Returns and Doubling Times

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Abstract

This paper examines the re-expression of returns in the time domain as doubling times. Several uses for doubling times are suggested including truth in lending. The expected time for an investment to double can be calculated from a time series of doubling times either using harmonic means, or simulation. A normal distribution for returns yields doubling times with an inverse Gaussian distribution and this holds approximately for any return distribution. Doubling times provide an alternative calculus for portfolio optimisation. The minimisation of either skewness, or the inverse of the shape parameter, for doubling times reproduces the Markowitz efficient frontier.

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Keywords: Returns, return transformation, doubling times, portfolio optimisation, risk metrics

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I. Introduction

This paper examines the re-expression of returns in the time domain. Traditionally investment performance is measured as the increment in wealth per unit of time. In this paper we turn the measurement round and ask how many units of time are required to give a unit increment in wealth. Thus returns are re-expressed as doubling times.

There are several reasons for the study of doubling times. Doubling times are an intuitively attractive way to express returns. This is evident from the development of rules of thumb for estimating doubling times, such as the rule of seventy-two. Despite this intuitive appeal, there has been little study of doubling times in finance. Therefore, one purpose of this paper is to present some of the properties of doubling times.

Transforming returns into the time domain provides a different perspective on returns. Viewing returns from a different perspective may stimulate new ideas and new insights that might not otherwise be obtained. Doubling times may also have advantages over returns in certain applications. We suggest three potential uses for doubling times, in relation to truth in lending, performance measurement, and capital budgeting. We demonstrate a fourth application in portfolio optimisation.

It is a simple matter to compute doubling times period by period. However, the mean of the period by period doubling times (the mathematical expectation) does not give the time over which the investor should expect to double their money (the expected doubling time). We present two approaches to computing the expected doubling time. An analytical approach shows that harmonic means can be used in estimating doubling times.

Formulas using harmonic means are given for discrete and continuously compounded rates of return and also for simple interest rates.

A Monte Carlo simulation method is also used to compute the expected doubling time. The results of the simulation approximate an inverse Gaussian distribution. We show analytically that if returns are normally distributed, then doubling times will follow an inverse Gaussian distribution. We also note that whatever the return distribution, the doubling time distribution will be approximately inverse Gaussian. We present formulas that transform the parameters of the normal return distribution to the parameters of the inverse Gaussian doubling time distribution. Using the properties of the inverse Gaussian distribution we demonstrate that minimising either the skewness, or the inverse of the shape parameter, of the doubling time distribution results in the Markowitz mean variance efficient frontier for returns.

The remainder of this paper is organised as follows. Section II outlines the history of doubling times and the possible applications to finance. Section III discusses methods for computing expected doubling times from a time series, or cross section, of doubling times. Section IV demonstrates the computation of doubling times for the Dow Jones U.S Total Stock Market Total Return Index and also presents an illustration of portfolio optimisation using doubling times. The conclusions of this study are presented in Section V.

II. Doubling Times History, Definition and Uses

For discrete returns the doubling time is approximated by the rule of 72, dividing 72 by the rate of return gives the approximate time for an investment to double in value.

Internet sources often credit Albert Einstein for the rule of 72;[1] but almost 500 years earlier it was mentioned by Luca Pacioli (1494) and quite possibly it predates Pacioli. When compounding is continuous, rather than discrete, the rule of 69 is used instead of the rule of 72.

Although the expression of returns as doubling times has been known for 500 or more years, there has been little academic work on equity market doubling times. Indeed, other than descriptions of the rules of 72 and 69, there is little on doubling times in the financial literature. However, the concept of doubling times has had more extensive use in other fields. Doubling times are applied to population growth as in Kendall (1949). Doubling times are also used in medicine; a common application is measuring the growth of a tumour, for example Hanks, D'Amico, Epstein and Schultheiss (1993). Half-lives, the converse of doubling times, are also used in other fields. Half lives are most commonly associated with radioactive decay, but can be applied to anything which decays. Medical applications involve nuclear medicine as in Hendee (1979). The study of population extinctions also makes use of half lives, Brooks, Pimm and Oyugi (1999).

Half-lives can also be defined as the number of periods required for the impulse response to a unit shock to a time series to dissipate by half. Such a metric is sometimes found when measuring the degree of mean-reversion or persistence in economic and financial time series. A notable example where half lives are used is in the study of the purchasing power parity, Rogoff (1996), Rossi (2005) and Kilian and Zha (2002).

Future values under continuous compounding are given by:

$$FV = PVe^{rt}$$

[1] See for example: www.investingpage.com/einsteins-rule-of-72.html

where FV is the Future Value, PV is the Present Value, r is the rate of return, and t time to maturity. By setting FV to 2 and PV to 1 and solving for time, t we obtain the doubling time, τ :

(1) $$\tau = \log(2)/r$$

A result which is approximated by the rule of 69 as the natural logarithm of 2 equals 0.6931.

Future values with discrete returns are given by:

$$FV = PV(1+r)^t$$

and τ is given by:

(2) $$\tau = \log(2)/\log(1+r)$$

This equation is approximated by the rule of 72.

For simple interest future values are given by:

$$FV = PV(1+rt)$$

and τ is given by:

(3) $$\tau = 1/r$$

While the formulas above differ, for any given asset they will all give the same doubling time, since no matter how the returns are expressed they all reflect the same underlying investment performance. We suggest three possible uses for doubling times below and later we show how doubling times can be applied to portfolio optimisation.[2]

Truth in Lending: Interest rates may expressed as simple or compound rates, compounding may be discrete or continuous, and interest rates may be expressed in nominal or effective terms. Whatever way the interest rate is expressed, there is only one doubling time. Thus doubling times could supply a standard benchmark for comparing loans and would probably have an intuitive appeal to consumers.

Performance Measurement: If doubling times are useful in truth in lending, this also suggests that they might provide a useful way to report performance to investors. For example, investment funds could be asked to report how long ago one dollar would need to have been invested with the fund in order to have doubled in value by the current date.

Capital Budgeting: The payback period continues to be very popular in capital budgeting despite its well known deficiencies.[3] The doubling time has the intuitive appeal of payback, but can also give decisions consistent with traditional DCF analysis. This would require computing the project IRR and converting it to a doubling time. The resulting doubling time would then be compared with the doubling time implied by the cost of capital. Clearly this is not a panacea for problems in project analysis. The computations

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[2] We doubt that this is an exhaustive of the possibilities, but it serves to illustrate the potential usefulness of doubling times.

[3] An Australian capital budgeting survey by Truong, Partington and Peat (2008) contains a convenient summary of surveys from around the world, which shows that payback continues to be a popular metric in project analysis.

are more complex than those required for the traditional payback measure and might sometimes involve dealing with multiple roots in the IRR equation. However, the doubling time provides a convenient and intuitive way of communicating a DCF result to managers with little understanding of finance.

III. Computing Expected Doubling Times

The computations that underlie the following results were based on continuously compounded returns and hence doubling times were computed using equation (1). As noted above, had we used discrete returns and equation (2) or simple interest rates and equation (3) identical doubling times would have been obtained. A plot of doubling times against continuous returns was generated from equation (1) and is given in Figure 1. It is immediately evident from Figure 1 that negative doubling times (half lives) are a mirror image of positive doubling times. It is also evident that there is a discontinuity at zero. As returns approach zero, half lives tend to minus infinity and positive doubling times tend to plus infinity.

Figure 1 suggests that in forming the parameters of doubling time distributions there will be a problem in handling cases with zero returns due to infinite doubling times. Care is also needed in combining doubling times. For example, consider a case where the first year's return has a half life represented by minus five years and the second year's return has a doubling time represented by plus five years. The arithmetic mean gives a doubling time of zero years. However, an investor who experiences this combination of half-life and doubling time will not instantly double her money. In general the arithmetic mean of doubling times is not the same as the doubling time that investors experience. Investors are naturally interested in the doubling times they actually experience and we describe below how this parameter can be computed from the time series of doubling times.

FIGURE 1 ABOUT HERE

One approach to dealing with zero returns and infinite doubling times is to exclude such data from the analysis and report the incidence of zero returns. We do not expect zero returns to be a common problem for actively traded stocks, but it could happen, and is more likely as we move from monthly to daily data. For thinly traded stocks apparent observations of zero return are not uncommon in daily data. However, it is important to distinguish when zero returns really mean that the price has not changed, and when they mean that a change in price has not been observed because of thin trading. In the latter case the return is not really zero but missing.

A. Computing expected doubling times

Several approaches can be used in computing the expected doubling time. The simplest and most obvious is to take the compound rate of return (geometric average) over the full set of return observations. Converting that compound return to a doubling time will give the expected doubling time. This approach however, provides no information about the relation between the expected doubling time and the individual doubling times derived from each element of the time series of returns.

B. The expected doubling time in time series

We use two approaches to estimating the doubling time directly from the time series of doubling times. One approach is a numerical estimate using Monte Carlo simulation, the other approach is analytical.

In the simulation approach the returns are repeatedly re-sampled at random and a cumulative compound return factor computed. When this cumulative return factor sums to two the doubling point has been reached. The number of iterations to reach this point gives the doubling time and many repetitions of this process give a distribution of possible doubling times about the expected doubling time.

The analytical approach to computing the expected doubling time requires different formulae for discrete and continuous returns. The derivations are as follows,

C. *For continuous compounding:*

From equation (1)

$$\tau = \log(2) / r$$

If the rate of return is r_i for a period t_i, the equivalent rate R satisfies:

$$e^{RT} = e^{r_1 t_1 + r_2 t_2 \ldots + r_n t_n}$$

where $T = \sum_i t_i$

Taking the natural logarithm of both sides of the above equation leads to

$$RT = r_1 t_1 + r_2 t_2 \ldots\ldots + r_n t_n$$

So R is the weighted arithmetic mean of the r_i. Since $r = log(2)/\tau$, the above equation can be rewritten as:

$$T \log(2)/\tau = \sum_{i=1}^{n} t_i \log(2)/\tau_i$$

which simplifies to

(4) $$\tau = T\left(\sum_{i=1}^{n} t_i/\tau_i\right)^{-1}$$

In this case the expected doubling time is the time weighted harmonic mean of the individual doubling times τ_i.

D. For discrete compounding:

From equation (2)

$$\tau = \log(2)/\log(1+r)$$

Suppose the returns are $(r_1, r_2, ..., r_n)$ for n-periods. Then the equivalent rate R satisfies

$$(1+R)^n = \prod_{i=1}^{n}(1+r_i)$$

So $(1+R)$ is the geometric mean of the one period terms $(1+r_i)$. Since $(1+r)^\tau = 2$, the above equation can be rewritten as:

$$2^{n/\tau} = \prod_{i=1}^{n} 2^{1/\tau_i}$$

$$2^{n/\tau} = 2^{\sum_{i=1}^{n} 1/\tau_i}$$

which leads to

(5) $$\tau = n\left(\sum_{i=1}^{n}\left(1/\tau_i\right)\right)^{-1}$$

This implies that the expected doubling-time is just the harmonic mean of the individual one step doubling times τ_i.

E. For simple interest rates:

From equation (3)

$$\tau = 1/r$$

Suppose the simple interest rates are r_i for a period t_i. Then the equivalent rate R satisfies

$$R\sum_{i}^{n} t_i = \sum_{i}^{n} t_i r_i$$

Since $r = 1/\tau$, the above equation can be rewritten as:

$$\frac{1}{\tau}\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} t_i/\tau_i$$

which leads to:

(6) $$\tau = T\left(\sum_{i=1}^{n}\left(t_i/\tau_i\right)\right)^{-1}$$

where $T = \sum_i t_i$.

As for continuous compounding the expected doubling time is the time weighted harmonic mean of the individual doubling times τ_i.

F. The expected doubling time in cross-section

It is a simple matter to derive an analytical formula for computing doubling times from the cross section of doubling times. The return for a portfolio in cross section is:

$$R = \sum_i^n w_i r_i$$

Where R is the portfolio return, r_i is an individual asset return in the cross-section and w_i is the weight assigned to asset i in the cross-section. Since $r = log(2)/\tau$, the above equation can be re-written as:

$$\log(2) \big/ \tau = \sum_{i=1}^n w_i \log(2) \big/ \tau_i$$

which leads to:

$$\tau = \left(\sum_{i=1}^n \left(w_i \big/ \tau_i \right) \right)^{-1}$$

The expected doubling time is the weighted harmonic mean of the individual doubling times.

IV. Data and Results

The calculation of doubling times is illustrated with equity data obtained from Bloomberg. The equity data consisted of daily values for the Dow Jones U.S Total Stock Market Total Return Index from the 16th of October 1988 to the 13th of October 2013 which gave a total of 6521data points.

*A. Raw doubling time*s

Histograms of half lives and doubling times for the Dow Jones U.S Total Stock Market Total Return Index are given in Figure 2 for daily weekly and monthly observations. The great majority of doubling times and half lives are under ten years and it appears that a surprisingly large number of observations are at zero. This latter observation, however, is somewhat misleading. Panel B of Figure 2 gives plots that are scaled to magnify the observations centred on zero. Panel B illustrates that there is a discontinuity at zero.

FIGURE 2 ABOUT HERE

Table 1 provides descriptive statistics for the full data set and for the sub-samples of positive and negative observations, which represent doubling times and half lives respectively. The results are presented at daily, weekly and monthly frequencies. In Panel A the doubling times are reported in years. In Panel B doubling times are reported relative to the frequency of observation, thus the doubling times for daily observations are reported in days and so on. Infinite doubling times are omitted in computing the descriptive statistics, but the number of such instances is reported as the number of zero returns removed.

The problem of computing the expected doubling time as the arithmetic mean of individual doubling times is clearly demonstrated in Table 1. These daily, weekly, and monthly samples are all drawn from the same period (16th October 1988 to 13th October 2013). As these samples all started and finished on the same day, the overall change in wealth is identical regardless of the frequency of measurement. Consequently, the expected doubling time is independent of the frequency of measurement. However, the mean of the doubling times in Table 1 vary with the observation frequency. The mean doubling time based on weekly observations for the full sample is 0.795 years. In contrast, the mean doubling time based on monthly returns for the full sample is minus 1.906 years. Similar variation is also observed when considering the doubling times and half-lives separately. Investors are naturally interested in the doubling times that they can expect, and that they actually experience, but doubling times that vary with the frequency of return observation will not provide this information.

TABLE 1 ABOUT HERE

B. Expected doubling times

Table 2 provides descriptive statistics for the expected doubling time of the Dow Jones U.S Total Stock Market Total Return Index consistent with an investor's experience. The expected doubling times are derived from the analytical approach and from the simulation using daily, weekly and monthly observations. The expected doubling times are reported both in years and in units of time corresponding to the measurement interval. In all cases, for the analytical calculation, the expected time to double the investment is equal to 7.27 years.

The simulated distributions of doubling times are based on 5000 iterations. The expected doubling time is computed as the arithmetic average of the simulated distribution. The simulated expected doubling times are 7.37, 7.41 and 7.54 years when using the daily, weekly and monthly returns respectively. As the *t*-statistic in Table 2 shows these simulated values are not significantly different to the analytical value of 7.27 years. As subsequently discussed, the appropriate distributional assumption is not normality, but rather that the doubling times follow the inverse Gaussian distribution. Consequently, the *t*-statistic was computed under this assumption following the method of Chhikara and Folks (1989).

TABLE 2 ABOUT HERE

The Distribution of Doubling Times

A density plot for the daily, weekly and monthly simulated doubling times is given in Figure 3. Also in Figure 3, the cumulative density function for the simulated distribution is plotted against the cumulative density function for the inverse Gaussian distribution. These P-P plots suggest that the doubling times follow the inverse Gaussian distribution. The intuition for this result is as follows. The inverse Gaussian distribution describes the time it takes to reach a fixed positive level (the first passage time) for a process that follows Brownian motion with positive drift. This is analogous to the time a stock with positive drift and a stochastic component in returns takes to double its initial value.

FIGURE 3 ABOUT HERE

Thus far no assumption has been made about the underlying distribution of returns. However, if returns are normally distributed then it can be shown that doubling times will follow an inverse Gaussian distribution. Details are given in the appendix, where it is shown that the mean (μ_{DT}) and variance (σ_{DT}^2) of the doubling times are given by:

(7) $$\mu_{DT} = \frac{\log(2)}{\mu_r}$$

(8) $$\sigma_{DT}^2 = \frac{\log(2)\sigma_r{}^2}{\mu_r^3}$$

Where μ_r and σ_r^2 are the mean and variance of the underlying normally distributed returns.

Whatever the return distribution, as long as returns are independent with a finite variance, then by the central limit theorem cumulative returns will be approximately normal, see de la Granville (1998).[4] Thus, even if the underlying return distribution is not normal we expect that doubling times will approximately follow an inverse Gaussian distribution.

Traditionally, the inverse Gaussian distribution is not defined by the mean and variance but by the mean and shape parameter. The shape parameter is defined as

(9) $$\lambda = \mu_{DT}^3 / \sigma_{DT}^2$$

Where μ_{DT} and σ_{DT}^2 are the inverse Gaussian distribution's mean and variance respectively and the *DT* subscript denotes doubling times.

Using equations (7) and (8) the shape parameter for the doubling time distribution can be written in terms of returns as:

(10) $$\lambda_{DT} = \frac{\log(2)^2}{\sigma_r^2}$$

Thus we can write the distribution of doubling times as:

(11) $$\text{Doubling Times} \sim IG\left(\frac{\log(2)}{\mu_r}, \frac{\log(2)^2}{\sigma_r^2}\right)$$

[4] We thank Paul Dunmore for drawing this point to our attention.

We note that our results can be applied to tripling, quadrupling or any other n-tuple by using a scaling rule. From the derivation in the appendix, it can be seen that any n-tuple time is distributed as

$$n\text{ - tuple time} \sim IG\left(\frac{\log(n)}{\mu_r}, \frac{\log(n)^2}{\sigma_r^2}\right)$$

Given this, a scaling rule to transform the expected doubling time to the expected n-tuple time is defined as follows:

$$\mu_N = \mu_{DT} \frac{\log(N)}{\log(2)}$$

Where μ_N is the N-tuple time and μ_{DT} is the expected doubling time.

Similarly, to transform the shape parameter for doubling times to n-tuple times, the following scaling rule applies:

$$\lambda_N = \lambda_{DT} \frac{\log(N)^2}{\log(2)^2}$$

Where λ_{N} is the N-tuple time shape parameter and λ_{DT} is the doubling time shape parameter.

C. Portfolio Optimisation

In optimising a doubling time portfolio the objective is to minimise the expected doubling time on the assumption that investors wish to amass wealth as quickly as possible, but at the same time investors wish to avoid risk. While minimising the doubling time is clearly analogous to maximising returns, the choice of risk metric is less obvious. Either skewness, or the shape parameter, but not variance, can be used in determining the efficient frontier.[5]

We show below that it is possible to obtain the global minimum variance portfolio by minimising the inverse of the shape parameter, subject to portfolio weights summing to one. We also show that minimising the skewness, subject to portfolio weights summing to one, gives the tangency portfolio (assuming no risk free asset) of the Markowitz efficient frontier.[6] By the two fund theorem, the full efficient frontier can be generated as weighted combinations of the global minimum variance and tangency portfolios, Merton (1972). However, we also show below that either skewness, or the shape parameter, can be used to directly generate the full efficient frontier.

From equation (10) the inverse of the shape parameter is:

$$(\lambda_{DT})^{-1} = \frac{\sigma_r^2}{c} \quad \text{where } c = \log(2)^2 \tag{12}$$

Equation (12) shows that the inverse of the shape parameter is the variance of returns scaled by a constant. As such, if the inverse of the shape parameter is minimised, with the constraint that portfolio weights sum to one, the resulting portfolio will be the global minimum variance portfolio of returns.

To reduce the risk of waiting for long periods of time to double their investment, investors will desire a reduction in the skewness of doubling times. The skewness, γ, of the inverse Gaussian distribution is defined as:

$$\gamma = 3\left(\frac{\mu_{DT}}{\lambda_{DT}}\right)^{1/2} \tag{13}$$

[5] Variance is inappropriate as a risk metric given that the distribution is not symmetric.
[6] In the absence of a risk-free asset this tangency point is the portfolio with the highest ratio of return to risk.

Substituting equations (7) and (10) for μ_{DT} and λ_{DT} in equation (13) gives:

(14) $$\gamma = 3\left(\frac{\sigma_r^2}{\mu_r \log(2)}\right)^{1/2}$$

(15) $$\gamma = c\left(\frac{\sigma_r}{\sqrt{\mu_r}}\right)$$

where c is a constant = $3/\sqrt{\log(2)}$

Accordingly, minimising skewness involves minimising a risk to return ratio or, equivalently, maximising a ratio of return to risk.

Using data as given in Broadie (1993), five assets are selected with known means, variances and correlations as given by Tables 3 and 4 below.

TABLE 3 ABOUT HERE

Using the data in Table 3 solutions to the following constrained optimisations were obtained:

Inverse Shape Parameter.

(16) Minimise: λ^{-1}

Subject to: $\sum w_i = 1$

Where w_i is the weight of security i in the portfolio. Using equation (12), the optimisation problem defined in equation (16) can be redefined using the rate of return parameters as:

Minimise: $\dfrac{w'\Omega w}{(\log(2))^2}$

Subject to: $w'\iota = 1$

Where w is the vector of portfolio weights, ι is a vector of ones and Ω is the covariance matrix of the returns for the assets in the portfolio

Skewness.

(17) Minimise: γ

Subject to: $\sum w_i = 1$

The optimisation problem defined in equation (17) can be redefined in terms of rate of return parameters, by using equation (15):

$$\text{Minimise: } \sqrt[3]{\frac{w'\Omega w}{w'r(\log(2))}}$$

$$\text{Subject to: } w'\iota = 1$$

where w is the vector of portfolio weights, ι is a vector of ones, Ω is the covariance matrix for returns and r is the vector of returns.

For these minimisation problems the objective functions will have only one local minimum which corresponds to the global minimum. Accordingly, the weights can be found using the Nelder-Mead method[7]. The asset weightings are computed for each of the optimisation problems and the corresponding portfolio mean and variance are then plotted, along with the Markowitz efficient frontier, in mean-variance space. The results are shown in Figure 4. The small circles represent the results of the above optimisations and these circles plot at the global minimum variance portfolio and the tangency portfolio of the Markowitz efficient set.

It can be observed that while it is the skewness, or the shape parameter, for the portfolio's doubling time that is minimised, they have been expressed in terms of the rates return and the covariance matrix for the rates of return. This approach was taken because of the absence of a general analytical formula for combining inverse Gaussian distributions.

[7] The Nelder-Mead is a downhill Simplex method which although not as efficient as some gradient methods, is more robust.

The next step is to show that the whole Markowitz efficient frontier can be formed using either of the doubling time metrics. To construct the full efficient frontier the objective functions of the optimisations are modified as follows:

(18) Minimise: $\Gamma_{\lambda} \times \mu_{DT} + \lambda_{DT}^{-1}$

Subject to: $\sum w_i = 1$

Where Γ is a risk tolerance parameter. In order to trace the efficient frontier the optimisation is undertaken for a range of risk tolerance parameters.

To find the efficient frontier when using skewness as a risk metric, an analogous approach can be taken by solving the following optimisation problem across various risk tolerance levels:

(19) Minimise: $\Gamma_{\gamma} \times \mu_{DT} + \gamma$

Subject to: $\sum w_i = 1$

The two frontiers derived from the foregoing optimisations should lie on the Markowitz efficient frontier in the mean-variance plane, as defined by the following objective function:

(20) Maximise: $\Gamma_{\sigma} \times \mu_{r} - \sigma_{r}^{2}$

Subject to: $\sum w_i = 1$

The efficient frontiers resulting from the three foregoing optimisations are plotted in Figure 4 and match each other exactly.[8]

[8] While the portfolios along the efficient set are coincident across the three optimisations the values for the risk aversion parameters differ across the optimisations at each point. This is to be expected as the measures of risk differ. The risk aversion parameters using doubling time metrics are smaller than for the Markowitz analysis.

V. Conclusion

This study examined the properties of doubling times theoretically and by analysis of returns on the US Equity Market. A useful property of doubling times is that no matter how returns are reported there is only one doubling time. It was suggested that doubling times could have applications in truth in lending, in performance measurement and in capital budgeting. Computing doubling times period by period is comparatively simple, but using the time series of the period by period estimates to compute the expected doubling time is a little more challenging. This requires the use of either harmonic means of the doubling times, or a Monte Carlo simulation.

The results of the Monte Carlo simulation gave a distribution about the expected doubling time that closely approximated the inverse Gaussian distribution. If returns are assumed normal then the doubling time distribution will be inverse Gaussian, with a mean and variance that can be expressed in terms of the mean and variance of the return distribution. Even if returns are not normally distributed the doubling times will be approximately inverse Gaussian for independent returns with finite variance.

The objective of an investor with respect to doubling times is to minimise the time taken to double their wealth, but of course consideration must be given to risk. Given an inverse Gaussian distribution for doubling times there is a choice of two risk metrics for portfolio optimisation, these are skewness or the shape parameter, but not the variance. Portfolio optimisation can be accomplished in terms of doubling times. Minimising the inverse of the shape parameter gives the global minimum variance portfolio which would be obtained using the classical Markowitz framework. Minimising the skewness of doubling times gives the Markowitz tangency portfolio with the highest reward-to-risk ratio.

The skewness result has particular intuitive appeal. This appeal arises as investors would want to minimise positive skew in their investment doubling times. Long waiting periods until their investments double would naturally be undesirable. Furthermore, this doubling time optimisation problem only requires one parameter (skewness) to obtain the tangency portfolio, yet the Markowitz framework requires two parameters (mean and variance), thereby the use of doubling times results in a single parameter yet equivalent optimisation problem. These results provide a new perspective from which to view portfolio theory and an alternative calculus for generating the efficient frontier.

APPENDIX

This appendix demonstrates the relation between the doubling times distribution and the return distribution assuming that the return distribution is normal.

It is well known that compounding returns following

$$X(t) = X(0)e^{\int_0^t r(s)ds}$$

give the stochastic differential equation:

$$\frac{dX(t)}{dt} = r(t)X(t)$$

Now, a relationship from Lax (1966) shows that if:

$$\frac{dX(t)}{dt} = g\big(X(t)\big)w(t)$$

Then the transformation

$$Y(X) = \int^{X} g(X^{'})^{-1} dX^{'}$$

Yields the process satisfying

$$\frac{dY(t)}{dt} = w(t)$$

Using this relationship, the following transformation is obtained

$$Y(X) = \int^{X} \frac{dX^{'}}{X^{'}}$$

$$\therefore Y(t) = \log X(t) \qquad \text{(A1)}$$

Which yields the random process satisfying

$$\frac{dY(t)}{dt} = r(t) \qquad \text{(A2)}$$

If $E[r(t)dt] = \mu dt$ and $Var[r(t)dt] = \sigma^2 dt$, then (A2) can be written as a stochastic differential equation for a Weiner process with drift μ.

$$dY(t) = \mu dt + \sigma Z(t)\sqrt{dt} \qquad \text{(A3)}$$

where $Z(t)$ is a Gaussian process with $E[Z(t)] = 0$ and $Var[Z(t)] = 1$

The first passage time of this process is well documented; see Domine (1995), where the expected time to absorption and its variance are defined as:

$$E(T) = \frac{Y_B - Y_0}{\mu}, \qquad V(T) = \frac{(Y_B - Y_0)\sigma^2}{\mu^3}$$

Where Y_B is the absorbing barrier and Y_0 is the starting point for the process. This is also known to be an inverse Gaussian distribution

Recalling equation (A1), it is known that $Y(t) = \log X(t)$. So if the starting point X_0 = 1, and the absorbing point X_B = 2. Then the mean (μ_{DT}) and standard deviation (σ_{DT}^2) of the distribution of doubling times are given by:

$$\mu_{DT} = \frac{\log(2)}{\mu_r} \qquad (10)$$

$$\sigma_{DT}^2 = \frac{\log(2)\sigma_r{}^2}{\mu_r^3} \qquad (11)$$

Where μ_r and σ_r^2 are the mean and variance of the underlying normally distributed returns.

References

Broadie M., 1993, Computing Efficient Frontiers Using Estimated Parameters, *Annals of Operations Research*, 45, 21-58.

Brooks T. M., Pimm, S. L., Oyugi, J. O., 1999, Time lag between deforestation and bird extinction in tropical forest fragments, *Conservation Biology*, 13, pp 1140-1150.

Chhikara, R. S. and Folks L., 1989, *The inverse Gaussian distribution: Theory, methodology, and applications*, M.Dekker, New York.

de la Granville, O., 1998, The Long-Term Expected Rate of Return: Setting It Right, *Financial Analysts Journal*, November-December, pp. 75-80.

Hanks G .E., D'Amico A., Epstein B E., Schultheiss T. E., 1993, Prostate-specific antigen doubling times in patients with prostate cancer: a potentially useful reflection of tumor doubling time, *International Journal of Radiation Oncology/Biology/Physics*, vol 27,1, pp 125-127

Hendee W, R., 1979, *Medical Radiation Physics*, Year Book Medical Publishers, Chicago.

Kendall D., 1949, Stochastic Processes and Population Growth, Journal of the Royal Statistical Society, Series B (Methodological), Vol 11, 2, pp 230-282

Kilian, L., Zha, T., 2002, Quantifying the uncertainty about the half-life of deviation from PPP, *Journal of Applied Economics*, 17, pp 107-125

Merton, R., 1972, An analytic derivation of the efficient portfolio frontier, *Journal of Financial and Quantitative Analysis*, 7, September 1972, 1851-1872.

Pacioli, P., 1494, *Summa de Arithmetica* (Venice, Fol 181, n 44), cited at //en.wikipedia.org/wiki/Rule_of_72.

Rogoff, K., 1996, The purchasing power parity puzzle, *Journal of Economic Literature*, 34, pp 647-668.

Rossi, B., 2005, Confidence intervals for half-life deviations from purchasing power parity, *Journal of Business and Economic Statistics*, 23(4), pp 432-442.

Truong G. L., Partington G., Peat M., 2008, Cost-of-Capital Estimation and Capital Budgeting Practice in Australia, *Australian Journal of Management*, vol 33, 1, pp 95-121.

Table 1: Descriptive Statistics for Doubling Times and Half Lives

Panel A: All values have been reported where periods are measured in years

	Daily Data			Weekly Data			Monthly Data		
	combined	doubling times	half lives	combined	doubling times	half lives	combined	doubling times	half lives
mean	0.389665	1.965965	-1.51795	0.795025	4.241576	-3.85813	-1.90646	3.443991	-11.3311
variance	206.7839	322.156	60.57924	1110.223	1772.548	179.8466	969.4176	24.66128	2508.673
standard deviation	14.37998	17.9487	7.783266	33.32	42.10164	13.41069	31.13547	4.966013	50.08666
min	-330.382	0.024829	-330.382	-188.051	0.1091178	-188.051	-460.095	0.5296499	-460.095
max	680.028	680.028	-0.02778	1134.44	1134.44	-0.06684	42.3469	42.3469	-0.299
median	0.160355	0.51397	-0.50461	0.441371	1.003475	-1.02311	1.169778	1.929448	-2.18151
zero returns removed		358			8			0	
Data sample size		6162			1317			301	

Panel B: All values have been reported corresponding to the interval length of the data used

	Daily Data			Weekly Data			Monthly Data		
	combined	doubling times	half lives	combined	doubling times	half lives	combined	doubling times	half lives
mean	100.9673	509.4	-393.32	41.6361	222.135	-202.054	-22.8775	41.32789	-135.973
variance	13883337	21629347	4067251	3045015	4861579	493266.5	139596.1	3551.225	361248.9
standard deviation	3726.035	4650.73	2016.743	1744.997	2204.899	702.3293	373.6257	59.59215	601.0399
min	-85606.2	6.43354	-85606.2	-9848.4	5.71459	-9848.4	-5521.14	6.355799	-5521.14
max	176203.8	176203.8	-7.19819	59411.58	59411.58	-3.50025	508.1627	508.1627	-3.58762
median	41.55008	133.1771	-130.75	23.11502	52.55285	-53.5811	14.03733	23.15338	-26.1781
zero returns removed		358			8			0	

These tables present the descriptive statistics for the raw doubling times and half lives ($\tau = \log(2)/r$) of the Dow Jones U.S Total Stock Market Total Return Index over the period 16 October 1988 to 13 October 2013. These statistics do not capture the doubling times that investors expect and it can be seen that the mean doubling times vary depending on whether the returns are measured at daily, weekly, or monthly frequency.

Table 2: Descriptive Statistics: Doubling Time for the Dow Jones U.S Total Stock Market Total Return Index (Oct 1998 to Oct 2013)

	Daily Data	Weekly Data	Monthly Data
Analytical expected doubling time (Periods)	1885.46	380.85	87.04
Analytical expected doubling time (Years)	7.276	7.275	7.276
Simulated expected doubling time (Periods)	1911.72	388.38	90.48
Simulated standard deviation (Periods)	1295.58	256.11	52.68
Simulated expected doubling time (Years)	7.377	7.410	7.540
Simulated standard deviation (Years)	5.00	18.72	4.39
t–statistic	1.14	1.24	1.78

The table gives the analytical expected doubling times and the simulated expected doubling times together with the standard deviations of the simulated doubling times. The t-statistics are for the comparison of the analytical doubling times with the simulated doubling times.

Table 3: Asset mean returns and standard deviations and their equivalent doubling times

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
Mean (Months)	0.006	0.01	0.014	0.018	0.022
Standard Deviation (Months)	0.085	0.08	0.095	0.09	0.1
Expected Doubling Time (Months)	115.525	69.3147	49.5105	38.5082	31.5067
Doubling Time Standard Deviation (Months)	152.267	66.6044	47.7468	31.0275	25.514

Table 4: Correlation matrix for asset returns

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
Asset 1	1	0.3	0.3	0.3	0.3
Asset 2	0.3	1	0.3	0.3	0.3
Asset 3	0.3	0.3	1	0.3	0.3
Asset 4	0.3	0.3	0.3	1	0.3
Asset 5	0.3	0.3	0.3	0.3	1

Figure 1: Variation of Doubling Times with Returns

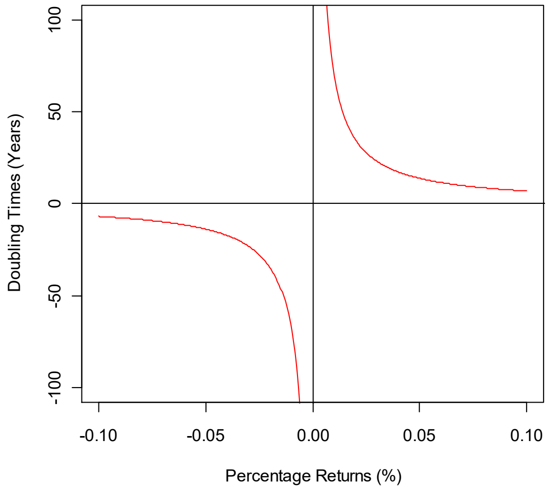
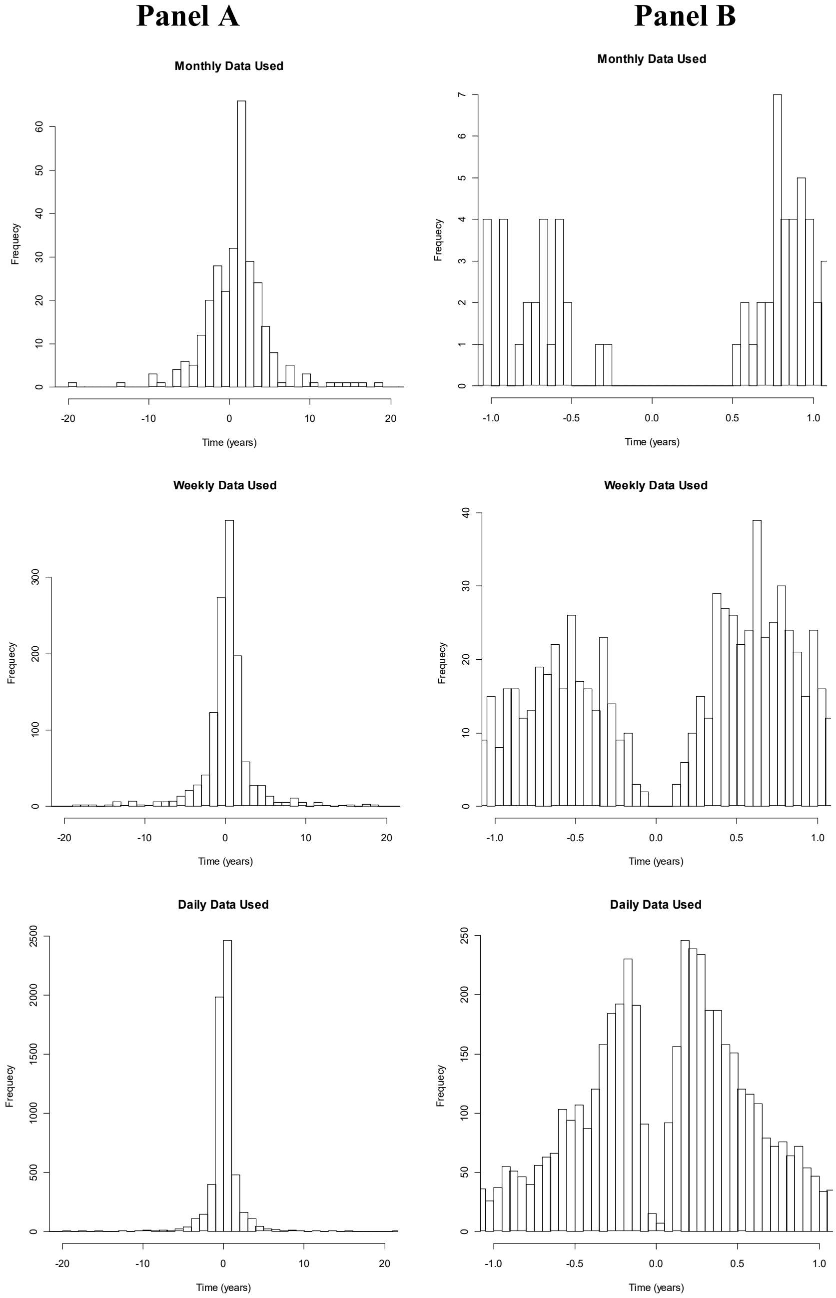


Figure 1 represents a plot of $\tau = \log(2) / r$ for continuously compounded returns varying from -10% to +10%

Figure 2: Histograms of Half Lives and Doubling Times for the Dow Jones U.S Total Stock Market Total Return Index.



Histogram of combined doubling times and half lives for the Dow Jones U.S Total Stock Market Total Return Index for the period 16 October 1988 to 13 October 2013. The analysis uses monthly, weekly and daily return observations. Panel A shows the full histogram, while Panel B shows a truncated and magnified version of the histogram in order to highlight the discontinuity at zero.

Figure 3: Doubling time density plot with corresponding P-P plot against an inverse Gaussian distribution.

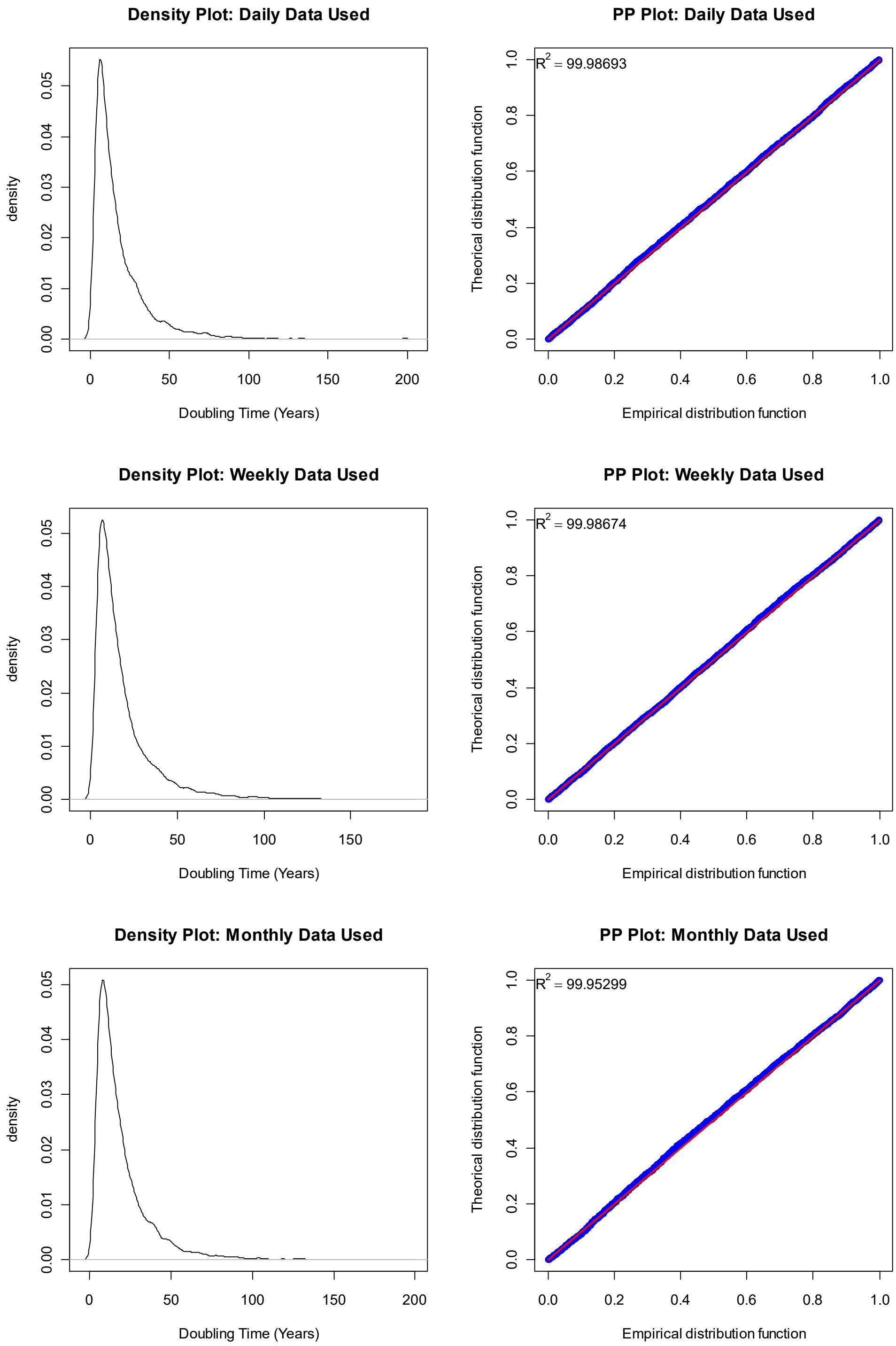
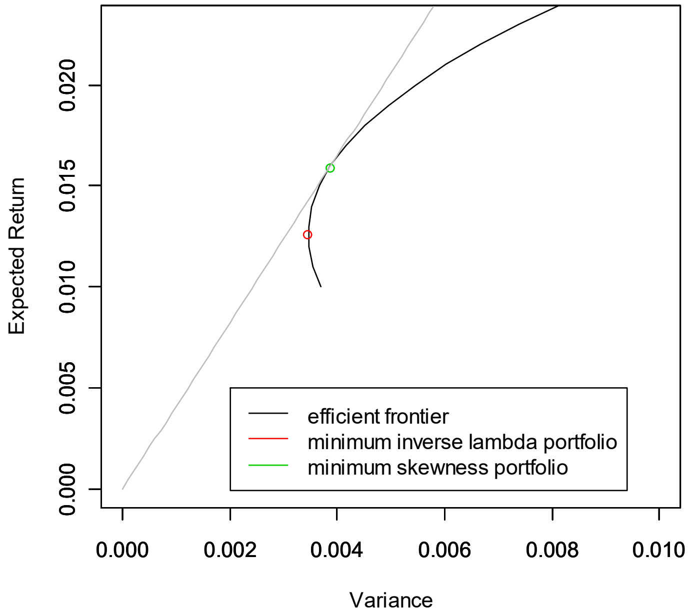


Figure 4: Efficient frontiers



This figure depicts the efficient frontier using the mean, variance and correlation parameters found in Tables 4 and 5. The efficient frontier was computed using the traditional Markowitz mean-variance framework (equation 20) and is shown as the black line. The optimisations based on the shape parameter and skewness of doubling times (equations 18 and 19 respectively) generate results that exactly match the Markowitz efficient frontier. The red circle represents the portfolio that minimises the inverse shape parameter and which lies at the global minimum variance portfolio. The green circle is the minimum skewness portfolio which is shown to lie on the tangency point between the origin and the frontier (as shown by the grey line).