Equipment Capital Budgeting with Technological Progress 1_15_11

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EFMA Conference June 2011

EFMA Classification Codes: 220, 430

Acknowledgements: We thank Alcino Azevedo, Mike Bowe, Michael Brennan, Geoffrey Evatt, Michael Flanagan, Ser-Huang Poon, Azfal Siddique and Richard Stapleton for comments on earlier versions.

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Abstract

We provide multi-factor real option models (and quasi-analytical solutions) for equipment capital budgeting when there is either anticipated or unanticipated technological progress. In the case of anticipated progress, with comparable input parameter values, the alert financial manager would wait longest before replacing equipment. For a supplier with the sole objective of selling immediately more capital equipment, the easiest but second best approach is to find myopic financial managers not aware of modern financial tools such as real options, or not believing in technological progress or the variability of revenues and costs attributable to incumbent capital equipment. However, the best approach for the supplier seems to be to announce improvements in new equipment at random times, so it becomes impossible to predict when or whether the launch of the next generation is going to occur. Then, financial managers have to treat technological progress as unanticipated, and so when technologically advanced products such as smart phones are launched, there are incentives for early replacement of old equipment.

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When a replaceable asset is installed, a traditional financial manager would normally assess its anticipated lifetime from a standard net present value (NPV) analysis for an infinite replacement chain. This solution, though, is only strictly applicable for like-for-like replacements, but there are many assets with embedded technological progress that violate this assumption, including vehicles and aircraft with higher fuel efficiency, robotic machine tools with greater functionality, mobile phones and computer-based products with faster and novel facilities. The presence of technological progress means that the evaluated ex-ante lifetime may not coincide with its ex-post value. Now, the economic lifetime for an asset depends not only on its own deterioration rate, but also on the technological progress embedded in the succeeding asset because the incumbent suffers implied obsolescence. Since the ex-post lifetime is likely to be variable in the presence of technological progress, an evaluation using the traditional NPV method is likely to be problematic because of its in-built assumption of an equal cycle time. Consequently, in this paper, we adopt a dynamic programming formulation for determining the optimal conditions signaling asset replacement because it avoids a cycle time framework. This approach is applied to a replaceable asset that is subject to both revenue and operating cost deterioration, with technological progress that is anticipated or unanticipated.

The effect of unforeseen technological progress on the replacement policy is originally analyzed by Caplan (1940). Building on the economic lifetime models of Hotelling (1925) and Preinreich (1939), he shows analytically that the consequence of unanticipated technological progress is to shorten the active life of the incumbent. If there is an unforeseen performance improvement in the succeeding asset or a more technologically advanced asset becomes available sooner than expected, then the incumbent becomes prematurely obsolete. If the increase in profit potential from replacing the incumbent more than compensates the loss in recovering the original investment, then the incumbent is replaced before its ex-ante lifetime has expired. In contrast, Stapleton, Hemmings and Scholefield (1972) apply numerical simulation to show that if technological progress is foreseen, the optimal time between successive replacements is lengthened. Although these authors adopt a dynamic programming formulation to avoid the equal life assumption, Elton and Gruber (1976) show that an equal life policy is optimum for assets with technological improvements. However, the shortcoming of these analyses is not only their restrictive focus on either anticipated or unanticipated technological progress, but also the absence of a simple operational rule for deciding the optimal conditions for replacing the incumbent.

Several authors have studied the adoption of technological innovations in a real options context, sometimes in a duopoly. Huisman and Kort (2003) assume a new technology has a greater "efficiency" than the existing technology, and firms determine outcomes in a strategic context. Huisman and Kort (2004) use a similar approach, except that the new technology becomes available for adoption at some unknown time in the future. Smith (2005) assumes technologies have different proportional cost savings, which may be complimentary. Azevedo and Paxson (2010) assume two technologies have different degrees of complimentarity, depending on the specific adopter, who learns by adopting early and then maintains a permanent pre-emptive advantage over competing firms. Pae and Hyun (2006) study technology patronage, where an early innovative firm builds an innovation brand image to retain and expand clientele. We simply provide a format for most of these types of technological innovation, through allowing either technological advances to involve different initial operating cost levels, and/or different initial operating cost deterioration rates, compared to like-for-like equipment replacements.

The aim of this paper is to create a single formulation that brings together the two models of anticipated and unanticipated change with their contrasting outcomes, so we can derive an operational rule for replacement and examine analytically the differential impact of technological progress on the replacement policy. This single formulation is developed from a dynamic programming framework by setting the volatilities to zero in the two-factor real option solution to the replacement timing boundary for an asset with deteriorating revenues and operating cost, Adkins and Paxson (2011). The dynamic programming framework avoids the equal life assumption, and since it frames the timing boundary in terms of thresholds, the initial levels for the attributes of the incumbent are irrelevant. By extending the number of factors to three, we can compare the differential effects of anticipated and unanticipated technological progress, while showing that the effect of foreseen technological progress is to prolong the economic life of the incumbent. Finally, the resulting solutions provide the basis for developing operational formulae for judging when the prevailing conditions are optimal for replacing the incumbent, depending on whether the technological progress is anticipated or unanticipated.

The paper is organized in the following way. In Section 1, we develop quasi-analytical solutions to the timing boundary for replacement with anticipated and unanticipated technological progress. These solutions assume that an optimal replacement occurs when the incremental value rendered by the replacement exceeds the re-investment cost. The behavior of the solutions is investigated in Section 2. Numerical analysis is used to illustrate model behavior. We show that while an anticipated technological progress prolongs the active life of the incumbent, unanticipated change shortens its active life. The final section is a conclusion.

1 Models of Replacement

We consider a durable productive asset, subject to both input and output decay, Feldstein and Rothschild (1974), whose efficiency diminishes progressively and deterministically with time. At any time, the revenue rendered by the asset, denoted by P, changes at a continuous rate θ_P, assumed to be negative, while its operating cost, denoted by C, changes at the continuous rate θ_C, assumed to be positive. When the incumbent attains a to be determined threshold, it is replaced by its succeeding asset at a constant re-investment cost of K.

The distinction between anticipated and unanticipated technological progress is critical to our analysis. Technological progress is interpreted as an improvement in the initial attribute levels for the succeeding asset relative to the incumbent. Unanticipated technological progress is represented as a jump in a favorable direction of one or more of the initial attribute levels for the succeeding asset relative to the incumbent. So, for example, an unexpected fall in the initial operating cost level for the succeeding asset relative to the incumbent is indicative of an unforeseen technological improvement in the asset performance. In contrast, a deterministic decline in the initial operating cost level for the succeeding asset is predictable, and because the improvement is foreseen, it is interpreted as anticipated technological progress.

In this section, we develop two new models, II and III, to illustrate the distinction between unanticipated and anticipated technological progress, respectively. We derive a replacement policy for each model as an operational rule for deciding whether the incumbent should or should not be replaced. It is characterized by a timing boundary, which is created from a quasi-analytical solution to the corresponding real replacement option with the underlying volatilities set equal to zero. But first, we specify the equal cycle time solution for the standard NPV model, depicted as Model I, because it acts as a benchmark.

1.1 Model I No Technological Progress

Model I characterizes the traditional representation for identifying the optimal cycle time between successive replacements, denoted by $\hat{T}_1$, which is determined from maximizing the value W for an infinite chain of identical assets. The revenue and operating cost levels for the incumbent at installation are denoted by P_0 and C_0, respectively. Following Lutz and Lutz (1951), the present value for any incumbent at installation V with lifetime T is given by:

$$V = \int_{t=0}^{T} \left(P_0 \mathrm{e}^{\theta_P t} - C_0 \mathrm{e}^{\theta_C t} \right) \mathrm{e}^{-rt} \mathrm{d}t\,, \tag{1}$$

where r denotes the appropriate continuous discount rate for risky projects. The optimal cycle time $\hat{T}$ is obtained from maximizing W, where:

$$W = V + \left(W - K \right) \mathrm{e}^{-rT}\,. \tag{2}$$

From (1) and (2), it is straightforward to derive the implicit solution for $\hat{T}_1$, which can be expressed as:

$$\frac{P_0\left(1-\mathrm{e}^{\theta_P \hat{T}_1}\right)}{r-\theta_P}-\frac{C_0\left(1-\mathrm{e}^{\theta_C \hat{T}_1}\right)}{r-\theta_C}=K+\left[\frac{-\theta_P P_0 \mathrm{e}^{\theta_P \hat{T}_1}}{\left(r-\theta_P\right)}+\frac{\theta_C C_0 \mathrm{e}^{\theta_C \hat{T}_1}}{\left(r-\theta_C\right)}\right]\frac{\left(1-\mathrm{e}^{-r\hat{T}_1}\right)}{r}. \qquad (3)$$

This states that the optimal cycle time occurs when the net incremental value rendered by the replacement, represented by the left hand side of (3), exactly balances the re-investment cost plus a positive amount. Denoting this positive amount as the incremental goodwill, it is measured as the weighted sum of the values for the revenue and operating cost for an incumbent at the optimal lifetime, adjusted by an annuity factor with an optimal lifetime horizon. For a replacement to be optimal, the rendered net incremental value has to exceed the re-investment cost. Given the initial revenue and operating cost levels for the incumbent, finding the optimal replacement time involves equating the LHS and RHS numerically, by changing the optimal time. Finally, the optimal thresholds for revenue $\hat{P}_1$ and operating cost $\hat{C}_1$ are given by $\hat{P}_1 = P_0\,\mathrm{e}^{\theta_P \hat{T}_1}$ and $\hat{C}_1 = C_0\,\mathrm{e}^{\theta_C \hat{T}_1}$, respectively.

1.2 Model II Unanticipated Technological Progress

A deterministic replacement policy can be derived from a suitable model that treats the factors as uncertain, but sets their volatilities equal to zero. Following Adkins and Paxson (2011), we formulate a two-factor, real-option replacement model for an asset that is subject to uncertainty in the magnitude of the input and output decay. We seek to find the threshold signaling the optimal replacement of the incumbent. This threshold is represented by a function of the trigger levels for the revenue and operating cost, denoted by $\hat{P}_2$ and $\hat{C}_2$, respectively, which divides the decision space into two mutually exclusive exhaustive regions of continuance and replacement. When plotted on this decision space, if the prevailing levels of the revenue and operating cost lie within the continuance region, then the optimum strategy is to continue with the incumbent, or if the prevailing levels belong to the replacement region, then replacing the incumbent is the optimal decision.

The function discriminating between continuance and replacement is obtained from the asset valuation function in conjunction with the economic boundary conditions. The valuation function, constructed from a dynamic programming framework, is expressed as:

$$F_2\left(P,C\right) = A_2 P^{\beta_2} C^{\eta_2} + \frac{P}{r-\theta_P} - \frac{C}{r-\theta_C}, \tag{4}$$

with coefficient A_2 and parameters β_2 and η_2. The function F_2 is composed of two elements. The term $A_2 P^{\beta_2} C^{\eta_2}$ is interpreted as the replacement option value, which being positive means that $A_2 > 0$. Since the incentive to replace the incumbent grows as the revenue decreases, but as the operating cost increases, we conjecture $\beta_2 < 0$ and $\eta_2 > 0$. The second element:

$$\frac{P}{r-\theta_P} - \frac{C}{r-\theta_C}$$

denotes the asset value in the absence of any optionality.

Value conservation at replacement requires that the incumbent value has to be exactly balanced by the net value for the succeeding asset. The incumbent value at replacement $F_2\left(\hat{P}_2, \hat{C}_2\right)$ is determined from the valuation function (4), defined at the threshold levels. When the succeeding asset is installed, its initial attribute levels are specified by P_I and C_I for revenue and operating cost, respectively, where the constraints $P_I \geq \hat{P}$ and $C_I \leq \hat{C}$ are imposed since both revenue and operating cost deteriorate over time. This specification implies that the attribute levels of the succeeding asset may be allowed to differ from those of the incumbent that it replaces. So for Model II, there is no underlying presumption that the incumbent is replaced by a replica with identical attribute levels. Consequently, we can model the presence of an unanticipated technological progress as the unexpected jump in the initial attributes, from C_0 to C_I where $C_0 > C_I$, or from P_0 to P_I where $P_0 < P_I$. As the value for the succeeding asset at installation is $F_2\left(P_I, C_I\right)$, its net value is $F_2\left(P_I, C_I\right) - K$, so the value matching relationship can be expressed as:

$$A_2 \hat{P}_2^{\beta_2} \hat{C}_2^{\eta_2} + \frac{\hat{P}_2}{r-\theta_P} - \frac{\hat{C}_2}{r-\theta_C} = A_2 P_I^{\beta_2} C_I^{\eta_2} + \frac{P_I}{r-\theta_P} - \frac{C_I}{r-\theta_C} - K . \tag{5}$$

Although value conservation is enforced by the value matching relationship, the requirement governing an optimal replacement is specified by the two smooth pasting conditions, one for each of the two factors. These can be expressed as:

$$A_2 \hat{P}_2^{\beta_2} \hat{C}_2^{\eta_2} = -\frac{\hat{P}_2}{\beta_2\left(r-\theta_P\right)} = \frac{\hat{C}_2}{\eta_2\left(r-\theta_C\right)} > 0 , \tag{6}$$

which affirms our conjecture that $\beta_2 < 0$ and $\eta_2 > 0$. Using (6) to eliminate A_2 from (5) yields:

$$\frac{P_I - \hat{P}_2}{r-\theta_P} - \frac{C_I - \hat{C}_2}{r-\theta_C} = K + \frac{\hat{C}_2}{\eta_2\left(r-\theta_C\right)}\left[1 - \frac{P_I^{\beta_2}}{\hat{P}_2^{\beta_2}} \frac{C_I^{\eta_2}}{\hat{C}_2^{\eta_2}}\right]. \tag{7}$$

We assume that for replacement P and C when there is a technological advance, the subsequent drifts are the same as for the initial equipment. Since $P_I^{\beta_2} < \hat{P}_2^{\beta_2}$ and $C_I^{\eta_2} < \hat{C}_2^{\eta_2}$, (7) asserts that for an optimal replacement to occur, the incremental net revenue generated by the replacement has to exceed the re-investment cost by a positive amount.

The third component of the model is the characteristic root equation. This condition on the parameters of the valuation function (4) ensures that it satisfies the underlying relationship describing the evolutionary behavior of the asset value. When the volatilities are set equal to zero, the characteristic root equation becomes:

$$\beta_2\theta_P + \eta_2\theta_C = r\,. \tag{8}$$

The three equations, (6), (7) and (8), constitute the model solution from which the optimal timing boundary can be found. The parameters β_2 and η_2 can be identified from (6) and (8) as:

$$\beta_2 = \frac{-\dfrac{r\hat{P}_2}{r-\theta_P}}{-\dfrac{\theta_P\hat{P}_2}{r-\theta_P}+\dfrac{\theta_C\hat{C}_2}{r-\theta_C}} < 0 \text{ and } \eta_2 = \frac{\dfrac{r\hat{C}_2}{r-\theta_C}}{-\dfrac{\theta_P\hat{P}_2}{r-\theta_P}+\dfrac{\theta_C\hat{C}_2}{r-\theta_C}} > 0.. \tag{9}$$

This suggests that the values for these parameters adjust according to the position along the timing boundary. By using (9), the two parameters can be eliminated from (6) to create a relationship between $\hat{P}_2$ and $\hat{C}_2$, which specifies an implicit timing boundary function that divides the total decision space into two regions of continuance and replacement. In this way, the timing boundary can be found for the deterministic replacement problem from the initial attributes for the succeeding asset, which avoids evaluating the optimal cycle time and knowing the initial attributes for the incumbent.

Under certain conditions, the solutions to Model I and II are identical. If we set $P_I = P_0$ and $C_I = C_0$, and note that $\hat{P}_2 = P_0\,\mathrm{e}^{\theta_P\hat{T}}$ and $\hat{C}_2 = C_0\,\mathrm{e}^{\theta_C\hat{T}}$, so:

$$\frac{P_I^{\beta_2}}{\hat{P}_2^{\beta_2}}\frac{C_I^{\beta_2}}{\hat{C}_2^{\beta_2}} = \mathrm{e}^{-(\beta_2\theta_P+\eta_2\theta_C)\hat{T}} = \mathrm{e}^{-r\hat{T}} ,$$

then, in conjunction with (6) and (8), it is straightforward to demonstrate that (7) and (3) are identical. The optimal cycle time determined from the infinite chain NPV model (2) is equal to the dynamic programming solution, but only if the initial attributes of the incumbent and the succeeding asset are equal. The standard optimal cycle time result can be conceived as a particular case of the dynamic programming solution.

A more critical distinction between the two is that while the initial attribute levels for the incumbent and its age are required to be known for implementing the NPV result, the dynamic programming solution only requires knowledge of the initial attribute levels for the succeeding asset as well as the prevailing revenue and operating cost levels. Consequently, the former relies on past information but the latter on current and future information. Due to its forward looking stance, the dynamic programming solution is more in tune with our conceptual understanding of finance.

1.3 Model III Anticipated Technological Progress

Both competitive forces and the threat of new technology are likely to motivate asset suppliers to continuously improve product performance. If these improvements are realized through continuous changes in the initial attributes, then over a period of time, we would observe falls in either the initial operating cost level or re-investment cost for the succeeding asset, or increases in its initial revenue level. We again adopt a dynamic programming framework, primarily because within its design, it allows the initial attribute to change with time but is unencumbered by a cycle time conceptualization. Specifically, we treat anticipated technological progress as being expressed through a time dependent initial operating cost level. Even though the focus of our enquiry is interpreting anticipated technological progress as a decline in the initial operating cost level, it is straightforward to reproduce the solution to the timing boundary for the cases where either one of the two attributes changes with time. A more complete model is described in Appendix A, from which the particular solutions can be derived.

We start by assuming that for the succeeding asset, the initial operating cost level, which is denoted by C_N, can be adequately expressed by a growth function with a continuous constant rate θ_N. This growth parameter is expected to be negative, since performance improvements are presumed to be embedded in the succeeding asset with C_N declining over time. The presence of a variable initial operating cost level in the model means that the value function, which is denoted by F_3, depends on three factors, the initial operating cost level as well as the prevailing levels for the revenues and operating cost. In the two-factor model (4), the replacement option value is expressed as a product power function of the two factors, revenues and operating costs. For the three-factor model under consideration, we similarly adopt a product power function but now of three factors, revenues, operating costs and initial operating cost level, to represent the replacement option value. So, the valuation function becomes:

$$F_3\left(P,C,C_N\right) = A_3 P^{\beta_3} C^{\eta_3} C_N^{\gamma_3} + \frac{P}{r-\theta_P} - \frac{C}{r-\theta_C}, \qquad (10)$$

where $A_3 P^{\beta_3} C^{\eta_3} C_N^{\gamma_3}$, with $A_3 > 0$, represents the option value with power parameters β_3, η_3 and γ_3. Again, the term $P/\left(r-\theta_P\right) - C/\left(r-\theta_C\right)$ denotes the asset value in the absence of any optionality. As before, we conjecture that β_3 is negative, and η_3 positive. We now consider the sign of γ_3. Since a stronger economic incentive exists for replacing the incumbent when the initial operating cost level is low rather than high, we would expect the replacement option to increase in value as C_N decreases, so we conjecture that the value of γ_3 should be negative.

The replacement event is signaled when the three factor levels, P, C and C_N, simultaneously attain their respective optimal threshold levels, $\hat{P}_3$, $\hat{C}_3$ and $\hat{C}_{3N}$. Collectively, these three optimal thresholds form the timing boundary, which is formulated as the relationship linking $\hat{P}_3$, $\hat{C}_3$ and $\hat{C}_{3N}$. The timing boundary is determined from the model solution, which is made up of the economic conditions signaling an optimal replacement, that is the value matching relationship and the smooth pasting conditions, plus the characteristic root equation.

Because value is conserved at replacement, the incumbent value $F_3\left(\hat{P}_3, \hat{C}_3, \hat{C}_{3N}\right)$ has to exactly balance the succeeding asset value $F_3\left(P_I, \hat{C}_N, \hat{C}_N\right)$, less the re-investment cost K. By using (10), the value matching relationship can be expressed as:

$$A_3 \hat{P}_3^{\beta_3} \hat{C}_3^{\eta_3} \hat{C}_N^{\gamma_3} + \frac{\hat{P}_3}{r-\theta_P} - \frac{\hat{C}_3}{r-\theta_C} = A_3 P_I^{\beta_3} \hat{C}_{3N}^{\eta_3+\gamma_3} + \frac{P_I}{r-\theta_P} - \frac{\hat{C}_N}{r-\theta_C} - K\,. \tag{11}$$

Replacement is optimal whenever the smooth pasting conditions are obtained. Associated with (11), there are three smooth pasting conditions, for P, C and C_N, respectively, which can be expressed as:

$$\beta_3 A_3 \hat{P}_3^{\beta_3} \hat{C}_3^{\eta_3} \hat{C}_{3N}^{\gamma_3} + \frac{\hat{P}_3}{r-\theta_P} = 0\,, \tag{12}$$

$$\eta_3 A_3 \hat{P}_3^{\beta_3} \hat{C}_3^{\eta_3} \hat{C}_{3N}^{\gamma_3} - \frac{\hat{C}_3}{r - \theta_C} = 0 , \tag{13}$$

$$\gamma_3 A_3 \hat{P}_3^{\beta_3} \hat{C}_3^{\eta_3} \hat{C}_{3N}^{\gamma_3} = \left(\eta_3 + \gamma_3\right) A_3 P_I^{\beta_3} \hat{C}_{3N}^{\eta_3+\gamma_3} - \frac{\hat{C}_{3N}}{r - \theta_C} . \tag{14}$$

We observe from (12) and (13) that $\beta_3 < 0$ and $\eta_3 > 0$. Also, since $P_I^{\beta_3} \hat{C}_{3N}^{\eta_3} < \hat{P}_3^{\beta_3} \hat{C}_3^{\eta_3}$ because $\hat{P}_3 < P_I$ and $\hat{C}_3 > \hat{C}_{3N}$, then from (14) $\gamma_3 < 0$. The three smooth pasting conditions justify our conjecture on the signs of the power parameters.

From (12) and (13), then:

$$\frac{\hat{P}_3}{\beta_3\left(r - \theta_P\right)} + \frac{\hat{C}_3}{\eta_3\left(r - \theta_C\right)} = 0 . \tag{15}$$

Along the optimal replacement boundary with $\hat{C}_{3N}$ constant, the ratio of the optimal threshold values, $\hat{P}_3 / \left(r - \theta_P\right)$ and $\hat{C}_3 / \left(r - \theta_C\right)$, has to be proportional to the absolute ratio of their power parameters, $-\beta_3 / \eta_3$.

By combining (13) and (14), A_3 can be eliminated from (11) to yield:

$$\frac{\hat{C}_3 - \hat{C}_{3N}}{r - \theta_C} = \frac{\eta_3 + \gamma_3}{\eta_3 + \gamma_3 - 1}\left[K - \frac{P_I - \hat{P}_3}{r - \theta_P}\right]. \tag{16}$$

In (16), the mark-up factor >1, so $\eta_3 > 1$ since γ_3 is treated as negative,

$$\frac{\eta_3 + \gamma_3}{\eta_3 + \gamma_3 - 1} > 1$$

provided $\eta_3 + \gamma_3 > 1$. It follows that an optimal replacement is justified when the operating cost value improvement $\left(\hat{C}_3 - \hat{C}_{3N}\right) / \left(r - \theta_C\right)$ equals the re-investment cost less the revenue value improvement $\left(P_I - \hat{P}_3\right) / \left(r - \theta_P\right)$, adjusted by the mark-up factor.

Also, A_3 can be eliminated from (11) by using (13) to yield:

$$\frac{P_I - \hat{P}_3}{r - \theta_P} + \frac{\hat{C}_3 - \hat{C}_{3N}}{r - \theta_C} = K + \frac{\hat{C}_3}{\eta_3 \left(r - \theta_C\right)} \left[1 - \frac{P_I^{\beta_3}}{\hat{P}_3^{\beta_3}} \frac{\hat{C}_{3N}^{\eta_3}}{\hat{C}_3^{\eta_3}}\right]. \tag{17}$$

Since $P_I^{\beta_3} \hat{C}_{3N}^{\eta_3} < \hat{P}_3^{\beta_3} \hat{C}_3^{\eta_3}$, replacement is optimal whenever the sum of the value improvements rendered by the replacement exceeds the re-investment cost. Both (16) and (17) can be interpreted as optimality conditions that relate a measure of the value improvement rendered by the replacement to the re-investment cost.

The final component of the model is the characteristic root equation:

$$\beta_3 \theta_P + \eta_3 \theta_C + \gamma_3 \theta_N - r = 0. \tag{18}$$

There are four constituent equations comprising Model III. These are (i) the reduced form smooth pasting condition, (15), (ii) and (iii) two reduced form value matching relationships, (16) and (17), and (iv) the characteristic root equation, (18). Explicit solutions for the three parameters β_3, η_3 and γ_3 are obtainable from (15), (16) and (18). By substituting these solutions in (17), we can eliminate β_3, η_3 and γ_3 to produce the timing boundary as the implicit relationship linking the thresholds $\hat{P}_3$, $\hat{C}_3$ and $\hat{C}_{3N}$.

2 Numerical Illustration

The respective implicit solutions to the timing boundary for Models II and III can be expressed analytically as long single equations, but because of the equation complexity, it is more convenient to discuss their behavior through a numerical illustration. Table 1 exhibits the base case data we use to illustrate the solution. In Table 1, we observe that the re-investment cost and the revenue level at installation are identical for Models I and II, but the initial operating cost levels are different. Since this difference is interpreted as an unanticipated technological improvement in the succeeding asset, by comparing these two models, we can investigate whether the optimal replacement policy is altered by an unanticipated technological progress embedded in the succeeding asset. Also, we observe for Model III that the initial operating cost level is set to decline over time, which is interpreted as a consequence of anticipated technological progress. Accordingly, a comparison of Models II and III enables the differential impact of anticipated and unanticipated technological progress on the replacement policy to be examined.

2.1 Model I

Using the values of Table 1, the optimal cycle time is evaluated from (3), and this together with revenue and operating cost thresholds are presented in Table 2.

2.2 Model II

The optimal timing boundary, which relates the two thresholds $\hat{P}_2$ and $\hat{C}_2$, is determined for Model II from (6), (7) and (8). The boundary is evaluated from the values in Table 1 and illustrated in Figure 1 for $\hat{P}_1 \le P_I$ and $\hat{C}_1 \ge C_I$. It is depicted by the locus AB, which divides the decision space between the continuance region above AB and the replacement region below AB. When the pair of prevailing levels for the revenue and operating cost is plotted on this figure, if it lies above AB then the optimal decision is to continue with the incumbent, if otherwise, then replacement is the optimal decision.

An incumbent with an operating cost level C_0 at installation has an ex-ante lifetime given by $\hat{T}_1$ with threshold levels $\hat{P}_1$ and $\hat{C}_1$. If, during its lifetime, a succeeding asset is launched with unanticipated technological advances such that its operating cost level at installation is $C_I < C_0$, then the incumbent becomes prematurely obsolescent. When the pair of thresholds $\hat{P}_1 = 64.22$ and $\hat{C}_1 = 31.03$ representing the ex-ante solution for Model I is plotted in Figure 1, the pair is observed to belong to the replacement region. This means that following the launch of a technologically more advanced asset, the incumbent should be replaced at a time earlier than $\hat{T}_1$, because of its premature obsolescence. However, if the owner persists with a replacement policy according to the ex-ante lifetime, then he is bearing the increasing inefficiencies of the incumbent while failing to fully capture the benefits of the succeeding asset.

The effect of technological obsolescence arises for an improvement in any of the initial attribute levels. Figures 2(a–c) reveal respectively that either a fall in the initial operating cost level or re-investment cost, or a rise in the initial revenue level, produces a more liberal replacement policy, leading in each case to a shortening of the economic life for the incumbent. An improvement jump in one of the initial attribute levels for the succeeding asset, which reflects an unanticipated outcome of technological progress, shifts the timing boundary upwards. This means that the net revenue lost by continuing with rather than replacing the incumbent is more than offset by the potential gain rendered by the improvement achieved by technological progress. Further, an upward shift in the timing boundary reduces the incumbent's economic life as the timing boundary draws closer to any pair of revenue and operating cost levels positioned in the continuance region. Adkins and Paxson (2011) establish this effect analytically for a stochastic replacement model, so it must also hold for the deterministic version.

2.3 Model III

The optimal timing boundary for Model III is determined from (15), (16), (17) and (18) as an implicit function linking the three factors $\hat{P}_3$, $\hat{C}_3$ and $\hat{C}_{3N}$. This three dimensional boundary is illustrated in Figure 3 as a set of boundaries, each for a constant $\hat{C}_{3N}$ value as indicated, and evaluated using the values of Table 1. In Figure 3, the lowest operating cost threshold for each timing boundary is observed to be different but equal to the indicated initial operating cost threshold $\hat{C}_{3N}$. This is a consequence of the solution, since we infer from (16) that the minimum value of $\hat{C}_3$ is $\hat{C}_{3N}$ and when $\hat{C}_3 = \hat{C}_{3N}$, $\eta_3 = -\gamma_3$ otherwise (17) would be violated.

The optimal decision is found from the timing boundary in the following way. At any point in time, we observe the prevailing initial operating cost level and set $\hat{C}_{3N}$ to equal this value. For this given $\hat{C}_{3N}$, we evaluate the timing boundary linking the thresholds, $\hat{P}_3$ and $\hat{C}_3$. If the pair of prevailing levels for revenue and operating cost when plotted on Figure 3 lies above the evaluated timing boundary and belongs to the continuance region, then the optimal decision is to continue with the incumbent. But, if the plotted pair lies below the evaluated timing boundary and belongs to the replacement region, then replacing the incumbent becomes the optimal decision.

The three timing boundaries displayed in Figure 3 are approximately parallel, and those with an indicated lower $\hat{C}_{3N}$ threshold are shifted vertically upwards. At replacement, value conservation requires a compensatory balance between the sacrificial value from not continuing with the incumbent and the net gain rendered by the succeeding asset, which is characterized by (17) that relates the incremental value due to replacement and the re-investment cost. If the net value gain for the succeeding asset does not adequately compensate for the loss incurred from not capturing the full value of the incumbent, then replacement is not economically justified. But, as the initial operating cost for the succeeding asset falls, its net value gain increases, so inevitably, a balance between the sacrificial loss and the net value gain is achieved, and replacement becomes economically justified. Moreover, the justification for replacement intensifies as the initial operating cost decreases more and more. A lower initial operating cost level implies that the incumbent can be retired at a higher net revenue, so replacement occurs at an earlier time, and this effect is reflected in the vertical shifts of the observed boundaries. Finally, the set of loci forming the boundary can be interpreted in terms of an unanticipated change. If the discrete fall in the prevailing level of the initial operating cost can be conceived as unforeseen, then Figure 3 reveals that an unanticipated fall in the initial operating cost level produces a shortening in the economic life for the incumbent.

A fall in the initial operating cost is anticipated when its level is predictable. Since the change in level is determined by the geometric rate θ_N, the initial operating cost level experiences a faster rate of decline as $-\theta_N$ increases, while for $\theta_N = 0$, there is no anticipated decline. Figure 4 illustrates four timing boundaries, each for a different θ_N value, but all for the same prevailing initial operating cost level $\hat{C}_N = 15$. When the anticipated decline rate is zero, Model III collapses to Model II, so the timing boundaries for these two models are identical. Figure 4 reveals that the timing boundaries are approximately parallel, with a vertical downward shift as θ_N becomes progressively more negative, but at a decreasing rate.

We observe from Figure 4 that an absolute increase in the decline rate for the initial operating cost level for the succeeding asset produces a downward shift in the timing boundary, so the time until retirement is prolonged for greater anticipated falls in the initial operating cost level. This suggests that when a decline in the initial operating cost level is anticipated, waiting has value. We know that at replacement, there has to exist a balance between the sacrificial loss from retiring the incumbent and the net value gain created by installing the succeeding asset. However, waiting produces two contrasting economic consequences. On one hand, it leads to an increase in the net value for the succeeding asset because of the anticipated decrease in its initial operating cost level. But waiting also means that the value sacrificed by retiring the incumbent can be more fully captured. Because these two contrasting consequences at replacement have to be balanced, the effect of an anticipated decline in the initial operating cost level is to increase the operating cost threshold for some given revenue threshold, or alternatively, to reduce the revenue threshold for some given operating cost threshold. Because the value sacrificed by the incumbent is more fully captured for anticipated technological progress, replacement is deferred and waiting has a value. We also observe from Figure 4 that the impact of anticipated technological progress on the replacement deferment wanes as the rate θ_N becomes increasingly more negative, so a doubling of the rate produces a less than proportionate downward shift. As the absolute value of θ_N increases, the time until the next optimal replacement also increases but at a decreasing rate.

The analysis of Model II and III identifies the critical distinction regarding the replacement policy under unanticipated and anticipated technological progress. If the technological progress is unanticipated, then hastening the replacement act is the optimal outcome, sometimes to the extent that replacement occurs as soon as the launch of the succeeding asset with its improvements is announced. In contrast, since an expected improvement in the attribute level for the succeeding asset entails a gain in value when the replacement is deferred, waiting has value under anticipated technological progress, so the time until the next replacement is prolonged. This feature is observable from the numerical illustration. For the incumbent with initial operating cost level $C_0 = 20$, replacement is hastened if it is announced that the succeeding asset has an initial operating cost level $C_I = 15$, but the replacement is deferred if it is also known that the anticipated improvement rate for this succeeding asset is described by $\theta_N = -10\%$.

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3 Conclusion

When technological progress is interpreted as an improvement in one or more of the initial attribute levels for the succeeding asset, particularly in the initial operating cost level, we show that unanticipated and anticipated change can be integrated in a dynamic programming formulation. The benefits of this approach are not only its avoidance of the shortcomings of a cycle time conceptualization, but its provision of the optimal replacement policy as a simple operational formula expressed in terms of thresholds. We show that the replacement policy is differentiated according to whether the technological progress is considered to be unanticipated or anticipated. An unanticipated improvement in the initial operating level has the potential to hasten the act of replacement, while the act of replacement is deferred whenever the technological progress can be fully anticipated. When the change is unanticipated, the full value from continuing with the incumbent is sacrificed for the improved benefits offered by the recently launched succeeding asset. In contrast, when the technological progress is anticipated, waiting has an advantage because of the incremental value rendered by the expected successive improvements in the succeeding asset.

The contrasting impact on the replacement policy of unanticipated and anticipated technological progress has distinct implications for owners and suppliers of replaceable assets. Owners are predisposed to capturing as much of the full value embedded in the incumbent as is economically viable, so they are motivated to search the historical records for patterns of anticipated change because of its effect on deferring replacement. However, any indications that prolongs the act of replacement is unlikely to be in the suppliers' interest. Selfishly, suppliers should announce improvements in their offerings at random times, so it becomes impossible to predict when the launch of the next generation is going to occur. Then, financial managers have to treat technological progress as unanticipated, and when a technologically advanced asset is launched, it hastens the act of replacement. In this arena of conflicting interests, the suppliers command the upper hand because of information asymmetry, since they can control the flow of technological progress and thereby influence the owners' replacement decisions.

Extensions of this paper include more general models, such as outlined in Appendix A, which allow for variable levels of revenues, operating costs and reinvestment costs for each succeeding asset, and, naturally, viewing these replacement models in a stochastic environment.

Appendix A: A General Model

In Model IV, the attribute levels at installation for the succeeding asset are allowed to vary. In addition to the model variables revenue P_4 and operating cost C_4, we also have the initial levels for the revenue and operating cost, P_{4N} and C_{4N}, respectively, and the re-investment cost K_N. If each of these five variables follows a geometric Brownian process, with drift rate θ_P, θ_C, θ_{P_N}, θ_{C_N} and θ_{K_N}, respectively, the solution to the valuation relationship for the asset including the replacement option, formed as a partial differential equation using Ito's Lemma, is:

$$F_4\left(P_4, C_4, P_{4N}, C_{4N}, K_{4N}\right) = A_4 P_4^{\beta_4} C_4^{\eta_4} P_{4N}^{\lambda_4} C_{4N}^{\gamma_4} K_{4N}^{\delta_4} + \frac{P_4}{r-\theta_P} - \frac{C_4}{r-\theta_C}. \qquad (19)$$

In (19), the asset value F_4 is composed of two elements, $A_4 P_4^{\beta_4} C_4^{\eta_4} P_{4N}^{\lambda_4} C_{4N}^{\gamma_4} K_{4N}^{\delta_4}$, with $A_4 > 0$, which denotes the replacement option value, and

$$\frac{P_4}{r-\theta_P} - \frac{C_4}{r-\theta_C}$$

which denotes the asset value in the absence of any optionality. Because it is economically more justifiable to replace the incumbent when the revenue level is low or the operating cost level is high, or when the initial revenue level is high or the initial operating cost level is low, or when the re-investment cost is low, the option value increases with the operating cost level and the initial revenue level, but decreases with the revenue level, initial operating cost level and the re-investment cost. Consequently, we expect $\beta_4 < 0$, $\eta_4 > 0$, $\lambda_4 > 0$, $\gamma_4 < 0$ and $\delta_4 < 0$.

The power parameters of F_4 have to satisfy the characteristic root equation. After converting the model to be deterministic by setting the various volatilities to equal zero, the characteristic equation becomes:

$$\theta_P \beta_4 + \theta_C \eta_4 + \theta_{P_N} \lambda_4 + \theta_{C_N} \gamma_4 + \theta_{K_N} \delta_4 - r = 0 \,, \tag{20}$$

where r is the appropriate risky discount rate.

The optimal threshold levels for P_4, C_4, P_{4N}, C_{4N} and K_N that signal replacement are denoted by $\hat{P}_4$, $\hat{C}_4$, $\hat{P}_{4N}$, $\hat{C}_{4N}$ and $\hat{K}_N$, respectively. Value conservation demands that the incumbent asset value immediately before replacement, $F_4\left(\hat{P}_4, \hat{C}_4, \hat{P}_{4N}, \hat{C}_{4N}, \hat{K}_{4N}\right)$, has to be balanced by the value of the succeeding asset at installation, $F_4\left(\hat{P}_{4N}, \hat{C}_{4N}, \hat{P}_{4N}, \hat{C}_{4N}, \hat{K}_{4N}\right)$, less the re-investment cost $\hat{K}_{4N}$. The value matching relationship becomes:

$$A_4 \hat{P}_4^{\beta_4} \hat{C}_4^{\eta_4} \hat{P}_{4N}^{\lambda_4} \hat{C}_{4N}^{\gamma_4} \hat{K}_{4N}^{\delta_4} + \frac{\hat{P}_4}{r - \theta_P} - \frac{\hat{C}_4}{r - \theta_C} = A_4 \hat{P}_{4N}^{\beta_4} \hat{C}_{4N}^{\eta_4} \hat{P}_{4N}^{\lambda_4} \hat{C}_{4N}^{\gamma_4} \hat{K}_{4N}^{\delta_4} + \frac{\hat{P}_{4N}}{r - \theta_P} - \frac{\hat{C}_{4N}}{r - \theta_C} - \hat{K}_{4N} \,. \tag{21}$$

The five smooth pasting conditions are the first order conditions for optimality. For P_4, C_4, P_{4N}, C_{4N} and K_N, the respective smooth pasting conditions can be expressed as:

$$\beta_4 A_4 \hat{P}_4^{\beta_4} \hat{C}_4^{\eta_4} \hat{P}_{4N}^{\lambda_4} \hat{C}_{4N}^{\gamma_4} \hat{K}_{4N}^{\delta_4} + \frac{\hat{P}_4}{r - \theta_P} = 0 \,, \tag{22}$$

$$\eta_4 A_4 \hat{P}_4^{\beta_4} \hat{C}_4^{\eta_4} \hat{P}_{4N}^{\lambda_4} \hat{C}_{4N}^{\gamma_4} \hat{K}_{4N}^{\delta_4} - \frac{\hat{C}_4}{r - \theta_C} = 0 \,, \tag{23}$$

$$\lambda_4 A_4 \hat{P}_4^{\beta_4} \hat{C}_4^{\eta_4} \hat{P}_{4N}^{\lambda_4} \hat{C}_{4N}^{\gamma_4} \hat{K}_{4N}^{\delta_4} = \left(\beta_4 + \lambda_4\right) A_4 \hat{P}_{4N}^{\beta_4} \hat{C}_{4N}^{\eta_4} \hat{P}_{4N}^{\lambda_4} \hat{C}_{4N}^{\gamma_4} \hat{K}_{4N}^{\delta_4} + \frac{\hat{P}_{4N}}{r - \theta_P} \,, \tag{24}$$

$$\gamma_4 A_4 \hat{P}_4^{\beta_4} \hat{C}_4^{\eta_4} \hat{P}_{4N}^{\lambda_4} \hat{C}_{4N}^{\gamma_4} \hat{K}_{4N}^{\delta_4} = \left(\eta_4 + \gamma_4\right) A_4 \hat{P}_{4N}^{\beta_4} \hat{C}_{4N}^{\eta_4} \hat{P}_{4N}^{\lambda_4} \hat{C}_{4N}^{\gamma_4} \hat{K}_{4N}^{\delta_4} - \frac{\hat{C}_{4N}}{r - \theta_C}, \tag{25}$$

$$\delta_4 A_4 \hat{P}_4^{\beta_4} \hat{C}_4^{\eta_4} \hat{P}_{4N}^{\lambda_4} \hat{C}_{4N}^{\gamma_4} \hat{K}_{4N}^{\delta_4} = \delta_4 A_4 \hat{P}_{4N}^{\beta_4} \hat{C}_{4N}^{\eta_4} \hat{P}_{4N}^{\lambda_4} \hat{C}_{4N}^{\gamma_4} \hat{K}_{4N}^{\delta_4} - \hat{K}_{4N}. \tag{26}$$

The coefficient A_4 can be eliminated from (21) - (26) in the following way. From (22) and (23):

$$\frac{\hat{P}_4}{\beta_4\left(r - \theta_P\right)} + \frac{\hat{C}_4}{\eta_4\left(r - \theta_C\right)} = 0. \tag{27}$$

From (24) and (25):

$$\frac{\hat{P}_{4N}}{\lambda_4\left(r - \theta_P\right)} + \frac{\hat{C}_{4N}}{\gamma_4\left(r - \theta_C\right)} = 0. \tag{28}$$

From (21) and (26):

$$\left(\frac{\hat{P}_{4N} - \hat{P}_4}{r - \theta_P}\right) - \left(\frac{\hat{C}_{4N} - \hat{C}_4}{r - \theta_C}\right) = \frac{\delta_4 - 1}{\delta_4} \hat{K}_{4N} > \hat{K}_{4N}. \tag{29}$$

From (21) and (22):

$$\left(\frac{\hat{P}_{4N} - \hat{P}_4}{r - \theta_P}\right) - \left(\frac{\hat{C}_{4N} - \hat{C}_4}{r - \theta_C}\right) = \hat{K}_{4N} - \frac{\hat{P}_4}{\beta_4\left(r - \theta_P\right)}\left[1 - \frac{\hat{P}_{4N}^{\beta_4} \hat{C}_{4N}^{\eta_4}}{\hat{P}_4^{\beta_4} \hat{C}_4^{\eta_4}}\right] > \hat{K}_{4N}. \tag{30}$$

From (21), (23) and (25):

$$\left(\frac{\hat{P}_{4N} - \hat{P}_4}{r - \theta_P}\right) - \left(\frac{\hat{C}_{4N} - \hat{C}_4}{r - \theta_C}\right) = \hat{K}_{4N} - \frac{1}{\eta_4 + \gamma_4}\left(\frac{\hat{C}_{4N} - \hat{C}_4}{r - \theta_C}\right) > \hat{K}_{4N}. \tag{31}$$

The five reduced form equations, (27) - (31), and the characteristic root equation (20) are sufficient to eliminate the five power parameters, and to yield the replacement boundary as an implicit function of $\hat{P}_4$, $\hat{C}_4$, $\hat{P}_{4N}$, $\hat{C}_{4N}$ and $\hat{K}_N$.

Table 1

Base Case Data

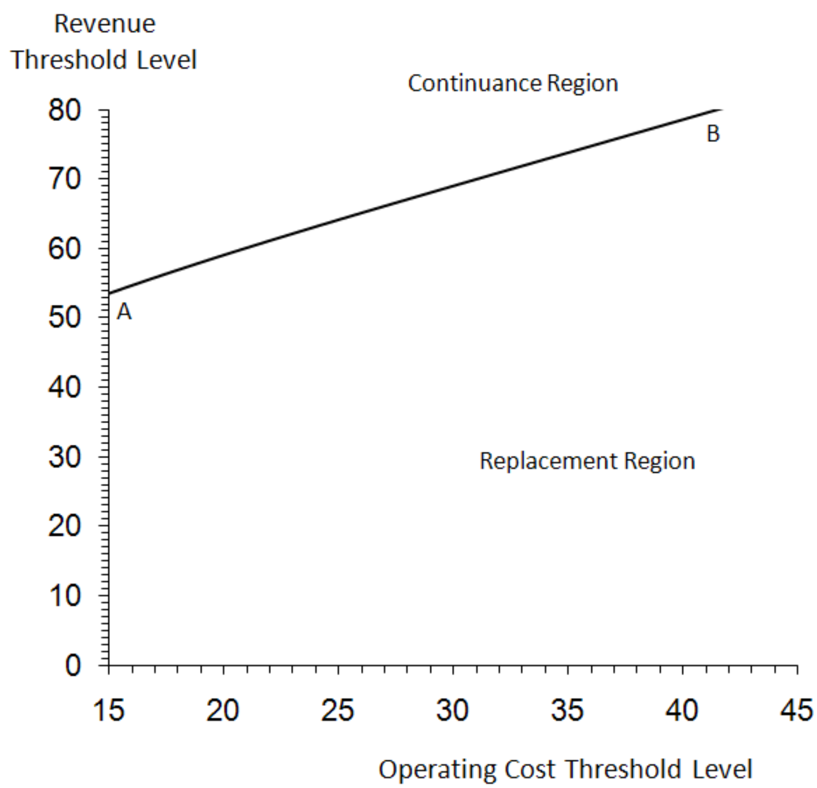
Description	Parameter	Value
Re-investment Cost	K	100
Initial Revenue Level for Incumbent	P_0	80
Initial Operating Cost Level for Incumbent	C_0	20
Initial Revenue Level for Succeeding Asset	P_I	80
Initial Operating Cost Level for Succeeding Asset	C_I	15
Revenue Growth Rate	θ_P	4%
Operating Cost Growth Rate	θ_C	-2%
Initial Operating Cost Growth Rate	θ_N	-5%
Relevant Discount Rate	r	12%

Table 2
Optimal Solution for Model I

Optimal Cycle Time	$\hat{T}_1$	10.99
Optimal Revenue Threshold	$\hat{P}_1$	64.22
Optimal Operating Cost Threshold	$\hat{C}_1$	31.03

The optimal cycle time $\hat{T}_1$ is evaluated from (3) using the values of Table 1. The thresholds $\hat{P}_1$ and $\hat{C}_1$ are determined from $\hat{P}_1 = P_0\, \mathrm{e}^{\theta_P \hat{T}_1}$ and $\hat{C}_1 = C_0\, \mathrm{e}^{\theta_C \hat{T}_1}$.

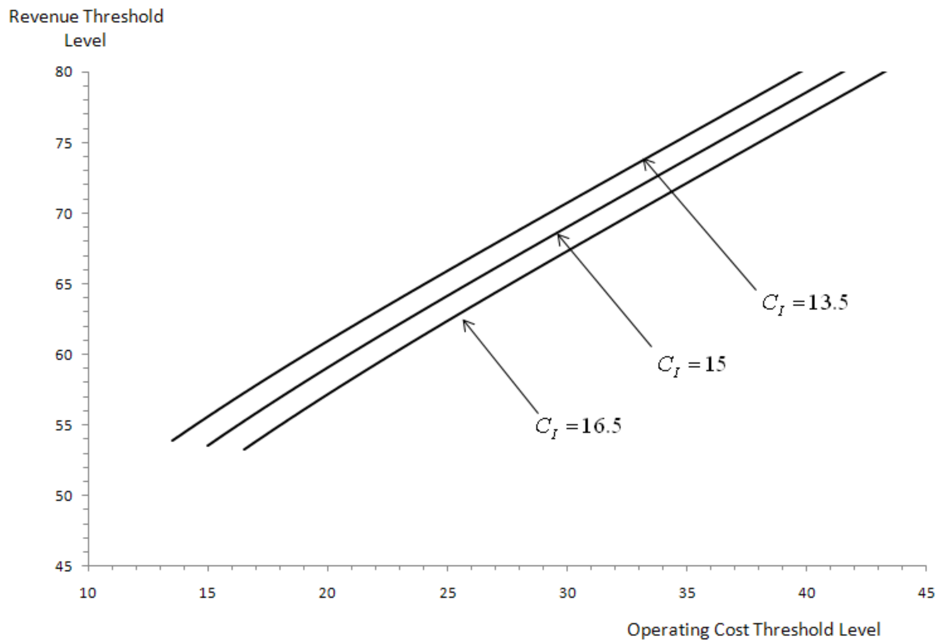
Figure 1
The Timing Boundary for Model II



The timing boundary AB is found as the solution to (6), (7) and (8) for $\hat{P}_2 \le P_I$ and $\hat{C}_2 \ge C_I$, using the values in Table 1. When the constraint $\hat{C}_2 \ge C_I$ is lifted, the minimum value of $\hat{C}_2$ is zero, with $\hat{P}_2 = 34.10$, $\beta_2 = -6$ and $\eta_2 = 0$. Representative values along AB are presented in the following table:

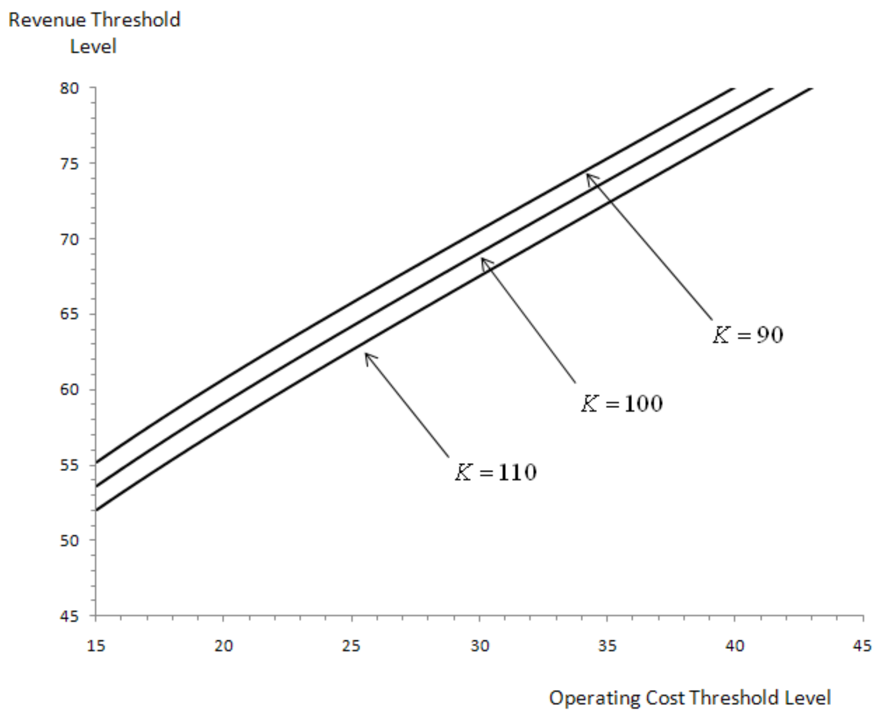
$\hat{C}_2$	$\hat{P}_2$	β_2	η_2
15.0	53.57	-3.0301	1.4849
20.0	59.10	-2.7467	1.6267
25.0	64.19	-2.5390	1.7305
30.0	69.07	-2.3808	1.8096
35.0	73.87	-2.2571	1.8715
40.0	78.65	-2.1582	1.9209

Figure 2(a)
The Effect of Variations in the Initial Operating Cost Level on the Timing Boundary



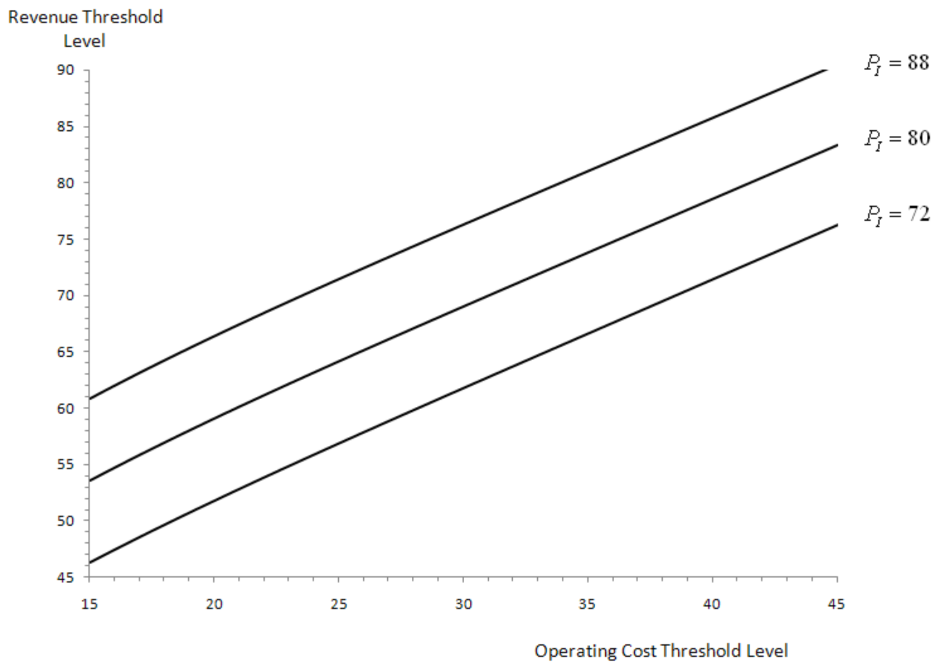
The timing boundary for each indicated C_I level is evaluated according to Figure 1, using the values of Table 1 except for C_I. The range for each boundary is $\hat{C}_2 \geq C_I$.

Figure 2(b)
The Effect of Variations in the Re-investment Cost Level on the Timing Boundary



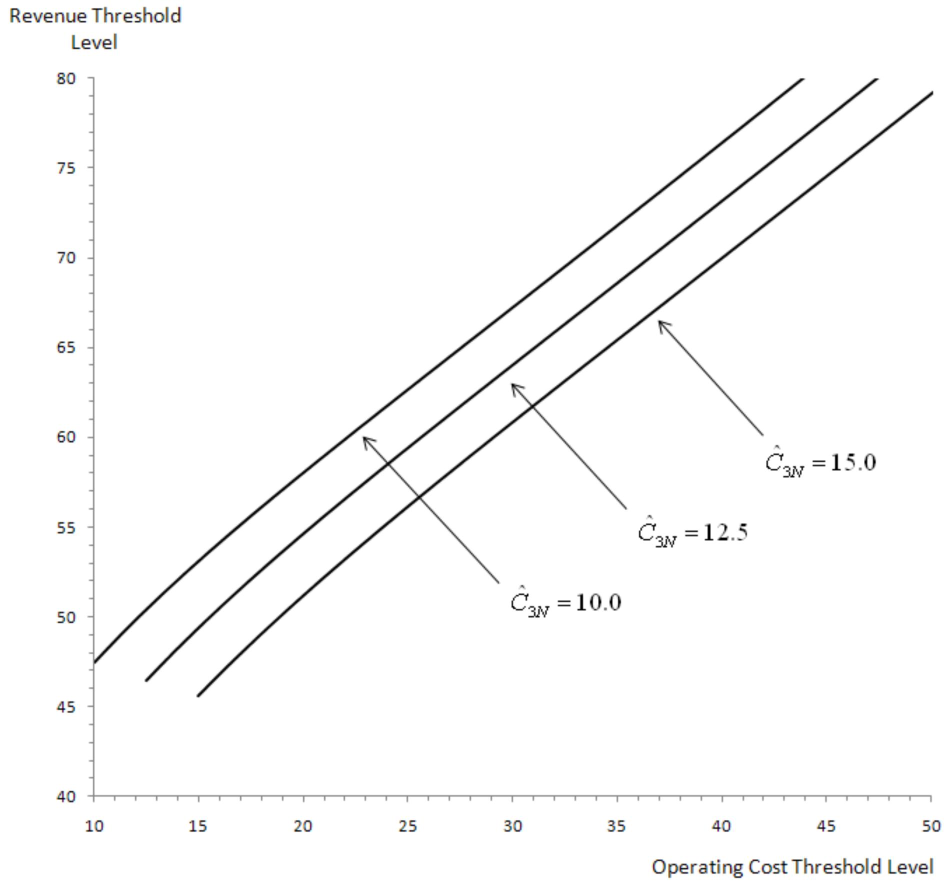
The timing boundary for each indicated K level is evaluated according to Figure 1, using the values of Table 1 except for K.

Figure 2(c)
The Effect of Variations in the Initial Revenue Level on the Timing Boundary



The timing boundary for each indicated P_I level is evaluated according to Figure 1, using the values of Table 1 except for P_I.

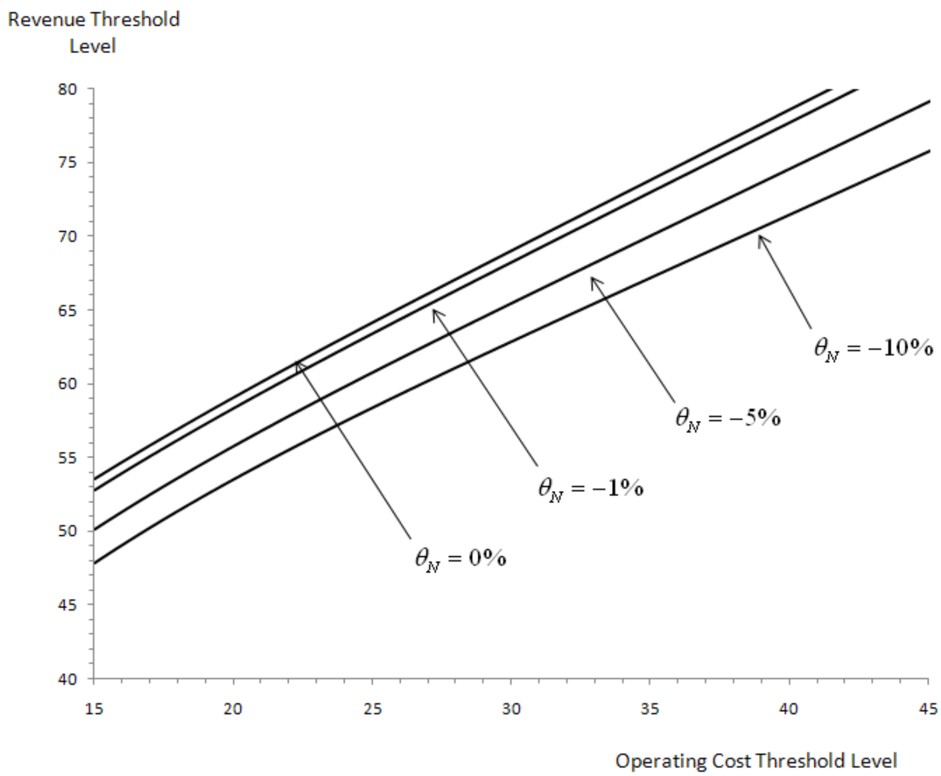
Figure 3
The Effect of Variations in the Initial Operating Cost Level on the Timing Boundary



The timing boundary is found as the solution to (15), (16), (17) and (18).

Figure 4

The Effect of Variations in Initial Operating Cost Growth Rate on the Timing Boundary



The timing boundary is found as the solution to (15), (16), (17) and (18).

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