	0.0833	3.4068	3.3996	3.3964	3.4161	3.4162
0.3	0.1	110	0.0833	10.4878	10.4868	10.4858	10.5698	10.5702
0.3	0.1	90	0.25	1.9313	1.9278	1.9253	1.9459	1.9453
0.3	0.1	100	0.25	5.2957	5.2853	5.2654	5.3568	5.3577
0.3	0.1	110	0.25	11.7623	11.7416	11.7264	11.9599	11.9674
0.3	0.1	90	0.5	N/A	3.1369	3.1183	N/A	3.1903
0.3	0.1	100	0.5	N/A	6.6253	6.5771	N/A	6.8042
0.3	0.1	110	0.5	N/A	12.6971	12.6289	N/A	13.1566
0.3	0.3	90	0.0833	1.1818	1.1837	1.1855	1.1885	1.1885
0.3	0.3	100	0.0833	4.2732	4.2628	4.2580	4.2765	4.2771
0.3	0.3	110	0.0833	10.9638	10.9563	10.9515	11.0203	11.0215
0.3	0.3	90	0.25	2.7411	2.7345	2.7297	2.7553	2.7560
0.3	0.3	100	0.25	6.2952	6.2520	6.2256	6.3101	6.3176
0.3	0.3	110	0.25	12.4817	12.4410	12.4129	12.6178	12.6299
0.3	0.3	90	0.5	3.9915	3.9279	3.8890	3.9603	3.9752
0.3	0.3	100	0.5	7.7467	7.5799	7.4876	7.6723	7.7087
0.3	0.3	110	0.5	13.6657	13.4913	13.3576	13.7426	13.8299

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