Performance Measurement for Option Portfolios in a
Stochastic Volatility Framework

Rainer Baule¹, Oliver Entrop², and Sebastian Wessels³

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¹University of Hagen, Universitätsstraße 41, 58084 Hagen, Germany, rainer.baule@fernuni-hagen.de
²University of Passau, Innstraße 27, 94032 Passau, Germany, oliver.entrop@uni-passau.de
³University of Hagen, Universitätsstraße 41, 58084 Hagen, Germany, sebastian.wessels@fernuni-hagen.de.
Abstract

Measuring the performance of stock portfolios that include options is challenging due to options’ nonlinearity in the underlying and their exposure to volatility risk. Our contribution to the literature is twofold: First, we provide a theoretically rigorous derivation of the time-variable factor loadings in a multi-factor model under stochastic volatility according to Heston (1993) when an option factor is included. We show that (i) any option factor is suitable if discrete returns are considered in instantaneous time and that (ii) the option factor’s loading equals the fraction of the vola elasticities of the portfolio and of the option factor while the option factor’s underlying elasticity enters the factor loading of the underlying. Second, in applications however, time has to be discretized and factor loadings are usually estimated in a single regression over a certain horizon, which regularly leads to a bias in performance measurement. We run a simulation analysis to analyze the size of this bias when different option factors from the common literature are used and propose a two-step procedure to keep the bias small.
1 Introduction

The field of performance measurement, in particular for managed funds and portfolios, has recently been gaining attention in the financial literature. The idea of performance measurement is rather elementary. It deals with the question of how well a portfolio performs in comparison to a certain (multi-factor) benchmark. Beginning with the pathbreaking paper by Jensen (1967), the CAPM-based differential return measure (Jensen’s alpha) has been extended in a number of directions, the most prominent being the multi-factor models proposed by Fama and French (1993), Carhart (1997), and Fama and French (2015).

This paper deals with the performance measurement of portfolios and funds including options. Basically, there are two major issues with such portfolios: First, option prices are nonlinear functions of the standard risk factors, in particular of the market. Also the self-financing factor portfolios such as the High-Minus-Low or Small-Minus-Big portfolios introduced by Fama and French (1993) are not suited to cover option portfolios. For instance, Grinblatt and Titman (1989) and Leland (1999) showed that based on Jensen’s alpha a portfolio manager, by holding a long position in a certain index and selling a call option on the index, will be incorrectly classified as a superior performer.

Second, option prices are exposed to risk factors - in particular to volatility risk - that do not affect direct investments in the underlying asset. There is empirical evidence that such additional risk factors exist. Buraschi and Jackwerth (2001) show that options are non-redundant securities in the sense that their payoffs cannot be fully replicated by a portfolio of the underlying and the risk-free asset. Bakshi and Kapadia (2003) demonstrate that delta-neutral portfolios of long positions in call options and short positions in the underlying systematically underperform the market. They interpret this underperformance as
evidence for a negative volatility risk premium.

Thus, for an adequate performance measurement of portfolios including options, both the nonlinearity and the existence of additional risk factors have to be taken into account. A seminal approach has been suggested by Glosten and Jagannathan (1994) in a general asset-pricing framework. While their approach is theoretically capable of addressing various risk factors, they perform tests in a simple Black-Scholes setting with the underlying price as the only source of risk. They thus concentrate on the nonlinearity issue, applying several sorts of nonlinear functions, including splines and other kinds of piecewise defined functions.

Whaley (2002) advocates the use of the official CBOE S&P 500 BuyWrite Index (BXM) as an additional factor. The index tracks a strategy of buying the S&P 500 and writing monthly at-the-money call options on the index. Such an additional factor covers both nonlinearity and volatility risk. The factor is being used by subsequent papers dealing with the performance measurement of funds including options, for example Natter et al. (2016).

As will become clear in this paper, a crucial property of the BXM is the fact that the overlapping period between the expiry of the $n$-th written option and the writing of the $(n+1)$-st option is zero. Hence, the option is not prematurely settled at the market, but actually expires. Bauer et al. (2009) propose a different approach, that of constructing an option-based factor by buying (or selling) a two-month option, settling this option after one month (by re-selling or re-buying it on the market), and buying (or selling) a new two-month option. They use their approach to measure the performance of option portfolios of individual investors. Similar factors are also used by Agarwal and Naik (2004), Titman and Tiu (2011), and Teo (2009) to assess the risk of hedge funds.
This paper analyzes the performance of option portfolios in a framework with stochastic volatility. To be more precise, the underlying index follows the process suggested by Heston (1993). Within this framework, we derive an analytical result that justifies the standard approach of performance measurement with a linear regression of portfolio returns on the risk factors. When the regression coefficients are allowed to be time-variable, the expression is theoretically exact for any choice of one option-based risk factor additional to the market factor. This result is not surprising, as the risk space is spanned by two factors in the Heston (1993) framework—however, it only holds for discrete (simple) returns. The regression is biased for continuous (log) returns as Nielsen and Vassalou (2004) have already shown in a Black-Scholes framework. We extend their results and analytically calculate the bias in the case of stochastic volatility.

The theoretical discussion is complemented by a simulation study. While theoretically, each option-based factor yields the exact representation of the portfolio returns with the adequate corresponding time-varying beta, the usual estimation of a constant beta may lead to different results and thus to potential biases. Thus, in a practical application, the choice of an adequate option-based factor is crucial. We perform a kind of horse race with several option-based factors, including the BXM factor used by Whaley (2002) and the rolling factor introduced by Bauer et al. (2009), for a number of test portfolios. As a general result, the mentioned lack of an overlapping period is an unfavorable property of the BXM factor, as it is not completely capable of reducing the bias of a one-factor approach. We suggest an adjustment of the BXM factor that leads to far less bias in the estimation of alphas.

Furthermore, we advocate choosing an individual factor for the respective application. This “optimized” factor is obtained via a two-step procedure: In the first step, test regres-
sions for a range of option-based factors with different option maturities are performed. The individual factor is chosen based on a minimization of the root mean squared error obtained from the respective regressions, which is then used for the actual performance measurement in the second step. In particular for option portfolios with a high percentage of option holdings, this optimized factor further substantially reduces the bias of the performance measure.

The paper proceeds as follows: In Section 2, we derive an expression for the excess returns on option portfolios within the Heston framework. In Section 3, the derived representation for portfolio returns under stochastic volatility is linked to the use of option-based factors. Section 4 presents the simulation study. Section 5 concludes.

2 Portfolio Returns in a Stochastic Volatility Framework

2.1 Theoretical Framework

The theoretical part of this paper is meant to supply a theoretical derivation of the standard approach to measuring the performance of portfolios including option holdings. To do so, we determine the specific return structure on a given option portfolio $P$, applying methods from option pricing theory and stochastic analysis. Afterwards, the obtained return structure is linked to the use of option-based factors.

As common literature in asset pricing and performance measurement addresses nonlinearity of option holdings only with regard to movements in the underlying, we extend this consideration to a further risk factor and take into account the option pricing model proposed by Heston (1993). We generally understand an option portfolio to be one that combines market holdings and certain European call or put options. In this framework,
the underlying of the option(s) held in the option portfolio is assumed to be the market portfolio. This assumption may seem somewhat restrictive, though it engages the derivation of a closed-form expression for the portfolio returns.

The theoretical framework is guided by the assumption that the risk space is spanned by the price of the market portfolio and the market portfolio’s variance. Hence the dynamics of those risk factors are modeled. The price of the market portfolio, $M_t$, is driven by a constant drift and a time-varying volatility process:

$$dM_t = \mu_M M_t dt + M_t \sqrt{v_t} dW_{M,t}. \quad (1)$$

Specifically, the market variance, denoted by $v_t$, follows a stochastic square root process:

$$dv_t = \kappa(\theta - v_t) dt + \eta \sqrt{v_t} dW_{v,t}, \quad (2)$$

where $\theta > 0$ corresponds to the long-run mean of the variance and $\kappa > 0$ controls the speed by which the variance returns to the long-run mean. $\eta > 0$, often referred to as the “volatility of the volatility”, determines the variance of $v_t$. In order to simplify the further expressions, the drift-rate of the markets variance dynamics is denoted as $\mu_{v,t} := \kappa(\theta - v_t)$. $W_M$ and $W_v$ are correlated Wiener processes with correlation $\rho$. For simplicity, no dividends are taken into account.

Acknowledging the Heston model, each given option portfolio follows the Heston PDE:

$$rP_t = \frac{1}{2} v_t M_t^2 \frac{\partial^2 P_t}{\partial M_t^2} + \rho \eta v_t M_t \frac{\partial^2 P_t}{\partial M_t \partial v_t} + \frac{1}{2} \eta^2 v_t^2 \frac{\partial^2 P_t}{\partial v_t^2} + rM_t \frac{\partial P_t}{\partial M_t}$$

$$+ \left[ \mu_{v,t} - \lambda(M, v, t) \right] \frac{\partial P_t}{\partial v_t} + \frac{\partial P_t}{\partial t}, \quad (3)$$

where $P_t$ denotes the price of the option portfolio $P(t, M_t, v_t)$ in $t$. The function $\lambda(M, v, t)$ can be interpreted as the volatility risk premium.\(^1\)

\(^1\)Heston (1993) assumes $\lambda(M, v, t)$ to be a linear function in the variance.
2.2 Discrete Portfolio Returns

Based on the theoretical framework described above, expressions for the discrete and continuous excess returns on the option portfolio $P$ can be derived. We distinguish between discrete and continuous portfolio returns, as these types of returns lead to different return structures on portfolio $P$. Discrete portfolio returns over an infinitesimal time-step are defined by $r_{P,t}dt = \frac{dP_t}{P_t}$.

Applying the Lemma of Itô (1944) in a bivariate framework leads to a basic structure of the portfolio returns:

$$r_{P,t}dt = \frac{1}{P_t} \frac{\partial P_t}{\partial t} dt + \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} dM_t + \frac{1}{P_t} \frac{\partial P_t}{\partial v_t} dv_t$$

$$+ \frac{1}{2 P_t} \left[ \frac{\partial^2 P_t}{\partial M_t^2} dM_t^2 + 2 \frac{\partial^2 P_t}{\partial M_t \partial v_t} dM_t dv_t + \frac{\partial^2 P_t}{\partial v_t^2} dv_t^2 \right]$$

$$= \left[ \frac{1}{2 v_t M_t^2} \frac{1}{P_t} \frac{\partial^2 P_t}{\partial M_t^2} + \rho \eta v_t M_t \frac{1}{P_t} \frac{\partial^2 P_t}{\partial M_t \partial v_t} + \frac{1}{2} \eta^2 v_t \frac{1}{P_t} \frac{\partial^2 P_t}{\partial v_t^2} + \mu M_t \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} 
+ \mu_v \frac{1}{P_t} \frac{\partial P_t}{\partial v_t} + \frac{1}{P_t} \frac{\partial P_t}{\partial t} \right] dt + M_t \sqrt{v_t} \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} dW_{M,t} + \eta \sqrt{v_t} \frac{1}{P_t} \frac{\partial P_t}{\partial v_t} dW_{v,t}. \tag{4}$$

As we aim at expressing the portfolio returns in a scheme which can be compared to the representations of the excess returns on portfolio $P$ obtained from common performance measurement models, the given expression needs to be transformed adequately to achieve a “$\beta$-representation”. The Heston-PDE is applied to resolve the partial differential of the portfolio value according to time $t$:

$$\frac{\partial P_t}{\partial t} = rP_t - \frac{1}{2} v_t M_t^2 \frac{\partial^2 P_t}{\partial M_t^2} - \rho \eta v_t M_t \frac{\partial^2 P_t}{\partial M_t \partial v_t} - \frac{1}{2} \eta^2 v_t \frac{\partial^2 P_t}{\partial v_t^2} - r M_t \frac{\partial P_t}{\partial M_t} - \mu_v \frac{\partial P_t}{\partial v_t} \left( \mu_v - \lambda(M,v,t) \right) \tag{5}$$

Inserting (5) in (4) leads to:

$$r_{P,t}dt = \left[ \mu M_t \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} + \mu_v \frac{1}{P_t} \frac{\partial P_t}{\partial v_t} + r - r M_t \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} - \left( \mu_v - \lambda(M,v,t) \right) \right] dt + M_t \sqrt{v_t} \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} dW_{M,t} + \eta \sqrt{v_t} \frac{1}{P_t} \frac{\partial P_t}{\partial v_t} dW_{v,t}. \tag{6}$$
Applying
\[
\frac{1}{P_t} \frac{\partial P_t}{\partial M_t} M_t \sqrt{\nu_t} dW_{M,t} = \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} dM_t - \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} M_t \mu_M dt
\] (7)
and
\[
\frac{1}{P_t} \frac{\partial P_t}{\partial v_t} \sqrt{\nu_t} dW_{v,t} = \frac{1}{P_t} \frac{\partial P_t}{\partial v_t} dv_t - \frac{1}{P_t} \frac{\partial P_t}{\partial v_t} \mu_{v,t} dt,
\] (8)
deduced from the dynamics of the market portfolio and the variance, the Wiener processes can be dissolved:
\[
x = r_{P,t} dt = \left[ r - r M_t \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} - \left( \nu_{v,t} - \lambda(M, v, t) \right) \frac{1}{P_t} \frac{\partial P_t}{\partial v_t} \right] dt
\] + \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} dM_t + \frac{1}{P_t} \frac{\partial P_t}{\partial v_t} dv_t.
\] (9)

In transition to relative values, let \( \beta_{P,M,t} := \frac{\partial P_t}{\partial M_t} \cdot \frac{M_t}{P_t} \) and \( \beta_{P,v,t} := \frac{\partial P_t}{\partial v_t} \cdot \frac{v_t}{P_t} \), such that:
\[
x = r_{P,t} dt - r dt = \beta_{P,M,t} \frac{dM_t}{M_t} + \beta_{P,v,t} \frac{dv_t}{v_t} - \beta_{P,M,t} r dt - \left( \nu_{v,t} - \lambda(M, v, t) \right) \frac{\beta_{P,v,t}}{v_t} dt
\] (10)
\[
\Leftrightarrow x = \beta_{P,M,t} \left( \frac{dM_t}{M_t} - r dt \right) + \beta_{P,v,t} \left( \frac{dv_t}{v_t} - \left( \nu_{v,t} - \lambda(M, v, t) \right) \frac{1}{v_t} dt \right).
\]

Thus, the discrete excess returns on portfolio \( P \) can be expressed as:
\[
x = \beta_{P,M,t} \left( r_{M,t} - r \right) + \beta_{P,v,t} \left( r_{v,t} - \left( \nu_{v,t} - \lambda(M, v, t) \right) \frac{1}{v_t} \right),
\] (11)
where \( r_{M,t} dt = \frac{dM_t}{M_t} \), and \( r_{v,t} dt = \frac{dv_t}{v_t} \).

The first term on the right side, \( \beta_{P,M,t} \left( r_{M,t} - r \right) \), captures the portion of the excess returns on portfolio \( P \) which can be explained by the excess returns on the market portfolio. Thus, \( \beta_{P,M,t} \) includes the portfolio delta \( \frac{\partial P_t}{\partial M_t} \) with regard to relative portfolio values. The second term on the right side, \( \beta_{P,v,t} \left( r_{v,t} - \left( \nu_{v,t} - \lambda(M, v, t) \right) \frac{1}{v_t} \right), \) covers the market’s variance risk premia. Here, \( \beta_{P,v,t} \) builds on the portfolio vega \( \frac{\partial P_t}{\partial v_t} \) based on relative portfolio values.
2.3 Continuous Portfolio Returns

In addition to discrete portfolio returns, an expression for continuous returns on portfolio $P$ is derived. Again, we consider a time-continuous setting, i.e., the continuous excess returns on the option portfolio $P$ can be written as: $r^c_{P,t}dt = d\log P_t$, regarding the log-process of the portfolio value. The entire derivation of the portfolio returns, summarized as follows, can be found in Appendix A. As in the case of discrete returns, the Lemma of Itô (1944) can be used to determine the structure of the continuous portfolio returns:

$$r^c_{P,t}dt = \left[ \frac{1}{2} M^2 v_t \frac{\partial^2 \log P_t}{\partial M_t^2} + \rho M_t v_t \frac{\partial^2 \log P_t}{\partial M_t \partial v_t} + \frac{1}{2} \eta^2 v_t^2 \frac{\partial^2 \log P_t}{\partial v_t^2} + \mu M_t \frac{\partial \log P_t}{\partial M_t} ight] dt + M_t \sqrt{v_t} \frac{\partial \log P_t}{\partial M_t} dW_{M,t} + \eta \sqrt{v_t} \frac{\partial \log P_t}{\partial v_t} dW_{v,t}. \quad (12)$$

Determining the partial derivatives, inserting the Heston PDE, and dissolving the Wiener processes lead to the following expression for the continuous returns on portfolio $P$:

$$r^c_{P,t} - r = \beta^P_{M,t} \left( r^c_{M,t} - r \right) + \beta^P_{v,t} \left( r^c_{v,t} - \left( \mu_{v,t} - \lambda(M, v, t) \right) \frac{1}{v_t} \right) + \frac{1}{2} v_t \left( \beta^P_{M,t} - \left( \beta^P_{M,t} \right)^2 \right) - \rho \eta \beta^P_{M,t} \beta^P_{v,t} + \frac{1}{2} \eta^2 v_t \left( \beta^P_{v,t} - \left( \beta^P_{v,t} \right)^2 \right), \quad (13)$$

where $r^c_{M,t}dt = d\log M_t$ and $r^c_{v,t}dt = d\log v_t$. Again, the relative portfolio delta and vega reveal the respective betas as elasticities: $\beta^P_{M,t} = \frac{\partial P_t}{\partial M_t} \frac{M_t}{P_t}$, $\beta^P_{v,t} = \frac{\partial P_t}{\partial v_t} \frac{v_t}{P_t}$.

Remarkably, compared to discrete portfolio returns, the continuous excess returns on portfolio $P$ include additional beta-correction terms $\frac{1}{2} v_t \left( \beta^P_{M,t} - \left( \beta^P_{M,t} \right)^2 \right) - \rho \eta \beta^P_{M,t} \beta^P_{v,t} + \frac{1}{2} \eta^2 v_t \left( \beta^P_{v,t} - \left( \beta^P_{v,t} \right)^2 \right)$. Hence, applying (11) based on continuous returns would lead to a bias exactly given by those beta-correction terms. Notably, if we assume the market’s variance to be constant, the continuous excess returns on portfolio $P$ are given by:

$$r^c_{P,t} - r = \beta^P_{M,t} \left( r^c_{M,t} - r \right) + \frac{1}{2} v_t \left( \beta^P_{M,t} - \left( \beta^P_{M,t} \right)^2 \right). \quad (14)$$
In comparison, the discrete portfolio returns are expressed as:

\[ r_{P,t} - r = \beta^P_{M,t} \left( r_{M,t} - r \right) \]  

which resembles the intertemporal capital asset pricing model utilizing time-varying betas. This corresponds to the observations of Nielsen and Vassalou (2004) who propose a modification of Jensen’s alpha in a time-continuous model within the Black-Scholes framework with constant volatility. Expression (13) extends their observations to a stochastic volatility framework, obtaining additional beta-correction terms that belong to the market portfolio’s variance.

3 Merging Portfolio Returns and Option-Based Factors

3.1 Option-Based Factors

Potentially, on the basis of (11) or (13), one could carry out performance measurement for traded portfolios and funds. E.g., applying discrete portfolio returns,

\[ r_{P,t} - r = \alpha_t + \beta^P_{M,t} \left( r_{M,t} - r \right) + \beta^P_{v,t} \left( r_{v,t} - \left( \mu_{v,t} - \lambda(M, v, t) \right) \frac{1}{v_t} \right), \]  

defines the corresponding performance measure \( \alpha_t \). \( \beta^P_{M,t} \left( r_{M,t} - r \right) + \beta^P_{v,t} \left( r_{v,t} - \left( \mu_{v,t} - \lambda(M, v, t) \right) \frac{1}{v_t} \right) \) corresponds to the fair value of the excess returns on \( P \), \( \alpha_t \) specifies the over- or underperformance of portfolio \( P \) compared to the fair value.

A direct estimation suffers from two problems: First, the variance \( v_t \), the drift rate \( \mu_{v,t} \), and the risk premium \( \lambda(M, v, t) \) are not observable on the market, and would have to be estimated first. Second, analytical portfolio deltas and vegas and thus time-varying beta factors cannot be calculated and thus would have to be additionally estimated. The usual
circumvention of these problems uses option-based factors to explain the influence of the variance on the portfolio.

In connection with performance measurement, option-based factors are ordinarily used to capture nonlinearity of option holdings. Based on the time-continuous framework considered in this paper, we provide a theoretical justification for applying option-based factors.

We use option-based factors to explain the influence of the market’s variance \( v_t \) on the returns on portfolio \( P \). The idea is to explain the excess returns on portfolio \( P \) by the excess returns on the market portfolio and the excess returns on a certain option-based factor. Nonlinearity of option holdings based on movements in the market is captured by a time-varying beta on the market. Nonlinearity based on movements in the variance is captured by time-varying betas on the option-based factor.\(^2\) Proposition 1 guarantees that within the Heston framework, any option-based factor is eligible to perfectly explain the variance’s influence on the excess portfolio returns if discrete portfolio returns are applied.

**Proposition 1.** Let us assume an option portfolio \( F \) that solves the Heston PDE. Further, the market portfolio is supposed to be the underlying of the option(s) included in \( F \).

Considering discrete portfolio returns,

\[
    r_{P,t} - r = \tilde{\beta}_{P,t}^{P} [r_{M,t} - r] + \tilde{\beta}_{F,t}^{P} [r_{F,t} - r]
\]

(17)

perfectly comprehends the excess returns on portfolio \( P \), if:

\[
     (i) \quad \tilde{\beta}_{M,t}^{P} = \beta_{M,t}^{P} - \frac{\beta_{v,t}^{P}}{\beta_{v,t}^{F}} \beta_{M,t}^{F}, \quad (ii) \quad \beta_{F,t}^{P} = \frac{\beta_{v,t}^{P}}{\beta_{v,t}^{F}},
\]

(18)

where \( r_{F,t} \) denotes the instantaneous factor return in time \( t \); \( \beta_{M,t}^{F} := \frac{\partial F_t}{\partial M_t} \cdot \frac{M_t}{F_t} \); and \( \beta_{v,t}^{F} := \frac{\partial F_t}{\partial v_t} \cdot \frac{v_t}{F_t} \).

\(^2\)Generally, the two factors are not orthogonal, so both factors will capture the aggregated portfolio sensitivity to the market.
Proof. See Appendix B.

Interestingly, the beta of the portfolio with respect to the option-based factor, $\beta_{P,t}$, only depends on the respective elasticities of the portfolio and the factor with respect to volatility as the second risk factor, $\beta_{P,v,t}$ and $\beta_{F,v,t}$, and are not related to the first risk factor, the market. In contrast, the beta of the portfolio with respect to the market in the two-factor model, $\tilde{\beta}_{M,t}$, is not identical with the sensitivity $\beta_{P,M,t} = \frac{\partial P_t}{\partial M_t} \frac{M_t}{P_t}$. Instead, it is adjusted by a term which depends on the elasticities with respect to volatility. The reason for this asymmetric structure is that the first factor, $r_{M,t} - r$, is exposed to market risk only, while the second factor, $r_{F,t} - r$, is exposed both to market risk and volatility risk. Accordingly, an analogous adjustment term for $\beta_{P,F,t}$, being $\beta_{P,M,t} \frac{\beta_{M,t}}{\beta_{M,v,t}}$, equals zero because $\beta_{M,v,t} = 0$.

Proposition 1 provides a theoretical justification for the use of option-based factors for the performance measurement of portfolios including options. Actually, any single factor would do the job of capturing both nonlinearity and variance risk, if the corresponding beta factors are allowed to vary over time. This finding reflects the setup of the Heston model with two risk factors—these two risk factors are captured by any other two factors that span the same risk space, which is the case for the market factor together with any volatility-sensitive (option-based) factor.

However, this result only holds for discrete returns. Applying continuous portfolio returns, an option-based factor that perfectly captures the relative portfolio deltas and vegas leads to an error term if the factor is not chosen to be portfolio $P$ itself:

**Proposition 2.** Consider portfolio $F$ from Proposition 1. Applying continuous portfolio returns, 

$$r_{P,t} - r = \tilde{\beta}_{M,t} \left[ r_{M,t} - r \right] + \beta_{P,F,t} \left[ r_{F,t} - r \right] + \zeta_t$$  \hspace{1cm} (19)
with an error term

\[
\zeta_t = \frac{1}{2} v_t \left( \tilde{\beta}_{PM,t}^P + (\beta_{PM,t}^P - \beta_{EM,t}^E) \beta_{EM,t}^E - (\beta_{PM,t}^P)^2 \right) \\
- \rho \eta \beta_{v,t}^P \left( \beta_{v,t}^P - \beta_{EM,t}^E \right) + \frac{1}{2} \eta^2 v_t \left( \beta_{v,t}^P \beta_{v,t}^E - (\beta_{v,t}^P)^2 \right),
\]

(20)

if (i) \( \tilde{\beta}_{PM,t}^P = \beta_{PM,t}^P - \frac{\beta_{EM}^P}{\beta_{v,t}^P} \beta_{EM,t}^E \), (ii) \( \beta_{PM,t}^F = \frac{\beta_{EM,t}^F}{\beta_{EM,t}^P} \), (iii) \( P \neq F \).  

(21)

Proof. See Appendix C.

\[ \square \]

3.2 Estimation of Portfolio Sensitivities

The previous findings refer to an analytical determination of the dynamic portfolio betas. As the structure of portfolios or funds including option components is unknown in general, measuring performance requires a non-analytical beta-estimation. Fu et al. (2012) developed a method for estimating option greeks when the closed-form solution for option prices is unknown by using random parameters and OLS regressions. Their algorithm produces accurate results and can be applied to options discontinuously as well as continuously traded. We follow their idea and the widespread practice in performance measurement and estimate the portfolio sensitivities using OLS regressions.

Applying discrete portfolio returns, the regression model is of type:

\[
r_{P,t} - r = \hat{\alpha}(t) + \hat{\beta}_{M,(t)}^P \left[ r_{M,t} - r \right] + \hat{\beta}_{F,(t)}^P \left[ r_{F,t} - r \right] + \hat{\epsilon}_t,
\]

(22)

where \( \hat{\beta}_{M,(t)}^P \) and \( \hat{\beta}_{F,(t)}^P \) are estimators for the discontinuous betas and \( \hat{\epsilon}_{P,t} \) is the error-term.

The designation \( (t) \) denotes that the model parameters could be chosen to be time-varying or constant. As observations are necessarily based on finite time intervals, both parameter choices lead to a discontinuous approximation of the betas considered in the theoretical framework of the paper.
In contrast to the theoretical setting discussed so far, an empirical estimation thus suffers from two sources of error: First, there is discretization error, as the continuous beta factors can only be estimated at discrete time intervals; often only once over the whole observation period, leading to a constant beta. Second, there is estimation error, as the factors cannot be observed directly, but must be estimated based on observed factor and portfolio returns. Obviously, given a fixed frequency of return observations, both errors depend negatively on each other: To reduce estimation error, the number of observed returns for an estimation has to be increased, leading to a longer estimation interval and thus to an increased discretization error. If the observation frequency is low (e.g., monthly returns), a single estimation of constant beta factors over the whole observation period will often be the best choice. In this case, the term \( \hat{\beta}_F \left[ r_{F,t} - r \right] \) is meant to capture the nonlinearity relating to both risk factors, the market and the variance.

In the next section, we analyze the suitability of several different option-based factors. Although irrelevant in a time-continuous framework, the appropriate choice of factor can be of high relevance when it comes to a discrete setting with estimation error. We look at monthly returns, as often done in performance measurement studies, and estimate constant betas over the whole observation period to minimize estimation error. A “good” factor should therefore be able to reflect the nonlinearity of the option portfolio also with respect to the market.

As a conclusion of Proposition 2, the use of option-based factors applying continuously compounded portfolio returns necessarily has to be avoided. Otherwise the beta-correction terms are erroneously attributed to the alpha, leading to a substantial bias in the performance measure.
4 Simulation Study

4.1 Methodology

The simulation study examines the performance of fixed option-based factors inferred from previous publications. As a fundamental part of the study, the processes of the market portfolio and the market’s variance are simulated by a Euler-Maruyama discretization of the corresponding Heston processes with condition $v_t \geq 0$.³ On the basis of those simulated processes, the respective options held in the investigated portfolios are priced.

The simulation of the market portfolio is driven by $\mu_M = 0.03$ and an initial value of $M_0 = 100$. The simulation of the market’s variance is built up on three parameter sets belonging to three different volatilities of market volatility. Set A represents a stage with a low volatility of volatility including the affiliated parameters. Set B represents a stage of a average volatility of volatility, whereas a phase of a high volatility of volatility is given by Set C. The respective variance parameters are obtained from a calibration of the Heston model on market prices of European call and put options on the DAX.⁴

As objects of observation we choose three different types of option portfolio. Each portfolio is long in the market portfolio and holds a certain portion of European call and put options


⁴For the calibration we applied daily settlement prices of European call and put options traded on the EUREX between may 2014 and december 2015, cutting the time to maturity at two years. The calibration is based on a minimization of the root mean squared error between theoretical and observed option prices. In order to determine a global minimum of the root mean squared error, we operate the differential evolution algorithm suggested by Storn and Price (1997). The obtained parameter sets are given by: Set A: $v_0 = 0.024$, $\theta = 0.044$, $\kappa = 1.212$, $\eta = 0.325$, $\rho = -0.835$, Set B: $v_0 = 0.046$, $\theta = 0.047$, $\kappa = 2.320$, $\eta = 0.468$, $\rho = -0.670$, Set C: $v_0 = 0.100$, $\theta = 0.042$, $\kappa = 9.300$, $\eta = 0.889$, $\rho = -0.949$. 
on the market portfolio. The structure of the observed portfolios is given by:

**Portfolio 1:** Long market, short call,

**Portfolio 2:** Long market, long put,

**Portfolio 3:** Long market, long call, long put.

For each option we consider a moneyness of 0.95, 1.00 and 1.05 with respect to the actual price of the market portfolio when the option is purchased.\(^5\) Hence we observe three types of options moneyness for each Portfolio 1, 2, and 3, leading to 9 portfolios in total. Each option has a time to maturity of one year. After half a year, the option is replaced by a new option with the same moneyness, again with a time to maturity of one year. Replacing each option takes place under the condition of the portfolio being self-financing. This rolling portfolio strategy occurs over a time period of ten years for each portfolio in order to have enough option prices available for performance measurement.

The options included in the portfolio are priced according to the Heston model. Hence the option prices do not include margins or other costs for the investor. As the option prices are given by the fair values, a performance measure should lead to an alpha of 0 for the considered option portfolios. An alpha unequal to 0 indicates a biased performance measure. The option pricing is based on the assumption of a constant risk-free rate \(r = 0.02\). In the course of measuring the portfolio performance, we refer to monthly portfolio returns as this is a common procedure in measuring fund performance.

First of all, considering Portfolios 1, 2 and 3, with three different strike prices of the included options, we review the quality of Jensen’s alpha (JA) and three fixed option-

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\(^5\)We encompass the area of strike prices where nonlinearity has the biggest influence on performance measurement.
based factors in stages of low, average and high volatility of volatility. Based on our theoretical findings that only one option-based factor is sufficient to capture the influence of the variance on the discrete excess returns on portfolio $P$, we test single option-based factors only. Applying monthly portfolio returns, we perform static regressions, obtaining constant alphas and betas. The rationale underlying this procedure is, as motivated before, that we prooved a single estimation of constant parameters leading to less bias than time-varying estimators obtained from rolling window regressions in the given setting. As a first benchmark, Jensen’s linear performance measure $JA$ can be expressed as:

$$r_{P_i,t} - r = \alpha_i + \beta_{P_i}^M \left[ r_{M,t} - r \right] + \epsilon_{i,t}. \quad (23)$$

For each option-based factor $F_j$, $JA$ is extended to the excess returns on the respective benchmark option portfolios:

$$r_{P_i,t} - r = \alpha_{ij} + \beta_{P_i}^M \left[ r_{M,t} - r \right] + \beta_{F_j}^P \left[ r_{F_j,t} - r \right] + \epsilon_{ij,t}. \quad (24)$$

The first fixed option-based factor employed in the study is descended from the study by Bauer et al. (2009) on the effects of option trading on individual investor performance in the Dutch market. The option-based factors are constructed as follows: At the end of each month, an at the money European call as well as an at the money put on the market index are bought. Both options have a time to maturity of two months. After one month the index options are sold and new at the money index options are purchased, again with a two-month time to maturity. This rolling strategy of buying and selling the options produces a time series of monthly returns. We aggregate the call and the put factor to a

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6As we are operating in a simulation setting we do not take the further developments proposed by Fama and French (1993), Fama and French (2015), and Carhart (1997) into account.
single option-based factor, named by CPF in the following, consisting of the aggregated call and put excess returns.

As a second basic benchmark considered in the simulation study, we refer to the excess returns on the BXM factor proposed by Whaley (2002). The BXM factor is an index that replicates a covered-call strategy on the S&P 500. More precisely, the index is long in the S&P 500 and each month sells a slightly out of the money European call on the S&P 500 with a time to maturity of one month. The factor is also used by Natter et al. (2016) in order to study the benefits of option use by mutual funds. Likewise, we replicate the BXM factor in our considerations by choosing the market portfolio to be the underlying of the respective call options entering the BXM factor. Again, we do not refer to market prices, but instead the calls are priced according to a Heston framework.

Recognizing the time to maturity of the benchmark options as an essential source of error, we find that a time to maturity of only one month leads to a poor performance of the option-based factor as the call is in the payoff when the new call is written. Because of this, we construct a further fixed option-based factor by selecting the structure of the BXM and replacing the time to maturity of the included calls by two months rather than one month. This adjusted BXM factor is denoted as BXM*.

The results of the implemented approaches are compared on the basis of the mean estimated annualized alpha over 10,000 simulations as well as the root mean squared error (RMSE). Considering that the option portfolios are priced in market equilibrium, a perfect model should measure no alpha. The higher the absolute value of alpha the less appropriate the model. The RMSE captures the scattering of the estimated portfolio returns.
4.2 Results

Table 1 reports the average annualized alpha and the RMSE of JA and the extensions by the fixed option-based factors CPF, BXM, and BXM* obtained from 10,000 simulations for the three test portfolios. In addition, the correlation of the respective option-based factors and the market factor is given. Each option included in the portfolios is set to be at the money when the option is purchased. The average option holdings for the three portfolios over all simulations are 10.0% for Portfolio 1, 6.5% for Portfolio 2 and 13.7% for Portfolio 3. Panel A, B, and C refer to the states of low, medium, and high volatility of volatility.

Insert Table 1 about here.

Table 1 reveals a negatively biased average annualized alpha for each approach for Portfolio 1 including a short position in a call on the market. For Portfolio 2 and 3, containing long positions in the respective options, the average annualized alphas are positively biased. All performance measures evince a recognizable relation between the option holdings of the portfolios and the occurring alphas. The higher the option holdings, the higher the absolute average annualized alpha (and thus the bias) for each approach. Considering JA, the bias generally increases as the volatility of volatility rises, since JA is not capable of capturing the influence of the volatility on the portfolio returns.

Noticeably, the BXM factor even underperforms the simple JA in each case. Likewise, CPF leads to a slightly higher bias than JA for each portfolio in states of low or medium volatility of volatility; however, CPF performs much better than BXM. The best performance is

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7The results for strike prices of 0.95 and 1.05 with respect to the market are presented in Tables 2 and 3.
obtained with the modified BXM* factor. Obviously, the rolling strategy of the original BXM factor, in which the option is rolled over at each expiry day, is not able to capture adequately the influence of the risk factors. Instead the rollover date should be at some intermediate timepoint when the option value is not yet its payoff value, as with the CPF and the BXM* factor.

Regarding the scattering however, the RMSE is reduced for each option-based factor compared to JA.

Tables 2 and 3 are the analogues of Table 1 for option strike prices of 0.95 and 1.05 times the market value, respectively. The results are almost identical. For strike prices of 0.95 (Table 2), all biases tend to decrease, while for strike prices of 1.05 (Table 3), they tend to increase. However, the ranking of the option-based factors remains the same as in Table 1, with the exception that the CPF performs slightly better than the BXM* in the stage of a high volatility of volatility.

4.3 Augmented Option Holdings

The previous results show a poorer performance of JA, CPF, BXM and BXM* for increasing option holdings. Stressing the influence of the option holdings, Table 4 presents the average annualized alphas and the RMSE of JA, CPF, BXM and BXM* for augmented option holdings. For that purpose, the holdings of the (long) put options in Portfolio 2 are increased to 50% of the total portfolio value. In addition to the results of at the money puts being included in Portfolio 2, the results of a put’s strike price of 0.95 (out of the money) and 1.05 (in the money) with respect to the actual value of the market portfolio are shown in Table 4.
To reiterate, Table 4 presents substantially increasing error measures for all approaches compared to the basic level of option holdings. The bias of JA and also the biases of the option-based approaches are so high that any assessment of the portfolio performance is questionable. In addition to the performance measures being highly biased, there is also a wide dispersion.

Comparing this to the previous results, the ranking of the considered approaches remains. BXM* is still the best overall measure, whereas BXM considerably underperforms even JA. Interestingly, CPF and BXM* show the relatively best performance in Panel C with high volatility of volatility.

4.4 Optimizing Option-Based Factor

Considering augmented option holdings of 50%, none of the option-based factors CPF, BXM, or BXM* can supply a satisfactory approximation of the portfolio returns, as shown by their considerably biased alphas and high scattering. Thus despite the theoretical notation that any option factor can perfectly explain the portfolio returns with time-varying beta coefficients, estimation error - particularly discretization error - can lead to severe biases. As discussed in Section 3.2, the option-based factor has to capture the nonlinearity of the portfolio returns with respect to the risk factors. Hence, discretization error and bias are small when the option-based factor and the portfolio exhibit a similar structure of nonlinearity. As this structure might be different for different portfolios, it is reasonable that a “one-factor-fits-all” approach is too narrow. Instead, different option-based factors should be chosen for different portfolios.

In this subsection, we suggest improving the approximation by assigning an individual-
ized option-based factor to each portfolio, in order to more precisely cover the individual portfolio deltas and vegas entering the respective portfolio betas. Basically, the idea is similar to the widely applied method of style analysis (e.g., Sharpe (1992), Brown and Goetzmann (1997)). We assume that we know nothing about the structure of the portfolio. The individualized factor, further denoted as optimizing factor, is determined in a pre-step before the actual performance analysis by an OLS-based procedure. Within this pre-step, several potential factors are tested. These test factors consist of aggregated monthly excess returns obtained from a rolling strategy of buying and selling at the money calls and puts, analogously to the CPF, but with different maturities ranging from $T = 3$ to $T = 33$.\footnote{We choose the structure of the CPF as the basis for the optimizing factor (OCPF), as the CPF is weakly correlated to the market factor and includes both types of standard options. The optimization of the OCPF only refers to a varying time to maturity, while the moneyness of the included options stays fixed. This is justified by the fact that the moneyness of the included options very strongly influences the correlation of the OCPF and the market factor. Only when the options are set at-the-money, the correlation can be kept low. Further, our observations exhibit a much larger influence of the time to maturity on the quality of the OCPF than the moneyness.} For each investigated portfolio, the specific factor is chosen by minimizing the RMSE (equivalently, maximizing the $R^2$) obtained from the regressions. The factor is called “optimizing call-put factor” (OCPF) henceforth.\footnote{Note that this approach also could be implemented in an empirical examination without increasing the effort tremendously.}

Figure 1 illustrates the influence of the chosen time to maturity entering the OCPF on the average annualized alpha and the RMSE. The graphs refer to Portfolio 2, which is long in the market and long in an at-the-money put on the market. Again, the option holdings of Portfolio 2 are increased to 50%.
Figure 1 exhibits a similar course of the curves representing the average annualized alpha and the RMSE. Both the average annualized alpha and the RSME reach their minimum for a time to maturity of nine months. Depending on the time to maturity, the OCPF alphas span the range from 0.0054 to 0.0791, which illustrates the relevance of choosing a fitting option-based factor. For times to maturity of one to thirty three months, the range of the RMSE is given by $[0.0439, 0.0697]$.

In order to assess the performance of the OCPF, we consider both the situation with augmented option holdings of Portfolio 2 and the base situation. Table 5 represents the performance of OCPF compared to CPF, assuming the basic level of option holdings (as in Table 1); Table 6 demonstrates the better fitness of OCPF over CPF in the setting with augmented option holdings (as in Table 4). We report both the average annualized alphas and the RMSE.

Table 5 confirms that an individualized parameter adjustment to the options included in the option-based factors consistently improves the goodness of the performance measure. As the option holdings considered in Table 5 are below 14%, the absolute improvement of OCPF over CPF is not very large. Accordingly, the extra effort required to implement an individualized option based-factor in connection with modest option holdings is worth weighing carefully.

However, observing portfolios with a high proportion of option holdings, Table 6 reveals that an individualized factor as OCPF substantially improves the approximation. While the range of the CPF alphas is given by $[0.0169, 0.0450]$, the OCPF alphas range from
0.0011 to 0.0108. Hence the maximum bias of the OCPF alphas is above the minimum bias of the CPF alphas, and the maximum bias of the CPF alphas is even as much as four times as high as the maximum bias of the OCPF alphas. The average bias of the OCPF alphas over all portfolios in Panel A, B, and C is 0.0058. Refining the approximation obtained from the OCPF by selecting the time to maturity of the included options at closer intervals, the results for the OCPF even could be improved.

5 Conclusion

Considering a stochastic volatility framework in line with Heston (1993), we determine an expression for the discrete as well as continuous excess returns on a given option portfolio. The obtained expressions for the excess returns on an option portfolio differ in that the continuous portfolio returns contain additional beta-correction terms while discrete returns do not. This is consistent with the observations of Nielsen and Vassalou (2004). They propose a modification of Jensen’s alpha in a time-continuous model. As Nielsen and Vassalou assume a constant volatility, we extend their observations to a stochastic volatility framework, and obtain additional beta-correction terms for continuous portfolio returns that belong to the market portfolio’s variance. Applying option-based factors as a proxy for the influence of the variance in continuous time, we demonstrate that a single option-based factor is able to capture perfectly the relative portfolio deltas and vegas in the case of discrete portfolio returns. Performing static regressions of the portfolio returns, obtaining constant portfolio betas, our approach sees option-based factors as explaining variable considering nonlinearity of option components with regard to the underlying and the volatility. Importantly, we advise against applying option-based factors to performance
measurement based on continuous portfolio returns, as the beta-correction terms that occur are not adequately captured, leading to biased alphas.

In a simulation study based on a stochastic volatility framework, we analyze how well different option-based factors approximate the nonlinearity of option components with respect to the underlying and the volatility. We show that the goodness of an option-based factor strongly depends on its structure and the choice of the time to maturity of the options included in the factor. The linear performance measure proposed by Jensen (1967) outperforms even two out of three fixed option-based factors in stages of a low and an average volatility of volatility. In particular, the BXM factor taken from Whaley (2002) substantially aggravates the results compared to Jensen’s alpha, as the included options attain the payoff, which prevents them from adequately covering the option portfolio deltas and vegas. Considering augmented option holdings of 50%, none of the generalized option-based factors is able to supply an efficient approximation of the portfolio returns, leading to considerably biased alphas and high scattering. We suggest improving the approximation by assigning an individualized option-based factor to each option portfolio, the aim being to cover more precisely the individual portfolio deltas and vegas entering the respective portfolio betas. The proposed individualized factor consists of aggregated returns on at-the-money calls and puts, allowing the time to maturity of the including options to vary. Applying a minimization of the RMSE, the fitting time to maturity is individually chosen. The results obtained in the study confirm the superiority of the individualized factor over the generalized option-based factors in all of the situations considered.
References


Appendix

A Continuous Excess Returns in a Heston Framework

Consider a time-continuous setting, i.e., the continuous excess returns on an option portfolio $P$ can be written as: $r_{P,t}^c dt = d \log P_t$. Like in the case of discrete returns, the Lemma of Itô (1944) can be used to determine the structure of the continuous portfolio returns:

$$r_{P,t}^c dt = \frac{\partial \log P_t}{\partial t} dt + \frac{\partial \log P_t}{\partial M_t} dM_t + \frac{\partial \log P_t}{\partial v_t} dv_t$$

$$+ \left[ \frac{1}{2} \frac{\partial^2 \log P_t}{\partial M_t^2} dM_t^2 + \frac{1}{2} \frac{\partial^2 \log P_t}{\partial v_t^2} dv_t^2 + \frac{\partial^2 \log P_t}{\partial M_t \partial v_t} dv_t dM_t \right]$$

$$= \frac{\partial \log P_t}{\partial t} dt + \frac{\partial \log P_t}{\partial M_t} \mu_M M_t dt + \frac{\partial \log P_t}{\partial v_t} \nu_t M_t dW_{M,t} + \frac{\partial \log P_t}{\partial v_t} \mu_{v,t} dt$$

$$+ \frac{\partial \log P_t}{\partial v_t} \eta \sqrt{v_t} dW_{v,t} + \frac{1}{2} \frac{\partial^2 \log P_t}{\partial M_t^2} M_t^2 dt + \frac{\partial^2 \log P_t}{\partial v_t^2} v_t + \frac{1}{2} \frac{\partial^2 \log P_t}{\partial M_t \partial v_t} \nu_t M_t dt + \frac{1}{2} \frac{\partial^2 \log P_t}{\partial v_t^2} \eta^2 v_t dt$$

$$+ \frac{\partial \log P_t}{\partial M_t} \rho M_t v_t dt + \frac{1}{2} \frac{\partial^2 \log P_t}{\partial M_t^2} M_t^2 dt + \frac{\partial^2 \log P_t}{\partial v_t^2} v_t + \frac{1}{2} \frac{\partial^2 \log P_t}{\partial M_t \partial v_t} \rho M_t dt$$

$$+ \mu_{v,t} + \frac{\partial \log P_t}{\partial v_t} dt + M_t \sqrt{v_t} \frac{\partial \log P_t}{\partial M_t} dW_{M,t} + \eta \sqrt{v_t} \frac{\partial \log P_t}{\partial v_t} dW_{v,t}.$$ 

The appearing partial derivatives are given by:

$$\frac{\partial \log P_t}{\partial t} = \frac{1}{P_t} \frac{\partial P_t}{\partial t}, \quad (26)$$

$$\frac{\partial \log P_t}{\partial M_t} = \frac{1}{P_t} \frac{\partial P_t}{\partial M_t}, \quad (27)$$

$$\frac{\partial \log P_t}{\partial v_t} = \frac{1}{P_t} \frac{\partial P_t}{\partial v_t}, \quad (28)$$

$$\frac{\partial^2 \log P_t}{\partial M_t^2} = \frac{\partial \left( \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} \right)}{\partial M_t} = \frac{\partial \left( \frac{1}{P_t} \right) \frac{\partial P_t}{\partial M_t}}{\partial M_t} + \frac{1}{P_t} \frac{\partial^2 P_t}{\partial M_t \partial M_t} = \frac{1}{P_t^2} \frac{\partial^2 P_t}{\partial M_t^2} - \frac{1}{P_t^2} \left( \frac{\partial P_t}{\partial M_t} \right)^2, \quad (29)$$

$$\frac{\partial^2 \log P_t}{\partial v_t^2} = \frac{\partial \left( \frac{1}{P_t} \frac{\partial P_t}{\partial v_t} \right)}{\partial v_t} = \frac{\partial \left( \frac{1}{P_t} \right) \frac{\partial P_t}{\partial v_t}}{\partial v_t} + \frac{1}{P_t} \frac{\partial^2 P_t}{\partial v_t^2} = \frac{1}{P_t^2} \frac{\partial^2 P_t}{\partial v_t^2} - \frac{1}{P_t^2} \left( \frac{\partial P_t}{\partial v_t} \right)^2, \quad (30)$$

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\begin{align*}
\frac{\partial^2 \log P}{\partial M_t \partial v_t} &= \frac{\partial}{\partial t} \left( \frac{\partial P_t}{\partial M_t} \right) = \frac{1}{P_t} \frac{\partial^2 P_t}{\partial M_t \partial v_t} - \frac{1}{P_t^2} \frac{\partial P_t}{\partial M_t} \frac{\partial P_t}{\partial v_t}.
\end{align*} \tag{31}

Inserting the partial derivatives:

\begin{align*}
r_{P,t} dt &= \left[ \frac{1}{2} M_t^2 v_t \left( \frac{1}{P_t} \frac{\partial^2 P_t}{\partial M_t^2} - \frac{1}{P_t^2} \frac{\partial P_t}{\partial M_t} \right)^2 + M_t v_t \eta P_t \frac{1}{P_t} \frac{\partial^2 P_t}{\partial M_t \partial v_t} - \frac{1}{2} \eta^2 v_t \frac{1}{P_t^2} \frac{\partial^2 P_t}{\partial v_t^2} - r M_t \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} \right] dt \\
&\quad + M_t \sqrt{v_t} \frac{\partial \log P_t}{\partial M_t} dW_{M,t} + \eta \sqrt{v_t} \frac{\partial \log P_t}{\partial v_t} dW_{v,t}. \tag{32}
\end{align*}

Applying:

\begin{align*}
\frac{1}{P_t} \frac{\partial P_t}{\partial t} &= r - \frac{1}{2} v_t M_t^2 \frac{1}{P_t^2} \frac{\partial^2 P_t}{\partial M_t^2} - \rho v_t M_t \frac{1}{P_t} \frac{\partial^2 P_t}{\partial M_t \partial v_t} - \frac{1}{2} \eta^2 v_t \frac{1}{P_t^2} \frac{\partial^2 P_t}{\partial v_t^2} - r M_t \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} \\
&\quad - \left( \mu_{v,t} - \lambda(M, v, t) \right) \frac{1}{P_t} \frac{\partial P_t}{\partial v_t}, \tag{33}
\end{align*}

obtained from the Heston-PDE, leads to:

\begin{align*}
r_{P,t} dt &= \left[ r - \frac{1}{2} M_t^2 v_t \frac{1}{P_t^2} \left( \frac{\partial P_t}{\partial M_t} \right)^2 - M_t v_t \eta P_t \frac{1}{P_t^2} \frac{\partial P_t}{\partial M_t} \frac{\partial P_t}{\partial v_t} - \frac{1}{2} \eta^2 v_t \frac{1}{P_t^2} \left( \frac{\partial P_t}{\partial v_t} \right)^2 \\
&\quad + \frac{1}{P_t} \frac{\partial P_t}{\partial M_t} \left( \mu_M M_t - r M_t \right) + \frac{1}{P_t} \frac{\partial P_t}{\partial v_t} \lambda(M, v, t) \right] dt \\
&\quad + \sqrt{v_t} \frac{M_t}{P_t} \frac{\partial P_t}{\partial M_t} dW_{M,t} + \eta \sqrt{v_t} \frac{1}{P_t} \frac{\partial P_t}{\partial v_t} dW_{v,t}. \tag{34}
\end{align*}

In order to resolve the occurring Wiener processes, the dynamics of \( \log M_t \) and \( \log v_t \) are derived. The dynamics of \( \log M_t \) follow the dynamics:

\begin{align*}
\log M_t &= \frac{\partial \log M_t}{\partial M_t} dM_t + \frac{1}{2} \frac{\partial^2 \log M_t}{\partial M_t^2} dM_t^2 \\
&= \frac{1}{M_t} dM_t + \frac{1}{2} \left( \frac{1}{M_t^2} - \frac{1}{M_t^2} \right) dM_t^2 \\
&= \mu_M dt - \frac{1}{2} v_t dt + \sqrt{v_t} dW_{M,t}, \tag{35}
\end{align*}

which is equivalent to:

\begin{align*}
\sqrt{v_t} dW_{M,t} &= d\log M_t - \mu_M dt + \frac{1}{2} v_t dt. \tag{36}
\end{align*}
Analogously, the dynamics of \( \log v_t \) follow the process:

\[
d \log v_t = \frac{\partial \log v_t}{\partial v_t} dv_t + \frac{1}{2} \frac{\partial^2 \log v_t}{\partial v_t^2} dv_t^2
\]

\[
= \frac{1}{v_t} \mu_{v,t} dt - \frac{1}{v_t} \eta^2 dt + \frac{1}{v_t} \eta \sqrt{v_t} dW_{v,t},
\]

therefore:

\[
\eta \sqrt{v_t} dW_{v,t} = v_t d \log v_t - \mu_{v,t} dt + \frac{1}{2} \eta^2 dt.
\]

Employing (36) and (38) leads to the following expression for the continuous excess returns on portfolio \( P \):

\[
r_{c,P,t} dt - rd t = \frac{M_t}{P_t} \frac{\partial P}{\partial M_t} d \log M_t + \frac{v_t}{P_t} \frac{\partial P}{\partial v_t} d \log v_t - \frac{1}{2} M_t^2 v_t \left( \frac{\partial P}{\partial M_t} \right)^2 dt
\]

\[
- \rho \eta v_t M_t \frac{\partial P}{\partial v_t} dt - \frac{1}{2} \eta^2 v_t \frac{\partial P}{\partial v_t} \left( \frac{\partial P}{\partial v_t} \right)^2 dt + \frac{M_t}{P_t} \frac{\partial P}{\partial M_t} \left( \frac{1}{2} v_t - r \right) dt
\]

\[
+ \frac{1}{P_t} \frac{\partial P}{\partial v_t} \left( \lambda(M,v,t) - \mu_{v,t} + \frac{1}{2} \eta^2 \right) dt.
\]

Cutting out \( dt \), the excess returns on portfolio \( P \) can be expressed as:

\[
r_{c,P,t} - r = \beta_{P,M,t} \left( r_{c,M,t} - r \right) + \beta_{P,v,t} \left( r_{c,v,t} - \left( \mu_{v,t} - \lambda(M,v,t) \right) \frac{1}{v_t} \right)
\]

\[
+ \frac{1}{2} v_t \left( \beta_{P,M,t} - (\beta_{P,M,t})^2 \right) - \rho \eta \beta_{P,M,t} \beta_{P,v,t} + \frac{1}{2} \eta^2 \left( \beta_{P,v,t} - (\beta_{P,v,t})^2 \right),
\]

denoting \( d \log M_t \) as \( r_{c,M,t} dt \) and \( d \log v_t \) as \( r_{c,v,t} dt \). Further, define \( \beta_{P,M,t} = \frac{\partial P}{\partial M_t} M_t \), \( \beta_{P,v,t} = \frac{\partial P}{\partial v_t} v_t \).

\section*{B Proof of Proposition 1}

\textit{Proof.} As portfolio \( F \) solves the Heston PDE and the market portfolio is assumed to be the underlying of the option(s) included in \( F \), the discrete excess returns on \( F \) can be expressed as:

\[
r_{F,t} - r = \beta_{M,t}^F \left( r_{M,t} - r \right) + \beta_{v,t}^F \left( r_{v,t} - \left( \mu_{v,t} - \lambda(M,v,t) \right) \frac{1}{v_t} \right).
\]
Inserting for \( r_{F,t} - r \) in
\[
    r_{P,t} - r = \beta_{P,t}^M \left[r_{M,t} - r\right] + \beta_{F,t}^P \left[r_{F,t} - r\right] \tag{42}
\]
leads to:
\[
    r_{P,t} - r = \left( \beta_{P,t}^M + \beta_{F,t}^F \beta_{M,t}^F \right) \left[r_{M,t} - r\right] + \beta_{F,t}^P \beta_{v,t}^F \left[r_{v,t} - \left(\mu_{v,t} - \lambda(M,v,t)\right) \frac{1}{vt}\right] \tag{43}
\]
which equals (11) for \( \beta_{F,t}^P = \frac{\beta_{P,t}^F}{\beta_{v,t}^F}, \beta_{M,t}^F = \beta_{M,t}^P - \frac{\beta_{P,t}^F}{\beta_{v,t}^F} \beta_{M,t}^F \).

\[\square\]

**C Proof of Proposition 2**

**Proof.** As portfolio \( F \) solves the Heston PDE and the market portfolio is assumed to be the underlying of the option(s) included in \( F \), the continuous excess returns on \( F \) can be expressed as:
\[
    r_{F,t}^c - r = \beta_{M,t}^F \left[r_{M,t}^c - r\right] + \beta_{v,t}^F \left[r_{v,t}^c - \left(\mu_{v,t} - \lambda(M,v,t)\right) \frac{1}{vt}\right] + \frac{1}{2} \left(\beta_{M,t}^F - (\beta_{M,t}^F)^2\right) - \rho \eta \beta_{M,t}^F \beta_{v,t}^F + \frac{1}{2} \left(\beta_{v,t}^F - (\beta_{v,t}^F)^2\right). \tag{44}
\]
Inserting for \( r_{F,t}^c - r \) in
\[
    r_{P,t}^c - r = \beta_{M,t}^P \left[r_{M,t}^c - r\right] + \beta_{F,t}^P \left[r_{F,t}^c - r\right] \tag{45}
\]
leads to:
\[
    r_{P,t}^c - r = \left( \beta_{M,t}^P + \beta_{F,t}^F \beta_{M,t}^F \right) \left[r_{M,t}^c - r\right] + \beta_{F,t}^P \beta_{v,t}^F \left[r_{v,t}^c - \left(\mu_{v,t} - \lambda(M,v,t)\right) \frac{1}{vt}\right] + \frac{1}{2} \left(\beta_{F,t}^F \beta_{M,t}^F - (\beta_{F,t}^F \beta_{M,t}^F)^2\right) - \rho \eta \beta_{F,t}^P \beta_{M,t}^F \beta_{v,t}^F + \frac{1}{2} \left(\beta_{v,t}^F - (\beta_{v,t}^F)^2\right). \tag{46}
\]
For $\beta_{F,t} = \frac{\beta_{P,t}}{\beta_{v,t}}$, $\tilde{\beta}_{M,t} = \beta_{P,t} - \frac{\beta_{P,t}}{\beta_{v,t}} \beta_{F,t}$:

$$
\begin{align*}
\tilde{r}_{P,t} - r &= \beta_{M,t} \left[ r_{M,t} - r \right] + \beta_{v,t} \left[ r_{v,t} - \left( \mu_{v,t} - \lambda_{v,M,t} \right) \frac{1}{v_t} \right] \\
&+ \frac{1}{2} v_t \left( (\beta_{P,M,t} - \tilde{\beta}_{P,M,t}) - (\beta_{P,M,t} - \tilde{\beta}_{P,M,t}) \beta_{F,M,t} \right) - \rho \eta \beta_{v,t} \beta_{M,t} \\
&+ \frac{1}{2} \frac{\eta^2}{v_t} \left( \beta_{v,t} - \beta_{v,t} \beta_{F,v,t} \right). \\

\end{align*}
$$

(47)

Obtaining error terms $\zeta_t$ by subtracting (47) from (13):

$$
\begin{align*}
\zeta_t &= \frac{1}{2} \left( \beta_{M,t} + (\beta_{M,t} - \tilde{\beta}_{M,t}) \beta_{F,M,t} - (\beta_{P,M,t})^2 \right) \\
&- \rho \eta \beta_{v,t} \left( \beta_{M,t} - \beta_{F,M,t} \right) + \frac{1}{2} \frac{\eta^2}{v_t} \left( \beta_{v,t} \beta_{F,v,t} - (\beta_{v,t})^2 \right). \\

\end{align*}
$$

(48)
Figure 1. Average annualized alpha and RMSE of the CPF in dependence on the options’ time to maturity (in months) entering the CPF, for Portfolio 2 with augmented option holdings of 50%. The options included in Portfolio 2 are set to be at the money when the options are purchased. The time to maturity of the options included in the CPF varies from 1 month to 33 months.
Table 1. This table reports the average annualized alpha and the RMSE of JA, CPF, BXM and BXM* obtained from 10,000 simulations for Portfolios 1, 2 and 3. It also presents the correlation of the respective option-based factors and the market factor. Each option included in Portfolios 1, 2 and 3 is set to be at the money when the option was purchased. For each portfolio the average option holdings over all simulations are specified, given by 10.0% for Portfolio 1, 6.5% for Portfolio 2 and 13.7% for Portfolio 3. Panel A, B, and C refer to the volatility of volatility, based on the three parameter sets obtained from the calibration of the Heston model on traded DAX options.
### Table 2.
This table reports the average annualized alpha and the RMSE of JA, CPF, BXM and BXM* obtained from 10,000 simulations for Portfolios 1, 2 and 3. It also presents the correlation of the respective option-based factors and the market factor. The strike price of each option included in Portfolios 1, 2 and 3 is set to be at 0.95 with respect to the actual price of the market portfolio when the option was purchased. For each portfolio the average option holdings over all simulations are specified, given by 14.0% for Portfolio 1, 4.9% for Portfolio 2 and 14.6% for Portfolio 3. Panel A, B, and C refer to the volatility of volatility, based on the three parameter sets obtained from the calibration of the Heston model on traded DAX options.

<table>
<thead>
<tr>
<th></th>
<th>Market - Call</th>
<th></th>
<th>Market + Put</th>
<th></th>
<th>Market + Call + Put</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(av. option hold.: 14.0%)</td>
<td></td>
<td>(av. option hold.: 4.9%)</td>
<td></td>
<td>(av. option hold.: 14.6%)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\hat{\alpha})</td>
<td>RMSE</td>
<td>(\rho_{\alpha,\mu})</td>
<td>(\hat{\alpha})</td>
<td>RMSE</td>
<td>(\rho_{\alpha,\mu})</td>
</tr>
<tr>
<td>JA</td>
<td>-0.0007</td>
<td>0.0122</td>
<td>-</td>
<td>0.0007</td>
<td>0.0121</td>
<td>-</td>
</tr>
<tr>
<td>CPF</td>
<td>-0.0011</td>
<td>0.0107</td>
<td>0.2010</td>
<td>0.0012</td>
<td>0.0111</td>
<td>0.1979</td>
</tr>
<tr>
<td>BXM</td>
<td>-0.0020</td>
<td>0.0110</td>
<td>0.8259</td>
<td>0.0018</td>
<td>0.0114</td>
<td>0.8268</td>
</tr>
<tr>
<td>BXM*</td>
<td>-0.0007</td>
<td>0.0105</td>
<td>0.9115</td>
<td>0.0007</td>
<td>0.0110</td>
<td>0.9119</td>
</tr>
</tbody>
</table>
| Panel A: Low volatility of volatility
| JA             | -0.0013 | 0.0144 | - | 0.0014 | 0.0140 | - | 0.0025 | 0.0238 | - |
| CPF            | -0.0009 | 0.0125 | 0.1984 | 0.0016 | 0.0126 | 0.1985 | 0.0021 | 0.0212 | 0.1953 |
| BXM            | -0.0025 | 0.0131 | 0.8272 | 0.0028 | 0.0131 | 0.8267 | 0.0048 | 0.0220 | 0.8819 |
| BXM*           | -0.0001 | 0.0122 | 0.9045 | 0.0009 | 0.0125 | 0.9039 | 0.0011 | 0.0209 | 0.9049 |
| Panel B: Average volatility of volatility
| JA             | -0.0051 | 0.0146 | - | 0.0042 | 0.0144 | - | 0.0076 | 0.0247 | - |
| CPF            | 0.0002  | 0.0125 | 0.1711 | -0.0001 | 0.0130 | 0.1917 | -0.0005 | 0.0217 | 0.1803 |
| BXM            | -0.0075 | 0.0126 | 0.8227 | 0.0062 | 0.0131 | 0.8196 | 0.0119 | 0.0219 | 0.8205 |
| BXM*           | 0.0006  | 0.0123 | 0.8941 | -0.0003 | 0.0129 | 0.8920 | -0.0010 | 0.0214 | 0.8922 |
| Panel C: High volatility of volatility

Note: The table includes the average annualized alpha \(\hat{\alpha}\), root-mean-square error (RMSE), and the correlation \(\rho_{\alpha,\mu}\) between the option-based factor and the market factor for different portfolios and parameter sets.
Table 3. This table reports the average annualized alpha and the RMSE of JA, CPF, BXM and BXM* obtained from 10,000 simulations for Portfolios 1, 2 and 3. Additionally, it presents the correlation of the respective option-based factors and the market factor. The strike price of each option included in Portfolios 1, 2 and 3 is set to be at 1.05 with respect to the actual price of the market portfolio when the option was purchased. For each portfolio the average option holdings over all simulations are specified, given by 7.2% for Portfolio 1, 8.4% for Portfolio 2 and 13.4% for Portfolio 3. Panel A, B, and C refer to the volatility of volatility, based on the three parameter sets obtained from the calibration of the Heston model on traded DAX options.
Table 4. This table reports the average annualized alpha and the RMSE of JA, CPF, BXM and BXM* obtained from 10,000 simulations of augmented option holdings in Portfolio 2 (long market, long put). It further presents the correlation of the respective option-based factors and the market factor. The strike prices of the included options are set to be 1.0 (ATM-Puts), 0.95 (OTM-Puts) and 1.05 (ITM-Puts) with respect to the actual value of the market portfolio when the options were purchased. For each portfolio the average option holdings over all simulations are increased to 50.0% of the total portfolio value. Panel A, B, and C refer to the volatility of volatility, based on the three parameter sets obtained from the calibration of the Heston model on traded DAX options.

<table>
<thead>
<tr>
<th></th>
<th>Market + ATM Puts (av. option hold.: 50.0%)</th>
<th>Market + OTM Puts (av. option hold.: 50.0%)</th>
<th>Market + ITM Puts (av. option hold.: 50.0%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\tilde{\alpha}$</td>
<td>RMSE</td>
<td>$\rho_{\tilde{\alpha}/r}$</td>
</tr>
<tr>
<td><strong>Panel A: Low volatility of volatility</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>JA</td>
<td>0.0362</td>
<td>0.0727</td>
<td>–</td>
</tr>
<tr>
<td>CPF</td>
<td>0.0441</td>
<td>0.0556</td>
<td>0.1963</td>
</tr>
<tr>
<td>BXM</td>
<td>0.0493</td>
<td>0.0611</td>
<td>0.8270</td>
</tr>
<tr>
<td>BXM*</td>
<td>0.0395</td>
<td>0.0553</td>
<td>0.9119</td>
</tr>
<tr>
<td><strong>Panel B: Average volatility of volatility</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>JA</td>
<td>0.0389</td>
<td>0.0803</td>
<td>–</td>
</tr>
<tr>
<td>CPF</td>
<td>0.0393</td>
<td>0.0577</td>
<td>0.1925</td>
</tr>
<tr>
<td>BXM</td>
<td>0.0500</td>
<td>0.0679</td>
<td>0.8273</td>
</tr>
<tr>
<td>BXM*</td>
<td>0.0324</td>
<td>0.0576</td>
<td>0.9044</td>
</tr>
<tr>
<td><strong>Panel C: High volatility of volatility</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>JA</td>
<td>0.0607</td>
<td>0.0682</td>
<td>–</td>
</tr>
<tr>
<td>CPF</td>
<td>0.0212</td>
<td>0.0353</td>
<td>0.1836</td>
</tr>
<tr>
<td>BXM</td>
<td>0.0823</td>
<td>0.0437</td>
<td>0.8205</td>
</tr>
<tr>
<td>BXM*</td>
<td>0.0193</td>
<td>0.0383</td>
<td>0.8928</td>
</tr>
</tbody>
</table>
Table 5. This table reports the average annualized alpha, the RMSE and the correlation with the market factor of OCPF compared to CPF obtained from 10,000 simulations for Portfolios 1, 2 and 3. Each option included in Portfolios 1, 2 and 3 is set to be at the money when the option was purchased. For each portfolio the average option holdings over all simulations are specified, given by 10.0% for Portfolio 1, 6.5% for Portfolio 2 and 13.7% for Portfolio 3. Panel A, B, and C refer to the volatility of volatility, based on the three parameter sets obtained from the calibration of the Heston model on traded DAX options. 

\( T = (T_1, T_2, T_3) \) denotes the time to maturity in months of the options entering the OCPF chosen by the RMSE-minimizing algorithm. \( T_i \) denotes the time to maturity chosen for Portfolio \( i, i = 1, 2, 3. \)
Table 6. This table reports the average annualized alpha, the RMSE and the correlation with the market factor of OCPF compared to CPF obtained from 10,000 simulations of augmented option holdings in Portfolio 2 (long market, long put). The strike prices of the included options are set to be 1.0 (ATM-Puts), 0.95 (OTM-Puts) and 1.05 (ITM-Puts) with respect to the actual value of the market portfolio when the options were purchased. For each portfolio the average option holdings over all simulations are increased to 50.0% of the total portfolio value. Panel A, B, and C refer to the volatility of volatility, based on the three parameter sets obtained from the calibration of the Heston model on traded DAX options. \( T = (T_1, T_2, T_3) \) denotes the time to maturity in months of the options entering the OCPF chosen by the RMSE-minimizing algorithm. \( T_i \) denotes the time to maturity chosen for Portfolio \( i, i = 1, 2, 3 \).