On the valuation of and returns to project flexibility within sequential investment

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June 2010

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Abstract

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When tackling sequential investment problems, traditionally the first step is specification of the underlying process diffusion and differential equation. Solutions are then customized to suit conditions at boundaries, where different forms are stitched together in reverse order, i.e. backward in time, from a final often inflexible condition to the initial state. This makes it hard to change flexibility paths in order to investigate the value of a different sequence and work must also start afresh if a different diffusion is suggested. Moreover it is not easy to solve problems that have no final inflexible state to aim for. Since few projects lose all flexibility, these methods are not well suited to solving many realistic investment cases, particularly cyclical ones that consume and generate flexibility.

In this paper, we separate the flexibility sequencing from the choice of diffusion/differential equation. This is done using the structure within a mathematical graph to capture the investment sequences. In order to value flexibility, discount functions standardized for diffusion choices are placed within a matrix representing the sequence's graph. This is done in a manner that facilitates location of optimal, smooth pasted, policies.

Under a range of diffusion choices, for perpetual and cyclical investment sequences this allows project value and cost to be determined explicitly as a function of trigger points; even for situations with very many states, new insights are made concerning the valuation of and return to flexibility. It also facilitates the numerical location of trigger points as a function of investment costs, the direction in which problems are typically presented.

Keywords: real options, investment sequences, flexibility values, discount functions and bi–partite directed graphs.

1 Introduction

Great strides have been made in valuing both financial and operational flexibility; furthermore the delta hedging activity and attendant risk neutral valuation technique[1] has migrated from stock options to other tradeable assets with the result that many operational concerns, especially in the energy sector, have benefitted from the study of these so called real options.[2]

At the heart of real option valuation is the idea that operational flexibility can be valued in a similar manner to financial optionality. Whilst many have questioned the applicability of risk neutral valuation to corporate or operational situations (especially where the underlying risk asset may not be fully traded) this assumption has allowed progress where many papers, tailored to individual situations, have adopted this technique to solve a range of problems.

However, the complexity of the operational flexibilities that have been accommodated to date is very limited. This is because the valuation functions containing the embedded options are non–linear and are difficult to solve for general cases. Too often a phrase similar to "these equations are highly non–linear and cannot be solved analytically" appears where the best that can follow is a limited numerical investigation of properties for certain parameters. Even for the numerical methods provided for specific parameters and choices, it is often very difficult to see useful generalizations or heuristics.

For a range of multistage problems, in this article we advance solution techniques by producing tractable sytems that solve for investment quantities as a function of thresholds. This is accomplished via examining at each decision point, what flexibility is consumed and what is generated. Linking these functions together, we form a **valuation system** that is expressed in **matrix** form, where the passage of flexibility and decisions is marked on a **graph** by transitions from one valuation and flexibility state to another. Especially for those systems that contain recursive or circular flexibilities, this facilitates the solution of many simultaneous equations. We demonstrate this with a system of twelve levels and flexibility states (although larger ones are possible) whilst to date the most states that have been described in multistage projects is four or more typically two.

For each project stage and progression decision, we identify the key ingredients that allow them to be **modularized**. Using a **discount factor** approach (see Dixit, Pindyck and Sødal (1999) [7] and Sødal (2006) [8]), stages are delineated and then coupled together using **matrix methods**

[1] Black and Scholes (1973) [1], Merton (1973) [2] and Cox, Ross Rubinstein (1976) [3].

[2] The term dates from Myers (1977) [4] but the texts of Dixit and Pindyck (1994) [5] and Trigeorgis (1996) [6] have proved influential.

that embed the optimal control techniques of value matching and smooth pasting.

This paper proceeds as follows, in Section 2 we outline the two key horizons and flexibility types that real options paper have used and how they are both accommodated by a discount factor approach. In Section 3 we outline a typical sequence of flexible timing decisions and events and the notation required to embrace the matrix algebra. In Section 4 two way timing is accommodated whilst in section 5 components are assembled including finite maturity decisions. Section 6 presents a unifying numerical example (with parameters choices to come), while Section 7 concludes.

2 Sequential flexibility

When putting a value on the opportunity to enter into a project, the investor must assess what options exist after the first decision. Many project have several decision points at which flexibility is either forfeited or gained therefore these must be tracked carefully and the knock on impact of each on the next stage noted. Finally, no project or investment will last forever; an understanding of what happens at the end of the project's life is key to determining what final value if any can be reclaimed upon termination.

2.1 Measurement of time and flexibility indices

When moving from one project stage to another leads to progress, every time a decision is made, some flexibility is used up. Early in a project, little may be complete but maximum flexibility in terms of uncommitted decisions remain. Later in the project, although considerable progress may have been made, remaining flexibility will be much more limited.

Thus when examining exercise of an investment option at a potential threshold, the manager must chose a course of action that maximizes the project value including immediate and future flexibility values. The trade-off that must be made compares the benefits to immediate value of exercise (e.g. current project value less investment cost) less the current flexibility (or option) value to waiting. This has been well documented in the so called real options literature.

Most generally, we label P as the market value of a project which can be activated or suspended with investment cost (or divestment benefit) X. Since we are interested in particular as to the critical project values that trigger investment and divestment, the net benefit $P - X$ is of concern. On occasions when investment occurs we would expect $P > X$ and divestment with $P < X$. In order to render $P - X$ positive, we use a sign operator $\mathbf{\Omega}$ that is 1 on investment and -1 on divestment so that $\mathbf{\Omega}(P - X)$ remains positive.

Time is principally measured through the passage of a project's price process P_t from an initial P_0 level to subsequent thresholds P_1, P_2 etc. defined by $P_{t_1} = P_1, P_{t_2} = P_2$. For American style stopping problems, these thresholds may be **fixed in project value** and indexed sequentially P_0, P_1, P_2 at times $0 < t_1 < t_2$ (although sequences may vary $0 < t_2 < t_1$).

Alternatively for European style valuations, the **times can be fixed** $t = 0, T$ etc. in which case at a fixed time from its start point, the process is measured and compared to a threshold $P_T \lesseqgtr P_1$.

Either way the thresholds are numbered in what is likely to be the sequence in which they are encountered and this index is useful when relating thresholds to each other by stacking them into a vector. The occupation of the states and thresholds can occur at either random times or at a fixed time but with random level.

Between the transitions that will occur at these threshold, different flexibility states s will pertain. For instance between the start and first threshold P_0, P_1 the value of the flexibility that is created at time 0 and used at time t_1 is labelled $V_s(P)$; a function of the price process throughout this period but in particular it has two special values, one at the **beginning** of its life and one at the **end** $V_s(P_0)$, $V_s(P_1)$. After the P_1 transition (which occurs at time t_1) $V_s(P_1)$ ceases to exist and another flexibility state S will pertain, namely $V_S(P_1)$ at the start of its life.

It is the transition between, and joint valuation of, these sequential flexibilities V_s, V_S that is of concern to this paper.

2.2 Discount factors

Consider one of these values of future flexibility $V_s(P_T)$ when at time T it is may be exercised and converted into payoffs or other forms of flexibility dependent upon the value of the project process P_T at time T. Assuming the future flexibility value does not depend upon any interim cashflows[3], only a payoff, if we wish to determine its current value $V_s(P_0)$ as a function of an initial (time 0) project process P_0 we can treat it as a pure discount instrument as with financial options. Using risk neutral expectations and discounting at r, the continuous risk free rate, the condition for such a discount instrument

[3]Investment costs and benefits that generate and consume cash are accounted for separately.

with no interim cashflows is

$$V_s(P_0) = E_{P_0}^Q \left[e^{-rT} V_s(P_T) \right].$$

This expression is quite general but in particular we most often take expectations with respect to one of two possible random variables. Either[4] T could be random in which case a stopping threshold P_T is typically fixed, or T fixed in time with the uncertain variable being the project value at this fixed time P_T. This paper uses both forms of uncertainty and option pricing allowing it to combine multiple forms of real option flexibility.

In the notation of this paper, if the flex value contains one a fixed threshold $V_s(P_1)$, then the flexibility does not depend explicitly on time, only through changes in P; this is true with perpetual style American valuation. Otherwise with $V_s(P_T)$ its flexibility comes into play at fixed T units of time after the last transition and its value depends explicitly on both the state variable (P_T stochastic) and time (T deterministic) as with finite European style valuation.

2.3 Random stopping time, one fixed threshold,

Consider first the former case, where the stopping time T is random and determined by the time taken for P_0 to diffuse to P_T coincident with P_1, a threshold of choice which is known in advance and optimally chosen. Since the payoff to exercise is fixed, the expectation operator in the equation above applies to the random time alone

$$V_s(P_0) = E_{P_0}^Q \left[e^{-rT} V_s(P_1) \middle| P_T = P_1 \right] = E_{P_0}^Q \left[e^{-rT} \right] V_s(P_1) = D(P_0, P_1) V_s(P_1).$$

This says that the current value of flexibility is a discounted version of future flexibility, where the discount function[5] $D(P_0, P_1)$ does not depend on time, only on the proximity of P_0 to P_1, (when $D(P_1, P_1) = 1$). Furthermore it has an index which is used to indicate the **type** of flexibility that is in play until P_1 is hit.[6]

[4] Another possibility is that both are random but since this leads to untractable results, we limit our study here.

[5] We also use the growth function G which is reciprocal of the discount function

$$G_s(P_0, P_1) = D_s(P_0, P_1)^{-1}.$$

[6] In line with the form of flexibility, this subscript can also denotes the elasticity of the discount factor

$$\varepsilon(P_0) = \frac{P_0}{D_{01}(P_0, P_1)} \frac{\partial D_{01}(P_0, P_1)}{\partial P_0}.$$

It is the functional form of the discount factor that does carries the elasticity whilst specific values at beginning and end $V_s(P_0), V_s(P_1)$ should be treated as quantities to be determined. The values of flexibility in between can be recovered from knowledge of these discount functions and the two known flex values.

Note that $D(P_0, P_1)$ will have different forms for $P_0 \lesseqgtr P_1$.

2.4 Random stopping time, two fixed thresholds

We also anticipate that both forms of flexibility may be in play at the same time, one coming into effect at $P_1 > P_0$ and another at $P_2 < P_0$ in which case two way discount factors may be required[7]

$$V_S(P_0) = D(P_0, P_1, P_2) V_S(P_1) + D(P_0, P_2, P_1) V_S(P_2).$$

Here the extended discount factor $D(P_0, P_1, P_2)$ allows for diffusion from the first to the second threshold **conditional on not** touching the third at any prior time. The alternative outcome is represented in the present value condition by $D(P_0, P_2, P_1)$ which as a decreasing function of P_0 (has negative elasticity). The second type of discount function with two thresholds nest the first with one when the "knock out" conditions becomes irrelevant.[8]

Note that these functions take care of both the **discounting until and the probability of** threshold P_1, P_2 hitting. Although they require the elasticity of the flexibilities over the period to be known, since they are terminated at a level of choice, no knowledge of the way V_S then transforms into further flexibility is required for their valuation.[9] This last point is not true in the next case, where the payoff time is known but not the value of the payoff.

This quantity will be different for the option to open a project (or call, $\varepsilon > 1$) compared to closing (put, $\varepsilon < 0$). In the call case the stopping threshold must be above the initial price $P_1 > P_0$ and the discount function increases with P_0, while in the latter it must be the other way round; the threshold is below the current value, the discount function decreases in P_0 and the elasticity is negative. See Sodal (2006) [8].

[7] One a function of positive and one negative ε.

[8]

$$D_s(P_0, P_1) = D_S(P_0, P_1, 0) : D_s(P_0, P_2) = D_S(P_0, P_2, \infty)$$

$$D_S(P_0, P_1, P_1) = 1 : D_S(P_0, P_2, P_2) = 0 : D_S(P_0, P_1, P_1) = 0 : D_{021}(P_0, P_2, P_2) = 1$$

The betas or elasticities will be a weighted average of the elasticities of the two components.

[9] However, the elasticity at this payoff point is required to evaluate the first order condition.

3 Serial/double hysteresis

In this section we annotate a system for tracking switching decisions at different times and thresholds, that is to say decisions that can be timed to occur at an optimal level of production or cessation. Initially these correspond to infinite horizon (action can be postponed indefinitely) option valuation problems with early exercise, i.e. American style but later we can incorporate fixed time intervals with random value outcomes.

Ekern (1993) [9] evaluates the value of operational flexibility in a sequential investment/divestment situation. In particular his firm can "open or close" a project a limited number of times and therefore switch between "idle and operating" status. The remaining flexibility value depends on the number of limited switching opportunities; so these must be carefully counted and indexed. Since they may not come in a fixed sequence, we try and capture their magnitude in a hierarchy $P_4 > P_3 > P_2$ etc. We proceed to link value sequential states V_s, V_S together; thus $V_s(P), V_S\,(P)$ represents the flexibility values (as a function of the state variable P represent project value) at a transition P_1.

Following Dixit (1989) [10], Ekern (1993) [9] attaches a cost rate to project operation[10] as well as a capital entry cost. In addition to capital investment costs, here to simplify matters we roll all operational costs into a fixed sum that must be borne on activation. Not all of this PV of cost can be recuperated on cessation of activity. Although the operating costs can be spared, it is unlikely that this laying up can occur costlessly, i.e. a small residual cost rate that keeps the plant alive whilst dormant will still be present. We thus here use investment and divestment quantities[11] X_3, X_2 as **lump sum investment and operating costs** that must be expended upon the transition from idle to active, or partially regained upon the transition from active to idle.

There are three types of equation labels used in this section, i) transitions where one type of asset and flexibility is instantaneously turned into another, ii) discount equations where one type of asset/flex is represented by a discounted version of itself at a later date and iii) optimality, or first order

[10] Also these two papers use P to denote a **flow rate** (say lower case p), unlike this paper where P is a project value. In most models, the flow and stock value have a constant scale factor, the dividend yield δ $(P = p/\delta)$.

[11] The relationships between variables in this paper and cost flows $w = x$ in Dixit (1989) [10] and Ekern (1993) [9] is

$$X_3 = \frac{x}{r} + K : X_2 = \frac{x}{r} : X_3 > X_2.$$

conditions. It is important to label them differently so transition equations are labelled T1, discount D1 and optimality O1 etc. Others equations which are mixed may be numbered but have no specific label.

3.1 Flexibility and switching timeline

Table 1 reflects the time line and usage of decision flexibility. Transition thresholds (opening or closing in rows) at project values occur sequentially and are labelled in the lefthand column. Ongoing states (in remaining columns) can either be idle or active and the flexibility value in this region is labelled in its subscript, e.g. $V_i(P), V_a(P)$ represents the flexibility that exists whilst idle and active. Since the states re-occur later but with different transition costs, they have different labels V_I, V_A. At the two thresholds P_4, P_3 opening occurs whilst at P_2, P_1 closing occurs.

States are linked by (horizontal) transitions at which a net payoff $P - X$ or $X - P$ is realised along with the transfer from of one type of flexibility to another. Within states (vertical boxes) the project process P is allowed to follow its diffusion, control only occurs at the transit points indicated (note that although a sequence is implied here, on occasion this investment/divestment pattern can get stuck, either open with a very high price, or closed with a low one; this is indicated by a box having an open top or bottom).

Upon opening at P_4 or P_3, a value gain is derived from the project value P less the value of investment capital and running costs X; therefore the payoff upon opening is $P_4 - X_4$ or $P_3 - X_3$, upon suspension of activities, the gain is either $X_2 - P_2$, $X_1 - P_1$ but both these are considered before the loss and gain of flexibility.

If no further flexibility existed beyond P_4 then $V_i(P_4) = 0$ and the usage of $V_i(P)$ would not beget another option term

(Ekern (1993) presents finite switching).

However here to illustrate a recursive system we have drawn up a circularity where not only does V_i beget V_a but in turn V_a begets V_I and then V_A before returning to V_i.

By allowing for a difference between X_1 and X_3 (or X_0 and X_2) this generalises single (Dixit (1989) [10]) to double hysteresis allowing for **different cost rates with each mode of operation**.

For example if costs rates and required capital in active state a are higher than those in region A, then $X_4 > X_3$. Similarly, and most generally, the present value of spared operational costs and recovered capital at the closing thresholds may be different $X_1 <> X_2$ but the savings on closure at each point (of depressed project worth) will be the positive quantities $X_1 - P_1$ and

action, thr.	idle V_i	p.off	act. V_p	p.off	idle V_I	p.off	act. V_A
open, P_4	$V_i\,(P_4)$	$\stackrel{P_4-X_4}{\longrightarrow}$	$V_a\,(P_4)$				
	$\uparrow$		$\downarrow$				
open, P_3					$V_I\,(P_3)$	$\stackrel{P_3-X_3}{\longrightarrow}$	$V_A\,(P_3)$
	$\uparrow$		$\downarrow$		$\uparrow$		$\downarrow$
close, P_2			$V_a\,(P_2)$	$\stackrel{X_2-P_2}{\longrightarrow}$	$V_I\,(P_2)$		
	$\uparrow$						$\downarrow$
close, P_1	$V_i\,(P_1)$			$\stackrel{X_1-P_1}{\longleftarrow}$			$V_A\,(P_1)$

Table 1: Serial or double hysteresis flexibility values $V_{i,a,I,A}$ red before and blue after transitions (horiz arrows) occuring at $P_{4,2,3,1}$ with payoffs net of PV costs $X_{4,2,3,1}$ (vertical arrows are diffusions).

$X_1 - P_1$. We call this is double hysteresis.[12]

Now consider a decision to move forward one stage by investing or opening. Since there will be "no going back"[13], the irreversible flexibility used to gain $P_4 - X_4$ on exercise (the current project benefit P_4 less its cost X_4) must be considered against the change in flexibility. This flexibility used is $V_i(P_4)$ evaluated at this threshold but simultaneously closing flexibility $V_a(P_4)$ (evaluated at the same threshold again) is acquired. Thus opening flex has been transferred into a payoff and it attendant closing flex.

[12]The total investment quantities $X_{4,3,2,1}$ etc. can be related to an operational cost variable x, its perpetruity x/r and switching or net investment and divestment costs $K_4..K_1$ so

$$
\begin{aligned}
X_4 &= \frac{x}{r} + K_4 : X_3 = \frac{x}{r} - K_3 \\
X_2 &= \frac{x}{r} - K_2 : X_1 = \frac{x}{r} - K_1.
\end{aligned}
$$

Thus K_4 represents a frictional cost in moving from idle i to active a and K_3 from I to A, K_2, K_1 are also frictional costs in the sense that on closing, present value costs of $\frac{w}{r}$ will be spared but these may be offset by (other) closure costs, K_2, K_1.

[13]See also Bjerksund and Ekern (1990) [11]. Other systems, like those of Ekern (1993) [9], can be solved sequentially in reverse order and do not require the matrix inversion employed later.

3.2 Begin and end state labelling

The items used in this section's equations are stacked into vectors so

$$\begin{array}{cccccc} \text{thr.} & & \text{PV costs} & \text{end flex} & \text{beg flex} & \text{pay off} \\ \mathbf{P} & & \mathbf{X} & \mathbf{Ve} & \mathbf{Vb} & \mathbf{\Omega} \\ \begin{bmatrix} P_4 \\ P_3 \\ P_2 \\ P_1 \end{bmatrix} & \begin{bmatrix} X_4 \\ X_3 \\ X_2 \\ X_1 \end{bmatrix} = & \begin{bmatrix} \frac{x}{r} + K_{i-a} \\ \frac{x}{r} + K_{a-I} \\ \frac{x}{r} - K_{I-A} \\ \frac{x}{r} - K_{A-i} \end{bmatrix} & \begin{bmatrix} V_i(P_4) \\ V_I(P_3) \\ V_a(P_2) \\ V_A(P_1) \end{bmatrix} & \begin{bmatrix} V_a(P_4) \\ V_A(P_3) \\ V_I(P_2) \\ V_i(P_1) \end{bmatrix} & \begin{bmatrix} P_4 - X_4 \\ P_3 - X_3 \\ X_2 - P_2 \\ X_1 - P_1 \end{bmatrix} \end{array}$$

where x is the operational cost rate associated with the project, and r the risk free rate so that the PV of perpetual cost is x/r. Opening and closing frictions K are incurred on opening and closing, i.e. on opening the PV cost rate must be borne plus an additional amount whilst on closing, the saving is less than the PV operational cost.

3.3 Flexibility transitions

At the optimal transition threshold P_4 the value matching condition balances sacrificing the valuable option used against those gained. Not only is this true at P_4 but P_2 also although the next threshold involves closing so that the correct sign of the payoff must be included. These equations at the thresholds can be written in stacked form and represent instantaneous transitions all a function of the instantaneous threshold $\mathbf{P}$

$$\begin{array}{ccccc} \begin{bmatrix} V_i(P_4) \\ V_I(P_3) \\ V_a(P_2) \\ V_A(P_1) \end{bmatrix} & = & \begin{bmatrix} V_a(P_4) \\ V_A(P_3) \\ V_I(P_2) \\ V_i(P_1) \end{bmatrix} & + & \begin{bmatrix} P_4 - X_4 \\ P_3 - X_3 \\ X_2 - P_2 \\ X_1 - P_1 \end{bmatrix} \\ \mathbf{Ve} & = & \mathbf{Vb} & + & \mathbf{\Omega} \end{array} \quad \text{(T1)}$$

These, T1, are transition equations in individual and matrix form.

3.4 Discount matrix

Note that the flexibility values $V_i(P)$ etc. generate no cashflows of their own, they are discount instruments that capture the present value benefit of being able to optimally time the investment/divestment in the future. Before an investment threshold is reached and its latent value realised, each flexibility can be valued using the discount factor approach as a fraction of its **future**

self at a different threshold. This is what occurs within the boxes by the passage of time indicated by the vertical arrows in Table 1.

$$\begin{bmatrix} V_a(P_4) \\ V_A(P_3) \\ V_I(P_2) \\ V_i(P_1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & D_{42} & 0 \\ 0 & 0 & 0 & D_{31} \\ 0 & D_{23} & 0 & 0 \\ D_{14} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_i(P_4) \\ V_I(P_3) \\ V_a(P_2) \\ V_A(P_1) \end{bmatrix} \tag{D1}$$

$$\mathbf{Vb} = \mathbf{D} \quad \mathbf{Ve}$$

$$D_{1,2} = D(P_1, P_2) = E^Q_{P_1}\left[e^{-rT}\middle| P_T = P_2\right] \text{ etc.}$$

Each state having its own flexibility has a discount function for that flexibility, these are represented individually and collectively discount equations D1.

This matrix also has an inverse, which corresponds to growth factors. For the example at hand it is easy to visualise and solve (but with other problems, it becomes less intuitive).

$$\begin{bmatrix} V_i(P_4) \\ V_I(P_3) \\ V_a(P_2) \\ V_A(P_1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & D_{14}^{-1} \\ 0 & 0 & D_{23}^{-1} & 0 \\ D_{42}^{-1} & 0 & 0 & 0 \\ 0 & D_{31}^{-1} & 0 & 0 \end{bmatrix} \begin{bmatrix} V_a(P_4) \\ V_A(P_3) \\ V_I(P_2) \\ V_i(P_1) \end{bmatrix}$$

$$\mathbf{Ve} = \mathbf{G} \quad \mathbf{Vb}$$

3.5 System graph and matrix

Overall the eight variables of concern form a "bipartite, directed graph" (see[14] Wilson 85 [14]), that is to say that **Ve** can only change into **Vb** (with an attendant payoff) whilst **Vb** becomes **Ve** by the passing of a diffusion over time and it associated discount function.

$$\begin{bmatrix} \mathbf{Ve} \\ \mathbf{Vb} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Ve} \\ \mathbf{Vb} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Omega} \\ \mathbf{0} \end{bmatrix}$$

3.6 Value matching

Now we have two expressions for the beginning and end flexibility values $\mathbf{Vb}, \mathbf{Ve}$, these can be used to identify their value as a function of the the net payoff at thresholds (and also the growth or discount matrices)

$$[\mathbf{I} - \mathbf{D}]\,\mathbf{Ve} = \boldsymbol{\Omega} = [\mathbf{G} - \mathbf{I}]\,\mathbf{Vb}. \tag{F1}$$

[14]Nagae and Akamatsu 04 [12] also propose a graph structure whilst Nagae and Akamatsu 08 [13] employ a complementarity solution approach to real option problems.

This says that usage, i.e. change in (discounted end flexibility or grown beginning) equals "payoff" (net non–flex PV). Equation F1 determines the relative but not absolute values of $\mathbf{Vb}, \mathbf{Ve}$. In this equation it can be seen that for every $\mathbf{P}, \mathbf{X}$ combination, (assuming invertibility of relevant matrices) for arbitrary $\mathbf{X}$ the flexibility is determined uniquely but possibly not optimally. Therefore apart from the (eight) conditions used so far, another (four) conditions must be used to determine optimal flex values[15] or equivalently to determine optimal $\mathbf{X}$.

This is done by combining the transition and discounting equations, which have been constructed in a manner that facilitates smooth pasting at each threshold and therefore overall optimality. Equations C1 show the key variables required at each threshold.

$$\begin{array}{cccc} & \text{prev trans } P_1 & \text{curr trans } P_4 & \text{next trans } P_2 \\ \text{vm} & G(P_1, P_4) V_i(P_1) & = V_i(P_4) = P_4 - X_4 & = P_4 - X_4 + \\ & & + V_a(P_4) & D(P_4, P_2) V_a(P_2) \\ \text{sp} & \frac{\partial G(P_1,P_4)}{\partial P_4} V_i(P_1) & = & \frac{\partial D(P_4,P_2)}{\partial P_4} V_a(P_2) + 1 \end{array} \tag{C1}$$

3.7 Optimal flexibility

So far we have presented four equations in a matrix that describe **accurate** but not necessarily **optimal** valuation. Four first order conditions are required to pin these optimal thresholds down. These so called **smooth pasting** conditions ensure that the value of flexibility at each stage is maximised (conditional on the next stage level). In addition to the value matching condition, they also ensure continuity of both the local elasticity (Sødal (1998) [15]) and rate of return (Shackleton and Sødal (2005) [16]) of the total flexibility value either side of the decision point.

Stacking the first differential of each smooth pasted row into another vector expression, the last set of conditions for optimality can be found

$$\begin{array}{ccccccc} \text{vm} & \mathbf{G}\begin{bmatrix} V_i(P_4) \\ \dots \\ V_A(P_1) \end{bmatrix} & = & \mathbf{D}\begin{bmatrix} V_a(P_4) \\ \dots \\ V_i(P_1) \end{bmatrix} & + & \begin{bmatrix} P_4 - X_4 \\ \dots \\ X_1 - P_1 \end{bmatrix} & . \\ \text{sp} & \frac{\partial \mathbf{G}}{\partial \mathbf{P}}\mathbf{Vb} & = & \frac{\partial \mathbf{D}}{\partial \mathbf{P}}\mathbf{Ve} & + & \frac{\partial \boldsymbol{\Omega}}{\partial \mathbf{P}} & \end{array}$$

Partial differentiation holding other thresholds constant, this separation smooth pastes using discount factors; since $\mathbf{D}, \mathbf{G}$ have *no diagonal elements* can isolate for each threshold, i.e. for $\frac{\partial}{\partial \mathbf{P}}$ wrt $P_{4,3,2,1}$ we have $\frac{\partial \mathbf{DVe}}{\partial \mathbf{P}} = \frac{\partial \mathbf{D}}{\partial \mathbf{P}}\mathbf{Ve}$.

[15]This is to say that there are many possible payments and receipts $\mathbf{X}$ that are consistent with a given $\mathbf{P}, \mathbf{Vb}, \mathbf{Ve}$. Given separation of levels $\mathbf{P}$, not all of them however generate optimality of flexibility values $\mathbf{Vb}, \mathbf{Ve}$ which is still free variables.

$$\frac{\partial \mathbf{G}}{\partial \mathbf{P}}\mathbf{Vb} = \frac{\partial \mathbf{D}}{\partial \mathbf{P}}\mathbf{Ve}+\frac{\partial \boldsymbol{\Omega}}{\partial \mathbf{P}} \tag{O1}$$

This is a **second independent equation** which relates **Ve** to **Vb** and therefore given any **P** this allows determination of the optimal flex values.

$$\begin{aligned}
\frac{\partial \mathbf{D}}{\partial \mathbf{P}},\frac{\partial \mathbf{G}}{\partial \mathbf{P}} &= \begin{bmatrix} 0 & 0 & D_{4'2} & 0 \\ 0 & 0 & 0 & D_{3'1} \\ 0 & D_{2'3} & 0 & 0 \\ D_{1'4} & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & D_{14'}^{-1} \\ 0 & 0 & D_{23'}^{-1} & 0 \\ D_{42'}^{-1} & 0 & 0 & 0 \\ 0 & D_{31'}^{-1} & 0 & 0 \end{bmatrix} \\
\mathbf{Vb} &= \left[\frac{\partial \mathbf{G}}{\partial \mathbf{P}} - \frac{\partial \mathbf{D}}{\partial \mathbf{P}}\mathbf{G}\right]^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \mathbf{P}} \\
&= \begin{bmatrix} -D_{4'2}D_{42}^{-1} & 0 & 0 & D_{14'}^{-1} \\ 0 & -D_{3'1}D_{31}^{-1} & D_{23'}^{-1} & 0 \\ D_{42'}^{-1} & 0 & -D_{2'3}D_{23}^{-1} & 0 \\ 0 & D_{31'}^{-1} & 0 & -D_{1'4}D_{14}^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}
\end{aligned}$$

Also the final expressions for the other flex value and the attendant investment cost change that is optimal given the set **P** are

$$\begin{aligned}
\mathbf{Ve} &= \left[\frac{\partial \mathbf{G}}{\partial \mathbf{P}}\mathbf{D} - \frac{\partial \mathbf{D}}{\partial \mathbf{P}}\right]^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \mathbf{P}} : \mathbf{Vb} = \left[\frac{\partial \mathbf{G}}{\partial \mathbf{P}} - \frac{\partial \mathbf{D}}{\partial \mathbf{P}}\mathbf{G}\right]^{-1}\frac{\partial \boldsymbol{\Omega}}{\partial \mathbf{P}} \qquad (1) \\
\boldsymbol{\Omega} &= \mathbf{Ve} - \mathbf{Vb}.
\end{aligned}$$

If $\frac{\partial \mathbf{D}}{\partial \mathbf{P}},\frac{\partial \mathbf{G}}{\partial \mathbf{P}}$ are available in closed or numerical form, they greatly facilitate retrieval of optimal levels. This is because the functional form of the discount factors can be used to pin down the relationship between the beginning and end flex values and therefore can be substituted into the value equation to eliminate one variable set.

Equivalently if the functional forms of **Vb** as a function of **P** are known up to a free constant, this extra condition is the one that is required to pin down each such constants. Presenting it in these terms, "takes the constants out" and allow differentiation (elasticities) of discount functions to be used. The terms in **Ve** and **Vb** come out of the differentiation because by construction the differential is carried in $\frac{\partial \mathbf{D}}{\partial \mathbf{P}},\frac{\partial \mathbf{G}}{\partial \mathbf{P}}$.

4 Elasticity ladder

Now consider a system with three modes of operation; idle/power/full and the flexibility to ratchet up or down a value "ladder" of non–flex present

from/to thr.	idle V_i	p.off	power V_p	p.off	full V_f
power/full P_4			$V_p(P_4)$ $\uparrow$	$\overset{P_4-P_4^\gamma-X_4}{\longrightarrow}$	$V_f(P_4)$ $\downarrow$
full/idle P_3			$V_p(P_3)$ $\updownarrow$	$\overset{-P_3+P_3^\gamma+X_3}{\longleftarrow}$	$V_f(P_3)$
idle/power P_2	$V_i(P_2)$ $\uparrow$	$\overset{P_2^\gamma-X_2}{\longrightarrow}$	$V_p(P_2)$ $\downarrow$		
power/idle P_1	$V_i(P_1)$	$\overset{-P_1^\gamma+X_1}{\longleftarrow}$	$V_p(P_1)$		

Table 2: Elasticity ladder with flexibility states Idle, Power and Full flow $V_{i,p,f}$; red before and blue after transitions (horiz arrows) occuring at $P_{1,2,3,4}$ with payoffs net of investment/divestment costs $X_{1,2,3,4}$ (vertical arrows are diffusions).

values that depend on different powers[16] of an underling flow; idle 0, power P^γ and full P.

Again with four thresholds P_{1-4} and switching costs X_{1-4} this can admit a new investment/divestment graph (Table 2), one with two way discount factors (note that the elements within **Ve**, **Vb** have changed)

Now since in the power state, reversion to the off state is possible (at P_1) as well as elevation to the full state (at P_4), the discount matrix is populated with more elements, and in particular each row now contains complementary discount factors that are mutually exclusive and conditional upon each others non occurrence.

$$D_{132} = E_{P_1}^{Q}\left[e^{-rT}\middle| P_3 = P_T <> P_2\right] : D_{123} = E_{P_1}^{Q}\left[e^{-rT}\middle| P_2 = P_T <> P_3\right]$$

The first of these D_{132} indicates the PV factor at P_1 for the value of a dollar paid at P_3 if P_2 is not reached first, and the second D_{123} is the complementary condition.

$$\underset{\mathbf{Vb}}{\begin{bmatrix} V_p(P_1) \\ V_f(P_2) \\ V_p(P_4) \\ V_i(P_3) \end{bmatrix}} = \underset{\mathbf{D}}{\begin{bmatrix} 0 & D_{123} & 0 & D_{132} \\ 0 & 0 & D_{24} & 0 \\ 0 & D_{423} & 0 & D_{432} \\ D_{31} & 0 & 0 & 0 \end{bmatrix}} \underset{\mathbf{Ve}}{\begin{bmatrix} V_i(P_1) \\ V_p(P_2) \\ V_f(P_4) \\ V_p(P_3) \end{bmatrix}} \tag{D2}$$

[16] $0 < \gamma < 1$ is generally a sufficient convergence condition for the PV of the power of a diffusion $P^\gamma = E_P^Q \int_0^\infty p(t)^\gamma e^{-rt} dt$.

action, thr.	act. V_a	p.off	idle V_i
open, P_+	$V_a(P_+)$	$\overset{P_+-X_+}{\longleftarrow}$	$V_i(P_+)$
close, P_-	$V_a(P_-)$	$\overset{X_--P_-}{\longrightarrow}$	$V_i(P_-)$

Table 3: Standard hysteresis/perfect reversibility values $V_{i,a}$ red before and blue after transitions (horiz arrows) converging at $P_{+,-}$ net of costs $X_{+,-}$.

$$\begin{bmatrix} V_i(P_1) \\ V_p(P_2) \\ V_f(P_4) \\ V_p(P_3) \end{bmatrix} = \begin{bmatrix} V_p(P_1) \\ V_f(P_2) \\ V_p(P_4) \\ V_i(P_3) \end{bmatrix} + \begin{bmatrix} P_1^\gamma - X_1 \\ P_2 - P_2^\gamma - X_2 \\ P_4^\gamma - P_4 + X_4 \\ -P_3^\gamma + X_3 \end{bmatrix} \tag{T2}$$
$$\mathbf{Ve} = \mathbf{Vb} + \mathbf{\Omega}$$

The inverse discount matrix still exists but is harder to interpret since it has some negative elements.

$$\mathbf{D}^{-1} = \mathbf{G} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{D_{31}} \\ \frac{D_{432}}{D_{123}D_{432}-D_{132}D_{423}} & 0 & \frac{-D_{132}}{D_{123}D_{432}-D_{132}D_{423}} & 0 \\ 0 & \frac{1}{D_{24}} & 0 & 0 \\ \frac{-D_{423}}{D_{123}D_{432}-D_{132}D_{423}} & 0 & \frac{D_{123}}{D_{123}D_{432}-D_{132}D_{423}} & 0 \end{bmatrix}$$

However the logic can still be applied by differentiating $\mathbf{D}, \mathbf{G}$ line by line, then solving by $\mathbf{Vb} = \left[\frac{\partial \mathbf{G}}{\partial \mathbf{P}} - \frac{\partial \mathbf{D}}{\partial \mathbf{P}}\mathbf{G}\right]^{-1} \frac{\partial \mathbf{\Omega}}{\partial \mathbf{P}}$.

5 Reversible switching at common threshold

The solution system proposed here can also accommodate reversible switching at a common threshold. Consider the degenerate system below as the thresholds merge, $P_+ \to P_-$ we would expect the costs to align as well but the matrix may become invertible.

thr.	PV costs	end flex	beg flex	pay off	$\mathbf{\Omega}$elast
$\mathbf{P}$	$\mathbf{X}$	$\mathbf{Ve}$	$\mathbf{Vb}$	$\mathbf{\Omega}[\mathbf{P}-\mathbf{X}]$	$\frac{\partial \mathbf{\Omega}}{\partial \mathbf{P}}$
$\begin{bmatrix} P_+ \\ P_- \end{bmatrix}$	$\begin{bmatrix} X_+ \\ X_- \end{bmatrix}$	$\begin{bmatrix} V_i(P_+) \\ V_a(P_-) \end{bmatrix}$	$\begin{bmatrix} V_a(P_+) \\ V_i(P_-) \end{bmatrix}$	$\begin{bmatrix} P_+ - X_+ \\ X_- - P_- \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

In fact the key matrices will have non–zero determinant if $P_+ <> P_-$ and inversion will only be problematic numerically as the limit is approached.

However, analytical progress can be made before the limit is taken in which case it is possible to show that as $P_+ \to P_-$ discounting between thresholds disappears

$$\begin{aligned}
\mathbf{D},\mathbf{G} &\to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \det\left[\frac{\partial \mathbf{G}}{\partial \mathbf{P}} - \frac{\partial \mathbf{D}}{\partial \mathbf{P}}\mathbf{G}\right] \to 0 \qquad \text{but} \\
\lim_{P_{+,-}\to P} \mathbf{Vb} &= \begin{bmatrix} P\frac{a-1}{b^2-ab} \\ P\frac{1-b}{a^2-ab} \end{bmatrix} : (X_{+,-}) = X = \frac{\delta}{r}P = \left(\frac{\delta}{r}P_{+,-}\right)
\end{aligned}$$

and using L'Hôpital's rule the GBM system returns finite values $V_i(P)$, $V_a(P)$. This corresponds to a flow condition $\delta P \gtrless rX$ which relates the exercise threshold to "strike price".

However in practice, this situation can actually be tackled non–analytically (without further differentiation) using a fixed level of numerical precision.

6 Mixing other processes

Finally before showing specific examples, Dixit, Pindyck, Sødal (99) [7] detail other discount factors, e.g. for mean reverting processes (where H is the hypergeometric function)

$$\begin{aligned}
\frac{dP}{P} &= \eta\left(\overline{P} - P\right)dt + \sigma dZ : q(\theta) = \frac{1}{2}\theta^2(\theta - 1) + \eta\overline{P}\theta - r = 0 \\
D^H(P_1, P_2) &= \left(\frac{P_1}{P_2}\right)^\theta \frac{H\left(\frac{2\eta}{\sigma^2}P_1, \theta, 2\left(\theta + \frac{\eta\overline{P}}{\sigma^2}\right)\right)}{H\left(\frac{2\eta}{\sigma^2}P_2, \theta, 2\left(\theta + \frac{\eta\overline{P}}{\sigma^2}\right)\right)} \\
\begin{bmatrix} V_i(P_2) \\ V_a(P_1) \end{bmatrix} &= \begin{bmatrix} 0 & D(P_2, P_1) \\ D^H(P_1, P_2) & 0 \end{bmatrix} \begin{bmatrix} V_a(P_2) \\ V_i(P_1) \end{bmatrix}
\end{aligned}$$

Modular diffusions can be examined at the same stage as the flexibility "graph" and combined at will, e.g. GBM while idle but MR whilst operational, allowing the investment to have consequences for the process.

7 Examples under GBM (see xls)

Discount functions, their inverses and derivatives are well known under GBM

$$\begin{aligned}
\frac{dP}{P} &= (r - \delta) + \sigma dZ : q(\varepsilon) = \frac{1}{2}\varepsilon^2(\varepsilon - 1) + \varepsilon(r - \delta) - r \\
q(a, b) &= 0 : a > 1, b < 0 = \frac{1}{2} - \frac{r - \delta}{\sigma^2} \pm \sqrt{\left(\frac{r - \delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}
\end{aligned}$$

$$D\left(P_1, P_2 > P_1\right) = \left(\frac{P_1}{P_2}\right)^a : D\left(P, P_2 < P_1\right) = \left(\frac{P_1}{P_2}\right)^b$$

$$D\left(P_1, P_2 > P_1, P_3 < P_1\right) = \frac{\left(\frac{P_1}{P_2}\right)^a - \left(\frac{P_1}{P_3}\right)^b \left(\frac{P_3}{P_2}\right)^a}{1 - \left(\frac{P_3}{P_2}\right)^{a-b}}.$$

These can be used to evaluate the examples in Sections 3,4. This is done in Excel with analytical expressions for $\mathbf{D}, \mathbf{G}, \frac{\partial \mathbf{G}}{\partial \mathbf{P}}, \frac{\partial \mathbf{D}}{\partial \mathbf{P}}$. For other cases, numerical differentiation would also suffice (if stable) and indeed other discount factors might only be available in numeric form (e.g. Heston (1993) [17] affine diffusions).

8 Conclusions

Sødal et al. (1999, 2006) [7], [8] developed a useful discount factor approach. This can be extended and generalised to incorporate multiple levels and multiple processes.

This is achieved via an investment "graph" that separates flexibility states from discount functions and this often yields explicit solutions to more general and complex problems than have been tackled to date.

This breaks down two difficult and complex steps within the pricing framework deferring the diffusion/pde choice and allows investigation of the system flexibility separately to the diffusion choice.

Although there are many ways to potentially capture all the information associated with the flexibility paths, the one adopted here ensures a smooth pasting condition for optimality can be implemented using discount, growth matrices, their partial derivatives and inverses. This is key to making the solution work automatically.

The set assumes that thresholds are known and optimal investment costs are to be recovered. If these are not equal to target costs (i.e. if thresholds are required as output) then the system presented here can be used to iterate on thresholds until the required costs are achieved.

Finally, these threshold to cost conditions could also be used to infer hidden costs empirically, given an observed level of action. This offers empiricists a practical way forward to use and test real options theory.

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On the valuation of and returns to project flexibility within sequential investment

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Dec 2010

1 Overview

- Real options; maximising project/firm NPV with uncertainty but flexibility; developed alongside financial options but less advanced theory and empirics
- Black Scholes (73) [1], Merton (73) [2], Cox Ross Rubinstein (76) [3]
- Link to operations research; using correctly determined discount rates
- Myers (77) [4], Brennan Schwartz (85) [5], Dixit Pindyck (94) [6] and Trigeorgis (96) [7]
- Interactions are more important for real options (than traded), especially for multi–stage and network investments
- Links to costly reversibility (Abel and Eberly 96, [8]), Q theory and marginal cost of capital (Hayashi 82 [9], Abel et al. 96 [10])
- Discount factor approach; Dixit, Pindyck, Sødal (99) [11], Sødal (06) [12]

1.1 Approaches to date, + this paper

– Choose diffusion, obtain pde, solve functions, identify boundary conditions but customisation is hard analytically

– PV investment costs X treated as input, thresholds P output; $P(X)$

– Hard to solve large systems with many levels, very often "no closed form solution" with numerics only

+ We graphically unpick flexibility sequence and components so that...

+ The diffusion/pde choices are separated from, and can occur after, the flexibility modelling

+ Gives matrix solutions for $X(P)$; input thresholds – output costs and

+ Offers new methods and insights for modular valuation of flexibility

1.2 Variables and notation

$V_s(P_n)$ The flexibility or timing value (excl. operating costs/benefits) associated with state s at levels $P_{n,n+1}$ defined either at the beginning $V_s(P_n)$ or end $V_s(P_{n+1})$ of state s

X_n The PV perpetual operating costs ($= x/r \pm K_n$) incurred (spared) by the activation (cessation) of a project at...

P_n a project value threshold ($= p_n/\delta$) where investment (divestment) reaps a payoff (cost) to activation (cessation); an input in this paper

D_{P_1,P_2,P_3} The discount function associated with a diffusion from P_1 to P_2 without hitting P_3 (growth function $G = D^{-1}$ also used) and ...

r, δ, σ Risk free, conv/div yield, diffusion volatility etc.

2 Discount factor approach

- Dixit, Pindyck, Sødal (99), Sødal (06) [12]; P_1 viewpoint, risk neutral expected present value of \$1 at random stopping time/level $P_2 > P_1 > P_3 = 0$

$$\begin{aligned} D_{P_1,P_2,P_3=0} &= E_{P_1}^{Q}\left[e^{-rT} \middle| P_T = P_2 \right] \\ \varepsilon\left(P_1\right) &= \frac{P_1}{D_{P_1,P_2}} \frac{\partial D_{P_1,P_2}}{\partial P_1} \end{aligned}$$

- The convex function D depends on diffusion characteristics (solves pde subject to boundary conds) and has elasticity ε (linked to rate of return)
- It can conform to either the call (up, in) or put (down, out) options (for GBMs ε is constant at either a, b which solve a quadratic $q(\varepsilon) = 0$)
- Value maximisation implies rate of return minimisation

3 Serial/double hysteresis

- With frictions $(\pm K_{1,2})$, inv thresholds $P_{1,2}$ separate with a hysteresis inaction zone. Extend single Dixit 89 [13] and finite serial Ekern 93 [14] to double hysteresis with four input costs $X_1..X_4$; traditionally requires four option constants and thresholds $P_1..P_4$ from eight conditions
- Assume $P_1..P_4$, identify flex states $V_i, V_a, ..$ but separate their eight beginning and end values $V_i(P_1), V_a(P_4)..,V_i(P_4), V_a(P_2)..$ adding four conditions
- Values $V_s\left(P_n\right), V_s\left(P_{n+1}\right)$, form a bipartite, directed graph.* At P_4 flex $V_i\left(P_4\right)$ is sacrificed for payoff plus new flex $V_a\left(P_4\right)$, i.e.
- $V_i\left(P_4\right) = P_4 - X_4 + V_a\left(P_4\right)$ etc.

*See Wilson 85 [15]. Without discount factors, Nagae and Akamatsu 04 [16] proposed a graph structure whilst 08 [17] employed complementarity conditions to solve real option problems.

act.n, thr.s	idle V_i	p.off	act. V_a	p.off	idle V_I	p.off	act. V_A
open, P_4	$V_i\,(P_4)$	$\overset{P_4-X_4}{\longrightarrow}$	$V_a\,(P_4)$				
	$\uparrow$		$\downarrow$				
open, P_3					$V_I\,(P_3)$	$\overset{P_3-X_3}{\longrightarrow}$	$V_A\,(P_3)$
	$\uparrow$		$\downarrow$		$\uparrow$		$\downarrow$
close, P_2			$V_a\,(P_2)$	$\overset{X_2-P_2}{\longrightarrow}$	$V_I\,(P_2)$		
	$\uparrow$						$\downarrow$
close, P_1	$V_i\,(P_1)$			$\overset{X_1-P_1}{\longleftarrow}$			$V_A\,(P_1)$

Table 1: Serial–double hysteresis flex values $V_{i,a,I,A}$ red before and blue after (dis)investment (horizontal conversions) occuring with payoffs at thresholds $P_1 < P_4 > P_2 < P_3 > P_1$ net of PV costs $X_{4,2,3,1}$ (diffusions are vertical)

3.1 Stacked variables in vectors

thr.	PV costs	end flex	beg flex	pay off $\mathbf{\Omega} =$
$\mathbf{P}$	$\mathbf{X}$	$\mathbf{Ve}$	$\mathbf{Vb}$	$\pm\mathbf{1}[\mathbf{P} - \mathbf{X}]$

$$\begin{bmatrix} P_4 \\ P_3 \\ P_2 \\ P_1 \end{bmatrix} \quad \begin{bmatrix} X_4 \\ X_3 \\ X_2 \\ X_1 \end{bmatrix} = \begin{bmatrix} \frac{x}{r} + K_{i-a} \\ \frac{x}{r} + K_{I-A} \\ \frac{x}{r} - K_{a-I} \\ \frac{x}{r} - K_{A-i} \end{bmatrix} \quad \begin{bmatrix} V_i(P_4) \\ V_I(P_3) \\ V_a(P_2) \\ V_A(P_1) \end{bmatrix} \quad \begin{bmatrix} V_a(P_4) \\ V_A(P_3) \\ V_I(P_2) \\ V_i(P_1) \end{bmatrix} \quad \begin{bmatrix} P_4 - X_4 \\ P_3 - X_3 \\ X_2 - P_2 \\ X_1 - P_1 \end{bmatrix}$$

- Each vectors' components depend on the same thresholds in $\mathbf{P}$, i.e. the common (dis)investment conversion point
- Payoffs $\mathbf{\Omega} = \pm\mathbf{1}[\mathbf{P} - \mathbf{X}]$ are non–flex value changes; at $P_{4,3}$ the project is received and $P_{2,1}$ lost (conversely with the running and switching costs $X_{4,3}$ and $X_{2,1}$); $\mathbf{\Omega}$ represents other payoffs later

3.2 Full (8×8) matrix of system graph

$$\begin{bmatrix} \mathbf{Ve} \\ \mathbf{Vb} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Ve} \\ \mathbf{Vb} \end{bmatrix} + \begin{bmatrix} \mathbf{\Omega} \\ \mathbf{0} \end{bmatrix}$$

- Flex values at beginning are separated from end by diffusion paths; but end values are separated from beginning by (dis)investment conversions
- Traditionally four inputs $\mathbf{X}$ with sufficient conditions (eight) to pin down four option constants and four output thresholds $\mathbf{P}$
- Here, from four inputs $\mathbf{P}$ we create and solve for eight flex values across $\mathbf{Ve}, \mathbf{Vb}$ also determining four outputs $\mathbf{X}$ from twelve conditions
- If necessary, iterate on $\mathbf{P}$ using numerical or analytical derivatives to target "input" values of $\mathbf{X}$. Need specification of discount or diffusion matrix $\mathbf{D}$.

3.3 Beginning flex as diffusion discount matrix op. of end flex

$$
\begin{aligned}
\mathbf{Vb} &= \mathbf{D} \quad \mathbf{Ve} \\
\begin{bmatrix} V_a(P_4) \\ V_A(P_3) \\ V_I(P_2) \\ V_i(P_1) \end{bmatrix} &= \begin{bmatrix} 0 & 0 & D_{42} & 0 \\ 0 & 0 & 0 & D_{31} \\ 0 & D_{23} & 0 & 0 \\ D_{14} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_i(P_4) \\ V_I(P_3) \\ V_a(P_2) \\ V_A(P_1) \end{bmatrix} \\
D_{1,2} &= D_{P_1,P_2} = E^Q_{P_1}\left[e^{-rT} \middle| P_T = P_2 \right] : \mathbf{G} = \mathbf{D}^{-1} \\
\begin{bmatrix} V_i(P_4) \\ V_I(P_3) \\ V_a(P_2) \\ V_A(P_1) \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & D_{14}^{-1} \\ 0 & 0 & D_{23}^{-1} & 0 \\ D_{42}^{-1} & 0 & 0 & 0 \\ 0 & D_{31}^{-1} & 0 & 0 \end{bmatrix} \begin{bmatrix} V_a(P_4) \\ V_A(P_3) \\ V_I(P_2) \\ V_i(P_1) \end{bmatrix} \\
\mathbf{Ve} &= \mathbf{G} \quad \mathbf{Vb}
\end{aligned}
$$

3.4 (Dis)investment converts end flex to beg. flex plus payoff

$$\begin{bmatrix} V_i(P_4) \\ V_I(P_3) \\ V_a(P_2) \\ V_A(P_1) \end{bmatrix} = \begin{bmatrix} V_a(P_4) \\ V_A(P_3) \\ V_I(P_2) \\ V_i(P_1) \end{bmatrix} + \begin{bmatrix} P_4 - X_4 \\ P_3 - X_3 \\ X_2 - P_2 \\ X_1 - P_1 \end{bmatrix}$$

$$\mathbf{Ve} = \mathbf{Vb} + \mathbf{\Omega}$$

- These traditional value matching (vm) equations track project (dis)investment payoffs at $\mathbf{P}$ net of costs $\mathbf{X}$ using $\Omega = \pm 1[P - X]$, $\partial\Omega_n/\partial P_n$ tracks the elasticity of states at P_n
- Row–wise differentiation wrt P_n forms an elasticity change vector used later $\partial\mathbf{\Omega}/\partial P = [\partial\Omega_1/\partial P_1, .., \partial\Omega_4/\partial P_4]'$
- $\mathbf{Vb} = \mathbf{D}\mathbf{Ve}$ ($\mathbf{Ve} = \mathbf{G}\mathbf{Vb}$) condition compensates for $\mathbf{Ve}, \mathbf{Vb}$ separation

3.5 Value matching gives relative, not absolute, flex value

$$\text{Flex Usage} = \mathbf{Ve} - \mathbf{Vb} = \mathbf{\Omega} = [\mathbf{I} - \mathbf{D}]\,\mathbf{Ve} = [\mathbf{G} - \mathbf{I}]\,\mathbf{Vb}$$

- When flexibility is exercised, value matching (vm) holds, i.e. usage or gain in flex value equals net non–flex payoff Ω
- The PV change of each flex value (end less discounted or grown less beginning) is also the same net payoff
- Through value matching, thresholds $\mathbf{P}$ (present in $\mathbf{D}, \mathbf{G}$) control **relative or differential,** but **not absolute** flexibility values ($\mathbf{X}$ still free)
- Which $\mathbf{P}, \mathbf{X}$ combination ensures maximum flex value? Optimal $\mathbf{X}$ given $\mathbf{P}$ depends on a first order smooth pasting (sp) condition.

3.6 Smooth pasting diffusions at conversions

- Many ways to stack variables†, here we chose $\mathbf{D},\mathbf{G}$ to ease smooth pasting (sp); vm optimality implies equivalence of partial e.g. wrt P_4

	prev conv P_1	curr conv P_4	next conv P_2
vm	$G(P_1,P_4)\,V_i(P_1) =$	$V_i(P_4) = P_4 - X_4 + V_a(P_4)$	$= P_4 - X_4 + D(P_4,P_2)\,V_a(P_2)$
P_4 sp	$\frac{\partial G(P_1,P_4)}{\partial P_4} V_i(P_1)$	$=$	$\frac{\partial D(P_4,P_2)}{\partial P_4} V_a(P_2) + 1$

- This separation ensures P_4 and other smooth pastings because $\mathbf{D},\mathbf{G}$ have <u>no diagonal</u> elements, i.e. row wise differentiation of $[\mathbf{DVe}]$ matrix simplifies so $\partial\left[\mathbf{DVe}\right]/\partial P = \partial\mathbf{D}/\partial P\mathbf{Ve}$
- Also $\partial\left[\mathbf{GVb}\right]/\partial P = \partial\mathbf{G}/\partial P\mathbf{Vb}$ now tackle $\mathbf{GVb} = \mathbf{DVe} + \mathbf{\Omega}$

†Shackleton Wojakowski (01) [18] solve GBM constants and level ratios with a different matrix

3.7 Smooth pasting indicates absolute flex value

$$
\begin{array}{lccccc}
\text{vm} & \mathbf{G}\mathbf{Vb} & = & \mathbf{D}\mathbf{Ve} & + & \mathbf{\Omega} \\
 & \mathbf{G}\begin{bmatrix} V_i\,(P_4) \\ \ldots \\ V_A\,(P_1) \end{bmatrix} & = & \mathbf{D}\begin{bmatrix} V_a\,(P_4) \\ \ldots \\ V_i\,(P_1) \end{bmatrix} & + & \begin{bmatrix} P_4 - X_4 \\ \ldots \\ X_1 - P_1 \end{bmatrix} \\
\text{sp} & \partial\mathbf{G}/\partial P\mathbf{Vb} & = & \partial\mathbf{D}/\partial P\mathbf{Ve} & + & \partial\mathbf{\Omega}/\partial P
\end{array}
$$

- Elasticity and rate of return equalization at (dis)investment conversions[‡]
- This third extra sp (rate of return) restriction, solves the three optimal unknowns: beg/end flex $\mathbf{Vb}$, $\mathbf{Ve}$ and optimal costs $\pm\mathbf{X}$ as a function of

[‡]Shackleton Sødal 05 [19], $\mathbf{X}$ has zero elasticity

<u>inputs</u>: levels $\mathbf{P}$, discount $\mathbf{D}$, growth $\mathbf{G}$ and profits $\pm\mathbf{P}$

$\mathbf{Vb} =$	$\mathbf{Ve} =$	$\pm\mathbf{1}[\mathbf{P}-\mathbf{X}] =$
$\left[\frac{\partial \mathbf{G}}{\partial P} - \frac{\partial \mathbf{D}}{\partial P}\mathbf{G}\right]^{-1} \frac{\partial \mathbf{\Omega}}{\partial P}$	$\left[\frac{\partial \mathbf{G}}{\partial P}\mathbf{D} - \frac{\partial \mathbf{D}}{\partial P}\right]^{-1} \frac{\partial \mathbf{\Omega}}{\partial P}$	$\mathbf{Ve} - \mathbf{Vb}$

3.8 Matrix solution $\mathbf{Vb} = \left[\frac{\partial \mathbf{G}}{\partial P} - \frac{\partial \mathbf{D}}{\partial P}\mathbf{G}\right]^{-1} \frac{\partial \Omega}{\partial P} =$

$$\begin{bmatrix} -\frac{\partial D_{42}}{\partial P_4} D_{42}^{-1} & 0 & 0 & \frac{\partial D_{14}^{-1}}{\partial P_4} \\ 0 & -\frac{\partial D_{31}}{\partial P_3} D_{31}^{-1} & \frac{\partial D_{23}^{-1}}{\partial P_3} & 0 \\ \frac{\partial D_{42}^{-1}}{\partial P_2} & 0 & -\frac{\partial D_{23}}{\partial P_2} D_{23}^{-1} & 0 \\ 0 & \frac{\partial D_{31}^{-1}}{\partial P_1} & 0 & -\frac{\partial D_{14}}{\partial P_1} D_{14}^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\frac{\partial \mathbf{D}, \mathbf{G}}{\partial P} = \begin{bmatrix} 0 & 0 & \frac{\partial D_{42}}{\partial P_4} & 0 \\ 0 & 0 & 0 & \frac{\partial D_{31}}{\partial P_3} \\ 0 & \frac{\partial D_{23}}{\partial P_2} & 0 & 0 \\ \frac{\partial D_{14}}{\partial P_1} & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & \frac{\partial G_{14}}{\partial P_4} \\ 0 & 0 & \frac{\partial G_{23}}{\partial P_3} & 0 \\ \frac{\partial G_{42}}{\partial P_2} & 0 & 0 & 0 \\ 0 & \frac{\partial G_{31}}{\partial P_1} & 0 & 0 \end{bmatrix}$$

4 Elasticity ladder

- Now consider three modes of operation; idle/power/full and the flexibility to ratchet up or down a ladder of non–flex PVs

state	idle	power	full
non–flex value at P	0	P^{γ}	P

- $0 < \gamma < 1$ is generally a sufficient convergence condition for the PV of the power of a diffusion $P^{\gamma} \propto E_P^Q \int_0^{\infty} p(t)^{\gamma} e^{-rt} dt$
- Again with four thresholds P_{1-4} and switching costs X_{1-4} this admits a new (dis)investment graph (Table 2)

from/to thr.	idle V_i	Ω p.off	power V_p	Ω p.off	full V_f
power/full P_4			$V_p\,(P_4)$	$\overset{P_4-P_4^\gamma-X_4}{\longrightarrow}$	$V_f\,(P_4)$
			$\uparrow$		$\downarrow$
full/idle P_3			$V_p\,(P_3)$	$\overset{-P_3+P_3^\gamma+X_3}{\longleftarrow}$	$V_f\,(P_3)$
			$\updownarrow$		
idle/power P_2	$V_i\,(P_2)$	$\overset{P_2^\gamma-X_2}{\longrightarrow}$	$V_p\,(P_2)$		
	$\uparrow$		$\downarrow$		
power/idle P_1	$V_i\,(P_1)$	$\overset{-P_1^\gamma+X_1}{\longleftarrow}$	$V_p\,(P_1)$		

Table 2: Elasticity ladder with flexibility states (vertical diffusions) Idle, Power and Full $V_{i,p,f}$; red before and blue after conversions (horizontal) at thresholds $P_1 < P_2 < P_4 > P_3 > P_1$ with PV costs $X_{2,4,3,1}$.

$$\begin{bmatrix} V_f(P_4) \\ V_p(P_3) \\ V_p(P_2) \\ V_i(P_1) \end{bmatrix} = \begin{bmatrix} 0 & D_{43} & 0 & 0 \\ D_{341} & 0 & 0 & D_{314} \\ D_{241} & 0 & 0 & D_{214} \\ 0 & 0 & D_{12} & 0 \end{bmatrix} \begin{bmatrix} V_p(P_4) \\ V_f(P_3) \\ V_i(P_2) \\ V_p(P_1) \end{bmatrix}$$

$$\mathbf{Vb} = \mathbf{D} \quad \mathbf{Ve}$$

$$\mathbf{Ve} = \mathbf{Vb} + \mathbf{\Omega}$$

$$\begin{bmatrix} V_p(P_4) \\ V_f(P_3) \\ V_i(P_2) \\ V_p(P_1) \end{bmatrix} = \begin{bmatrix} V_f(P_4) \\ V_p(P_3) \\ V_p(P_2) \\ V_i(P_1) \end{bmatrix} + \begin{bmatrix} P_4 - P_4^\gamma - X_4 \\ P_3^\gamma - P_3 + X_3 \\ P_2^\gamma - X_2 \\ -P_1^\gamma + X_1 \end{bmatrix} : \frac{\partial \mathbf{\Omega}}{\partial P} = \begin{bmatrix} 1 - \gamma P_4^{\gamma-1} \\ \gamma P_3^{\gamma-1} - 1 \\ \gamma P_2^{\gamma-1} \\ -\gamma P_1^{\gamma-1} \end{bmatrix}$$

now $\mathbf{\Omega}$ tracks net $0, P^\gamma, P$ (dis)investments

$$D_{341} = E_{P_3}^{Q}\left[e^{-rT}\middle| P_T = P_4 <> P_1\right] : D_{314} = E_{P_3}^{Q}\left[e^{-rT}\middle| P_T = P_1 <> P_4\right]$$

$$\mathbf{D}^{-1} = \mathbf{G} = \begin{bmatrix} 0 & -\frac{D_{214}}{-D_{214}D_{341}+D_{241}D_{314}} & \frac{D_{314}}{-D_{214}D_{341}+D_{241}D_{314}} & 0 \\ \frac{1}{D_{43}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{D_{12}} \\ 0 & \frac{D_{241}}{-D_{214}D_{341}+D_{241}D_{314}} & -\frac{D_{341}}{-D_{214}D_{341}+D_{241}D_{314}} & 0 \end{bmatrix}$$

- This uses two way discount factors with two payoffs (e.g. \$1, \$0 at P_4, P_1)
- The inverse diffusion $\mathbf{G}$ is somewhat harder to interpret (negative elements)
- But the logic can still be applied by differentiating $\mathbf{D}, \mathbf{G}$ line by line wrt $\mathbf{P}$, then $\mathbf{Vb} = \left[\frac{\partial \mathbf{G}}{\partial P} - \frac{\partial \mathbf{D}}{\partial P}\mathbf{G}\right]^{-1} \frac{\partial \mathbf{\Omega}}{\partial P}$

5 Section 3,4 examples under GBM (see xls)

Discount functions and their derivatives are well known under GBM

$$\frac{dP}{P} = (r-\delta)\,dt + \sigma dZ : q(\varepsilon) = \frac{1}{2}\varepsilon^2(\varepsilon - 1) + \varepsilon(r-\delta) - r$$

$$q(a,b) = 0 : a > 1, b < 0 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} \pm \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

$$D_{P_1,P_2>P_1} = \left(\frac{P_1}{P_2}\right)^a : D_{P_1,P_2<P_1} = \left(\frac{P_1}{P_2}\right)^b$$

$$D_{P_1,P_2>P_1,P_3<P_1} = \frac{\left(\frac{P_1}{P_2}\right)^a - \left(\frac{P_1}{P_3}\right)^b\left(\frac{P_3}{P_2}\right)^a}{1 - \left(\frac{P_3}{P_2}\right)^{a-b}}$$

6 Reversible switching at common threshold

thr.	PV costs	end flex	beg flex	pay off	elast chg
$\mathbf{P}$	$\mathbf{X}$	$\mathbf{Ve}$	$\mathbf{Vb}$	$\mathbf{\Omega}$	$\frac{\partial\Omega}{\partial P}$
$\begin{bmatrix} P_+ \\ P_- \end{bmatrix}$	$\begin{bmatrix} X_+ \\ X_- \end{bmatrix}$	$\begin{bmatrix} V_i(P_+) \\ V_a(P_-) \end{bmatrix}$	$\begin{bmatrix} V_a(P_+) \\ V_i(P_-) \end{bmatrix}$	$\begin{bmatrix} P_+ - X_+ \\ X_- - P_- \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

- Consider hysteresis converging to perfect reversibility. As $P_+ \to P_- = P$ discounting between thresholds disappears

$$\mathbf{D},\mathbf{G} \;\to\; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \det\left[\frac{\partial \mathbf{G}}{\partial \mathbf{P}} - \frac{\partial \mathbf{D}}{\partial \mathbf{P}}\mathbf{G}\right] \to 0 \qquad \text{but}$$

$$\lim_{P_{+,-}\to P} \mathbf{Vb} \;=\; \begin{bmatrix} P\frac{a-1}{b^2-ab} \\ P\frac{1-b}{a^2-ab} \end{bmatrix} : (X_{+,-}) = X = \frac{\delta}{r}P = \left(\frac{\delta}{r}P_{+,-}\right)$$

- Using L'Hôpital's rule the GBM system returns finite values $V_i(P)\,,\, V_a(P)$

action, thr.	act. V_a	p.off	idle V_i
open, P_+	$V_a\,(P_+)$	$\overset{P_+ - X_+}{\longleftarrow}$	$V_i\,(P_+)$
close, P_-	$V_a\,(P_-)$	$\overset{X_- - P_-}{\longrightarrow}$	$V_i\,(P_-)$

Table 3: Standard hysteresis/perfect reversibility values $V_{i,a}$ red before and blue after transitions (horiz arrows) converging at $P_{+,-}$ net of costs $X_{+,-}$.

- A flow condition $\delta P \gtrless rX$ now relates the exercise threshold to "strike flow rate" $p \gtrless x$ (see Shackleton Wojakowski 07 [20])
- In practice, can tackle non–analytically (without further differentiation) using a fixed level of numerical precision

7 Mixing other processes

Dixit, Pindyck, Sødal (99) [11] detail other discount factors, e.g. for mean reverting processes (where H is the hypergeometric function)

$$\begin{aligned}
\frac{dP}{P} &= \eta\left(\overline{P}-P\right)dt+\sigma dZ : q\left(\theta\right)=\frac{1}{2}\theta^{2}\left(\theta-1\right)+\eta\overline{P}\theta-r=0\\
D^{H}\left(P_1,P_2\right) &= \left(\frac{P_1}{P_2}\right)^{\theta}\frac{H\left(\frac{2\eta}{\sigma^2}P_1,\theta,2\left(\theta+\frac{\eta\overline{P}}{\sigma^2}\right)\right)}{H\left(\frac{2\eta}{\sigma^2}P_2,\theta,2\left(\theta+\frac{\eta\overline{P}}{\sigma^2}\right)\right)}\\
\left[\begin{array}{c} V_i\left(P_2\right)\\ V_a\left(P_1\right)\end{array}\right] &= \left[\begin{array}{cc} 0 & D\left(P_2,P_1\right)\\ D^{H}\left(P_1,P_2\right) & 0\end{array}\right]\left[\begin{array}{c} V_a\left(P_2\right)\\ V_i\left(P_1\right)\end{array}\right]
\end{aligned}$$

Modular diffusions can be examined at the same stage as the flexibility graph and combined at will, e.g. GBM while idle but MR whilst operational, allowing the investment decision to change the process.

8 Summary

- Sødal et al. (99, 06) [11], [12] developed useful discount factor approach
- A graph of (dis)investment uses a matrix to separate flexibility states yielding explicit solutions to modular problems
- Ensures smooth pasting condition for optimality can be implemented using discount, growth matrices, their partial derivatives and inverses
- Breaks down two difficult and complex steps deferring the diffusion/pde choice within the modelling framework
- Optimal investment costs can be recovered, if not equal to target costs (i.e. if thresholds are required as output) then iterate on thresholds
- Could also be used in the forward sense to infer hidden costs empirically, given an observed level of action

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