Time-varying risk aversion: An application to European optimal portfolios

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Abstract. Despite the influence of risk aversion in the optimal portfolio context, there are not many studies which have explicitly estimated the risk aversion parameter. Instead of that, researchers almost always choose random fixed values to reflect the common levels of risk aversion. However, the above could generate optimal portfolios, which not reflect the actual investor’s attitude towards risk. Otherwise, as it is well known, an individual is more or less risk averse according to the economic and political circumstances. Given the above, we model the risk aversion attitude so that it changes over time, in order to take into account the variability in agents’ expectations. Therefore, the aim of this paper is to shed light on the choice of the risk aversion parameter that correctly represents the investors’ behaviour. For that purpose, we build optimal portfolios for different types of investment profiles in order to compare whether is better to use a constant risk aversion parameter or a dynamic one. In particular, our proposal is based on estimating the time-varying risk aversion parameter as a derivation of the market risk premium. For that purpose, we implement several statistical univariate and multivariate models. Specifically, we use conditional variance and correlation models, such as GARCH (1, 1), GARCH-M (1, 1) and DCC-GARCH.

Keywords: Optimal portfolio, Time-varying risk aversion, Market risk premium, GARCH models
INTRODUCTION

In accordance with the mean-variance approach, we can partially order the set of investment opportunities, reducing the choice of investors to those portfolios located on the efficient frontier. However, with this approach, the investors cannot compare which alternatives are dominant among themselves; therefore, they are not allowed to select the investment portfolio that best meets their economic objectives. To find this portfolio, we must use a different criterion, incorporating the individual risk attitude. Although these preferences are very complex (they depend on, for instance, the age, gender, education level, and income of the individual), to make their implementation easier, they are represented by a single parameter that summarizes the personal level of risk aversion, the risk aversion parameter.

In spite of playing a key role in the optimal portfolio construction, there are few studies that have explicitly estimated the risk aversion of an investor. Instead, they choose random values to reflect the common levels of risk aversion. The equity literature on risk aversion developed based on the review by Arrow (1971), who affirmed that the risk aversion parameter should be approximately 1. Otherwise, in the equity context, several studies have been published that differ in their estimations of risk aversion. For instance, Mehra and Prescott (1985) argued that this parameter should be greater than 10. Moreover, Ghysels et al. (2005) affirmed the risk attitude should be between 1.5 and 2, on average, while Guo and Whitelaw (2006) established the mentioned parameter of 4.93.

However, common sense tells us that the use of fixed arbitrary values for this parameter could yield optimal portfolios that do not reflect the actual investor’s attitude towards risk. An individual is more or less risk averse according to the economic and political circumstances. Given that, it seems reasonable to model the risk aversion parameter so that it changes over time, to consider the variability in the agents’ expectations. In this context, there are some studies in the financial literature that refer to time-varying risk aversion. For instance, Kim (2014) proposes a consistent indicator of conditional risk aversion in consumption-based CAPM. Other studies have differed widely in their estimates of time-varying risk aversion, such as Dionne (2014), who aims to extend the concept of orders of conditional risk aversion to orders of conditional dependent risk aversion. However, our motivation follows the framework proposed by Frankel (1982) and revised by Giovanni and Jorion (1989) and Cotter
and Hanly (2010), which is based on estimating the risk aversion parameter as a derivation of the CRRA.¹

The aim of this paper is to examine the optimal parameter choice that provides a better representation of the investors’ attitude towards risk. In particular, our proposal is based on estimating the time-varying risk aversion parameter as a derivation of the CRRA and strongly related to the market risk premium context. We build a well-diversified portfolio from ten risky equities traded on the Eurostoxx-50 index. We introduce the time-varying modeling of probability distribution moments as we consider that optimal portfolios changes over time. We consider the application of conditional variance and correlation schemes such as GARCH (1, 1) and DCC-GARCH to obtain the optimal portfolios.

Otherwise, we estimate the CRRA from the market risk premium, which depends on the mean and variance of the market.² This estimation allows us to obtain the risk aversion attitude of an investor in a single number. However, in this research we are more interested in the time-varying risk aversion, not in a constant parameter. Thus, we model the market mean and variance through conditional models such as the GARCH (1, 1)-M specification. Further, we aim to assess whether it is better to work with a constant or changing risk aversion parameter. For that purpose, we build optimal portfolios for different types of investment profiles, a conditional one associated with the CRRA, and one based on constant risk aversion. Note that we assess the portfolios for different timeframes, ones related to calm periods and others related to economic recession. Finally, to conclude this research, we present the main results and conclusions obtained from the study. Some important implications are revealed, including the best fit of the dynamic risk aversion attitude to the economic and political circumstances. The above could suggests that time-varying risk aversion models perform better than constant ones.

¹ This term refers to the changes in relative risk aversion, which is a way to express the risk aversion attitude of an investor through his utility function.
² Note that we use the daily closing prices of the EuroStoxx-50 Index as the market portfolio, and the 3-month German Treasury Bills as the risk-free rate.
2. MEAN-VARIANCE APPROACH AND THE OPTIMAL PORTFOLIO PROBLEM

2.1 Markowitz approach

Markowitz (1952) was the first scholar to pay attention to the practice of portfolio diversification, which can be found in his work in “Portfolio Selection: Efficient Diversification of Investments,” published in 1959. This is where investors generally prefer to keep asset portfolios rather than individual assets, because they do not consider only the returns of these assets but also the risk thereof.

Within this framework, Markowitz proposed the minimum variance portfolio, which is a combination of risky assets that has the lowest level of risk between the different possible combinations of risky assets. In the original problem, the author fixed a specific expected return of the portfolio as a constraint. However, in this paper, we face the construction of the minimum-variance portfolio in a different way, without restricting short positions, i.e., those positions that would probably take on assets of higher risk (volatility). Thus, the proposed problem is as follows:

\[
\min \ w'Vw \\
\text{restricted to: } \sum_{i=1}^{n} w_i = 1;
\]

According to the minimum-variance approach, the set of portfolios we could build in the case of \( n \) risky assets that can be displayed as a cloud of points showing the set of investment opportunities, which is given by the market. Possible portfolios entirely cover a region of the mean-variance space and this region is convex.

2.2 Inclusion of the utility theory

In uncertainty contexts it is possible to reach the preferences representation of economic agents through the expected utility. In short, it is suggested that financial theory has developed utility functions to assess how good an investment is, according to its expected utility. When we represent investor preferences through utility functions, we are assuming that the decision maker has a well-defined utility function of his wealth, \( U(W) \). It is also
assumed that each individual chooses among different alternatives, maximizing the expected utility of his wealth.

Within this area, we must make a very important distinction. While risk depends on the specific characteristics of financial assets, the risk attitude depends on the individual preferences and, therefore, may be different for each type of agent. In fact, in accordance with the shape of the utility function, we can distinguish three types of attitudes toward risk: risk aversion, risk neutrality and risk appetite. To understand the risk attitude, it is essential to study the risk aversion coefficients of Arrow-Pratt (1971).

On the one hand, Arrow developed the absolute risk aversion coefficient for an individual with an associated utility function, as follows:

$$ARA = -\frac{U''(W)}{U'(W)}$$  \hspace{1cm} (2)

This aversion coefficient is positive if and only if the individual is risk averse, that is to say, if the individual shows a concave utility function $U''(W) < 0$. Actually, it is the concavity (second derivative) that reflects the risk aversion level, but it is necessary to adjust this measure by the first derivative of the utility function, to ensure that it does not change under linear transformations.

On the other hand, we can talk about the relative risk aversion coefficient, which measures aversion in percentage terms:

$$\text{CRRA} = -W \frac{U''(W)}{U'(W)} = W \text{ ARA}$$  \hspace{1cm} (3)

Intuitively, it seems clear that absolute risk aversion should be decreasing for most individuals, whereas relative aversion is generally decreasing.

In general, we assume that investors are risk-averse. For that reason, in this work, we only focus on the analysis of two risk-averse utility functions, such as quadratic and CARA functions. Given that, we use the first ones to obtain explicit forms for optimal portfolios and the second ones to help us model the risk aversion parameter over time.
2.3 Negative exponential utility functions

Analytically, the negative exponential utility function (CARA) is represented as follows:

\[ U(W) = -e^{-\alpha w}, \quad \alpha > 0 \quad (4) \]

where an increase in wealth \( W \) produces an equal diminishing utility level.\(^3\) Then, calculating the first and the second derivative of this expression, we can see that it is an increasing and concave function. Which makes sense, due to this is a function, which represents the preferences of risk averse individuals.

\[
\begin{align*}
U'(W) &= \alpha e^{-\alpha w} > 0 \quad \text{increasing function} \\
U''(W) &= -\alpha^2 e^{-\alpha w} < 0 \quad \text{concave function}
\end{align*}
\]

Although this function exhibits constant absolute risk aversion, it is widely used. This is because this function combined with the assumption of normality in returns of financial assets, allows explicit forms for optimal portfolios to be obtained. We show the above through the following expressions:

\[
\begin{align*}
ARA &= -\left(\frac{-\alpha^2 e^{-\alpha w}}{\alpha e^{-\alpha w}}\right) = \alpha \\
\frac{\delta ARA}{\delta W} &= 0 \quad ARA = \text{constant} \\
CRRA &= -W \left(\frac{-\alpha^2 e^{-\alpha w}}{\alpha e^{-\alpha w}}\right) = W\alpha \\
\frac{\delta CRRA}{\delta W} &= 0 \quad CRRA = \text{increasing}
\end{align*}
\]

\(^3\) This is often a feature of institutional investors.
2.4 Optimal portfolio construction. An extension of the CARA function

In this research, we analyze the optimal portfolio using the negative exponential utility function (CARA), which has been described in detail in the previous equations (equations 10, 11 and 12). Given that, if the final investor’s wealth follows a normal distribution with an associated mean $\mu$ and variance $\sigma^2$, then the moment-generating function of a normal distribution is used, as follows:

$$E[U(W)] = E[-e^{-aw}] = -e^{-aw\left(\frac{\mu}{2\alpha\sigma^2}\right)} = U\left(E(W) - \frac{\alpha}{2}\sigma^2(W)\right)$$  \hspace{1cm} (7)

On the other hand, analytically, the investor's problem is based on determining the weights of the risky equities that maximize the expected utility, given the constraint that these weights sum the unity$^4$:

$$\max_{w} U(E_p, \sigma_p^2) = (w' E - \frac{\alpha}{2} w' V w)$$

s. a. $\sum_{i=1}^{n} w_i = 1$  \hspace{1cm} (8)

Based on the above, we solve the Lagrangian problem, obtaining an expression for the optimal portfolio weights, as follows:

$$W_o = W_{mv} + \frac{1}{\alpha} \left(V^{-1} 1n - \frac{1}{1n' V^{-1} 1n} V^{-1} 1n\right)$$

$$W_o = W_{mv} + \frac{1}{\alpha} \left(V^{-1} E - \frac{1}{1n' V^{-1} 1n} V^{-1} n\right)$$  \hspace{1cm} (9)

where $V^{-1}$ is the 10x10 inverse covariance matrix, $1n$ is a 10x1 ones vector, $E$ is the 1x10 expected return vector and $\alpha$ is the individual level of risk aversion.

Graphically, the investor's optimal portfolio is represented as the intersection of the efficient frontier and the indifference curve (utility function). At this point, the slopes of both curves

$^4$ To analyze that problem, we rely on Gómez (2011).
are equal, so the rate at which we can exchange return for market risk is equal to the ratio at which the investor is willing to do it personally. Thus, the optimal portfolio represents the combination of assets that supports the efficient frontier and is also in the highest indifference curve.

2.5 Quadratic utility functions

Formally, the quadratic utility function is represented by the following expression:

\[
U(W) = aW - bW^2, \quad b > 0, W < \frac{a}{2b}
\]  

Then, calculating the first and second derivative we can show the main characteristics of this type of function:

\[
U'(W) = a - 2bW
\]

\[
> 0 \text{ if } W < \frac{a}{2b} \quad \text{not always an increasing function}
\]

\[
U''(W) = -2b < 0 \quad \text{concave function}
\]

\[
\frac{\delta ARA}{\delta W} > 0 \quad ARA = \text{increasing}
\]

\[
CRRA = -W \frac{(-2b)}{a - 2bW} = \frac{2b}{\left(\frac{a}{W}\right) - 2b}
\]

\[
\frac{\delta CRRA}{\delta W} > 0 \quad CRRA = \text{increasing}
\]

These functions have some important drawbacks. On the one hand, utility is not always increasing. On the other, the absolute risk aversion is always increasing. However, working with this utility function, the mean-variance approach is consistent with the criteria of maximizing the investor’s expected utility, which is something relevant in the CAPM context. The above is a key element in this work, as it relates the quadratic utility functions
with the market risk premium. Given the importance of the market risk premium, we analyze it more in depth in the quadratic utility context.

Within the quadratic utility functions framework, we must mention the CRRA. This term is more related to the measuring of changes in relative percentages invested in risky and risk-free assets. Moreover, this expression is a useful tool in financial contexts in the sense of helping us with the calculus of the risk aversion parameter.

Thus, we could affirm that it is possible to represent the risk aversion attitude of an investor in a single number by the CRRA expression. As previously mentioned, we only review the CRRA within the context of quadratic utility, following the approach revised by Giovanni and Jorion (1989), which is based on the estimation of the CRRA through the market risk premium. In this context of asset pricing, the size of the risk premium is determined by the aggregate risk aversion of investors and by market volatility, which is usually represented by the variance. Particularly, since we are analyzing the performance in European stock markets, we use the EuroStoxx-50 index as a proxy of the market. The proposed formula is as follows:

\[ E(R_m) - R_f = \alpha \sigma_m^2 \]

\[ \alpha(CRRA) = \frac{E(R_m) - R_f}{\sigma_m^2} \]  

(12)

3. A CONDITIONAL APPROACH TO BUILD TIME-VARYING PORTFOLIOS

First, we must remember that one of the main aims of this paper is the time-modeling of probability distribution moments, to make our optimal portfolios change over time. To obtain the above, and focusing on the optimal portfolio problem, equation (9), we propose the application of conditional variance and correlation schemes, such as GARCH (1, 1) and DCC-GARCH, to model the conditional moments included in the mentioned formula.

Otherwise, according to the previous section, the CRRA allows us to obtain the risk aversion attitude of an investor in a single number. However, in this research, we are more interested in the time-varying risk aversion, not in a constant parameter. Thus, we will model the market mean and variance through conditional models, such as GARCH-M (1, 1) and GARCH (1, 1) specifications. Once we have achieved the above, we obtain a number of risk aversion
parameters that fluctuate over time, due to the movements in the conditional market mean and volatility.

Further, we aim to assess whether it is better to work with a constant or changing risk aversion parameter. Thus, the idea is to build optimal portfolios for different types of investment profiles, the conditional ones associated with the CRRA, and other approaches based on constant risk aversion.

### 3.1 Optimal portfolio construction. Modeling variances and correlations

This section describes how to obtain the optimal portfolio weights, given a set of assets. We assume that the expected return of our portfolio follows a normal distribution function with a constant mean and a time-varying variance:

\[
E_{o,t} \sim N(E, V_t)
\]  

where \(E_{o,t}\) is the expected return of the optimal portfolio at a given moment of time, \(N\) is the multivariate normal distribution function, \(E\) is the fixed expected return vector and \(V_t\) is the dynamic covariance matrix, calculated by modeling conditional correlations and volatilities.

Focusing our study on the construction of dynamic portfolios, to obtain the optimal conditional weights, we must adapt equation (9) to a time-varying context:

\[
W_{o,t} = \frac{V_t^{-1}1_n}{1_n'V_t^{-1}1_n} + \frac{1}{\alpha} \left( V_t^{-1}E - \frac{1_n'V_t^{-1}E}{1_n'V_t^{-1}1_n}1_n'V_t^{-1}1_n \right)
\]

\[
W_o = W_{mv,t} + \frac{1}{\alpha} \left( V_t^{-1}E - \frac{1_n'V_t^{-1}E}{1_n'V_t^{-1}1_n}1_n'V_t^{-1}1_n \right)
\]  

(14)

where \(V^{-1}\) is the 10x10 inverse dynamic covariance matrix, \(1_n\) is a 10x1 ones vector, \(E\) is the 1x10 fixed expected return vector and \(\alpha\) is the time-varying individual level of risk aversion. Note that the expected return vector and the covariance matrix are expressed in annual terms.

Then, once we known the amounts invested in each of the selected assets, we can calculate the expected return and the volatility of the optimal portfolio, as follows:
\[ E_{o,t} = W'_{o,t}R_t \]
\[ \sigma_{o,t} = \sqrt{W'_{o,t}V_tW_{o,t}} \]  

where \( W_{o,t} \) is the vector which contains the dynamic weights invested in each of the studied equities, \( R_t \) is a vector composed by the assets' returns for each month of the market and \( V_t \) is the conditional covariance matrix.

Moreover, we need to determine the previous calculus of the time-varying covariance matrix. As described in section 2.4, “The optimal portfolio construction. An extension of the CARA function”, this matrix is given by the next expression:

\[ V_t = \begin{pmatrix} \sigma_{1t}^2 & \cdots & \sigma_{1,10t} \\ \vdots & \ddots & \vdots \\ \sigma_{10,1t} & \cdots & \sigma_{10,t}^2 \end{pmatrix} \]  

where the main diagonal elements are the variances of each of the selected assets, and the rest of the elements are the covariance between these equities. Note that, despite having the assets’ returns at monthly frequency, we must annualize them by multiplying the covariance matrix by \( \sqrt{12} \), to determine the optimal portfolio weights. In addition, we must multiply the expected return vector by 12.

Given that step, the question that arises in this context is the following: How can we make these probability distribution moments changing over time? The answer is by applying conditional variances and correlation moments. In particular, in the context of the optimal portfolio’s construction, we model these moments through the GARCH (1, 1) specification for the case of the variance, and DCC-GARCH for the correlation terms\(^5\).

Once we have obtained the conditional correlations, the conditional covariance matrix is obtained from the next expression:

\(^5\) These and other models are explained more in detail in section 2.5, “Methodology. Conditional probability distribution moments”. \( \)
\[ V_t = D_t \Gamma_t D_t \]

\[ V_t = \begin{pmatrix} \sigma_{1,t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{10,t} \end{pmatrix} \begin{pmatrix} 1 & \cdots & \rho_{1,10t} \\ \vdots & \ddots & \vdots \\ \rho_{10,1t} & \cdots & 1 \end{pmatrix} \begin{pmatrix} \sigma_{1,t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{10,t} \end{pmatrix} \]

(17)

where \( D_t \) is the conditional deviation matrix, in which the main diagonal is composed of the conditional deviations obtained from the GARCH (1, 1) model, and the rest of the matrix is composed of zeros. On the other hand, \( \Gamma_t \) is the dynamic correlation matrix obtained through the application of DCC-GARCH schemes. This matrix has its main diagonal composed by ones, and conditional correlations out of the mentioned diagonal.

3.2 Model A. Dynamic risk aversion and the market risk premium

As discussed, the risk aversion attitude is a key input in the estimation of optimal portfolios based on expected-utility maximization. However, in the equity literature, there are few studies that have explicitly calculated the risk aversion parameter. For that reason, we follow the approach revised by Giovanni and Jorion (1989), which is based on estimates of the observed risk aversion through a derivation of the CRRA, and applies it to generate utility maximizing based on the unleaded gasoline market. Specifically, as described in the last section, the derivation of the CRRA is based on the market risk premium. In this case, we use the mentioned estimation of the market risk premium, but for the EuroStoxx-50 data\(^6\). In addition, we adjust equation (17) to our purpose, trying to avoid negative values of the parameter when the market yields less than the risk-free asset. For that reason, our proposal is based on fixing the numerator of the mentioned formula as the maximum between the excess market return at each month of the market and its mean for the whole data period. The formula is as follows:

\[
\alpha = \frac{\text{Max}(E(R_{m,t}) - R_{f,t}, h)}{\sigma_{m,t}^2} \\
\text{where, } h = \frac{\sum_{t=1}^{n}(E(R_{m,t}) - R_{f,t})}{n}
\]

(18)

\(^6\) We use daily closing prices and we transform them into returns by applying logarithms, following the same procedure described in section 2.1.
Regarding the formula terms, $E(R_m) - R_f$ represents the market risk premium (the excess return on the market), $\alpha$ is the coefficient of relative risk aversion (CRRA) and $\sigma_m^2$ is the market variance. In particular, we use the daily closing prices of the EuroStoxx-50 index as the market portfolio, and 3-month German Treasury Bills at daily frequency as the risk-free rate. Then, we ascertain the returns in monthly.

In this case, we use the GARCH in mean schemes to estimate the $\alpha$ parameter. We have chosen these types of models because of their good statistical properties, which are well-known for modeling the mean and the variance simultaneously. In particular, we estimate the risk parameter through the GARCH-M $(1, 1)$ specification.

### 3.3 Model B. Constant risk aversion. An application of the Sharpe ratio

Focusing on the case of constant risk aversion, we must set a criterion for choosing an appropriate parameter according to the risk aversion attitude in Europe. In particular, our proposal is based on choosing several values according to each of the mentioned studies (section 2.3.3.). The chosen values are $1$ (Arrow), $11$ (Mehra and Prescott), $1.8$ (Ghysels et al) and $4.93$ (Guo and Whitelaw).

Once we have selected these values, we keep the ones that make the optimal portfolio have better performance each month according to the Sharpe ratio. We ascertain this ratio monthly, to avoid the noise frequency of this type of data, and we use the 3-month German Treasury Bills as the risk-free rate.

Then, we calculate the average of the optimal parameters obtained at each month in the market to reach a single risk parameter. Finally, the average parameter is found to be $8.51$, and it is used to model the optimal portfolios of the market at each time.

### 4. METHODOLOGY. CONDITIONAL PROBABILITY DISTRIBUTION MOMENTS

In this section, we show how to implement the conditional distribution models. According to the formulas, we find that the modelization of the current variance or correlation is expressed as a function that depends on the residuals of the previous period, among other
parameters. However, we aim to make it clear that we do not apply the autocorrelation adjustment in this research. The above means that we work with returns, despite having the whole models expressed in terms of residuals.

We estimate the model parameters through the application of the maximum log likelihood. Note that you can review the application of this technique for univariate and multivariate models in Appendix B.

4.1 Conditional variance/mean models

Knowing the volatility is important in financial markets. Investors are interested in the volatility of stock prices, as high volatility can mean huge losses or potential profits, and can consequently lead to greater uncertainty. Given the above, the question is, how could we model the volatility of the time series?

Regarding levels, a characteristic of most time series is that they are random walkers, i.e., they are not stationary. Moreover, in the form of first differences, they are usually stationary. As a result, the models are built with first differences. However, these differences often show wide variations, or "volatility", which makes us think that the variance of time series changes over time. In these cases, it is very useful to use the “Autoregressive conditional heteroscedasticity model” (ARCH), developed by Engle (1982). In this model, the unequal variance, may have an autoregressive structure, in which we observed that heteroscedasticity over different periods could be autocorrelated.

Since its discovery in 1982, the development of ARCH models have become a thriving area, with all types of variations from the original model. One of the most popular is the Generalized Autoregressive Conditional Heteroscedasticity model, proposed by Bollerslev (1986). The simplest version of the GARCH model is the GARCH (1, 1), which is the one we use in this paper to model the conditional volatility over time. In addition, we model the volatility through other conditional specifications, such as the GARCH-M models. Note that the application of the last one is important in this research, because it allows us to model the variance and the mean simultaneously.

4.1.1 GARCH (1, 1) specification

This model states that the current conditional variance depends not only on squared return of the previous period (as in ARCH (1)) but also on its conditional variance of the previous period. In fact, the GARCH (1, 1) model is much more similar to an ARCH (2).
Thus, the GARCH (1, 1) model is represented as

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

(18)

With the following constraints:

$$\alpha + \beta < 1$$
$$\omega > 0$$
$$\alpha \geq 0$$
$$\beta \geq 0$$

(19)

For regression purposes, it can be rewritten as

$$R_t = \gamma + \epsilon_t$$
$$\epsilon_t = \sigma_t \eta_t$$
$$\eta_t \sim N(0, 1)$$

(20)

Where $R_t$ denotes the assets’ return, $\epsilon_t$ is the residual and $\sigma_t^2$ is the assets’ variance. Moreover, following this scheme, the long-term variance can be calculated as follows:

$$\sigma^2 = \omega/(1 - \alpha - \beta)$$

(21)

### 4.1.2 GARCH-M (1,1) specification

One of the most important statements of financial theory is the relationship between risk and return. The CAPM model, for instance, implies a linear relationship between the expected return of the market portfolio and its variance. If the variance is not constant over time, then the conditional expected return of the market is a linear function of the conditional variance. Engle (1987) proposed the estimation of conditional variances by GARCH schemes and then that these estimations will be used in the conditional means' estimation. This is known as the GARCH-in-Mean (GARCH-M) model.
The GARCH-M scheme models the mean by making it dependent on the variance. In addition, the variance is modeled by a GARCH (1, 1) scheme, so we must estimate the conditional mean and variance of the process simultaneously.

The variance is modeled according to a GARCH (1, 1) scheme, as we describe before. For the mean, it will look similar to this:

$$\mu_t = \delta + \lambda \sigma_t^2$$

(22)

Where \(\delta\) is a constant and \(\lambda\) is a parameter to be estimated.

The GARCH regression model, adding an extra regressor as the standard deviation, for this scheme is

$$R_t = \delta + \lambda \sigma_t + \varepsilon_t$$

$$\varepsilon_t = \sigma_t \eta_t \quad \eta_t \sim N(0, 1)$$

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

(23)

where \(R_t\) denotes the Eurostoxx-50 return, \(\varepsilon_t\) is the residual, \(\sigma_t^2\) is the Eurostoxx-50 variance and \(\lambda\) is the CRRA. Further, in this model the \(\lambda \sigma_t\) term could be interpreted as the risk premium.

4.2 Conditional correlation models
4.2.1 DCC GARCH specification

The dynamic conditional correlations GARCH model is defined as:

$$q_{ij,t+1} = \omega + \alpha (\eta_{i,t} \eta_{j,t}) + \beta q_{ij,t}$$

(24)

With constraints:
\[ \bar{\omega} = (1 - \alpha - \beta)\rho_{ij} \]
\[ \alpha > 0 \]
\[ \beta > 0 \]  

(25)

where \( \eta_{i,t} \) and \( \eta_{j,t} \) are the standardized returns of the chosen assets, obtained from the GARCH (1, 1) model.

In addition, to normalize the conditional correlation, we use the following expression:

\[ \rho_{ij,t+1} = \frac{q_{ij,t+1}}{\sqrt{q_{ii,t+1}}\sqrt{q_{jj,t+1}}} \]

(26)

Furthermore, to initialize the calculus of the correlation coefficient by the DCC GARCH model, we must impose two initial conditions:

\[ q_{ij,1} = \frac{\sum_{t=1}^{T} \eta_{i,t}\eta_{j,t}}{T} \]
\[ q_{ii,1} = q_{jj,1} = 1 \]

(27)

5. DATA ANALYSIS AND PORTFOLIO CONSTRUCTION

5.1 The data sample

As seen in the market, the investment outlook is changing. Although a few years ago investors preferred fixed incomes, today it seems to be changing, and the focus is radically moving towards a trend of investment in equities, which grows daily. However, why is this change occurring in the markets? This is occurring because public debt offers very low interest rates, which causes investors to prefer to take the risks of other instruments, such as equities, due to the expectation of higher returns in the future. Given the above, it seems obvious to think that rational investors aim to focus their portfolios on equities, and not on
debt. On this basis, in this section, we select the optimum ratio of equities to be included in a portfolio. In particular, for reasons of financial diversification, we build our portfolios with ten risky stocks traded on the EuroStoxx-50 Index, because it is the most representative indicator in the European stock markets.

As we mentioned in the last section, the main purpose of the research is to generate a different kind of time-varying portfolio for two sample periods. The first one is a calm period, which runs from 01/01/2004 to 31/12/2007, and the second timeframe is a more stressed one, from 01/09/2008 to 31/08/2012, to consider how the portfolios have changed during the crisis period. To show the empirical results of this research, the daily closing prices of some European equities and indicators are used as the sample data. Then, we analyze them in greater detail.

We use a set of several assets selected from the EuroStoxx-50 Index, at daily frequency, to ascertain the different terms and parameters of the conditional portfolio weights. Regarding to the composition of the optimal portfolios, we assess them from Table 1. The chosen assets meet with two important requirements proposed by Tapon and Vitali (2013) to ensure a well-diversified portfolio. These requirements are “number of assets” (ten equities) and “different sources” (they come from very different sectors or branches of business).

On the other hand, we also use daily frequency data to estimate the time-varying risk aversion parameter as a derivation of the market risk premium. The conditional portfolio model is constructed applying of the EuroStoxx-50 index at daily frequency as a proxy of the time-varying risk aversion attitude.

We analyze these portfolios over two different periods. The calm period, which runs from 01/01/2004 to 31/12/2007, has 1043 work days, and the second timeframe (stress period) includes data from 01/09/2008 to 31/08/2012 (1045 work days). We obtained these data from the DataStream database.

5.2 Key elements of financial time series

Regarding to top left graph of Figure 1, we observe that stock price changes over time, in other words, stock price returns are being modeled. There are two ways to convert prices

---

8 We review the diversification problem in Evans and Archer (1968) and Tapon and Vitali (2013). These studies show that 8, 9 and 10 stocks are sufficient to reach a well-diversified portfolio.
into returns, they can either be converted to periodically compound returns or to continuously compound returns (See equation 28). Note that in this paper, we use the second option.

\[ R_t = \ln \left( \frac{S_t}{S_{t-1}} \right) \]  

(28)

where \( S_t \) is the stock price at time \( t \), \( S_{t-1} \) is the previously traded stock price in the market and \( R_t \) is the stock return. Note that in Figure 1, we calculate the returns at a daily frequency, but for the rest of this paper, we ascertain them by month, i.e., by taking log-differences in traded stock prices every 22 business days.

[INSERT FIGURE 1 HERE]

In the first two graphs (located at the top of Figure 1), the ENI stocks move similar to a random walk, with a stochastic trend or unit root. It would also be unsteady. For the mean, as in most financial returns, it is constant over time. Moreover, regarding the variance, the series shows random variability, because we can observe groupings or "volatility clusters". This term refers to those moments in time when there is high volatility, which tend to be followed by periods of high volatility. Otherwise, low volatility moments are followed by a succession of low volatility periods. To collect these groupings of volatility, we must use conditional heteroscedasticity models (GARCH, exponential smoothing, etc.).

Then, from the kernel graph, we can observe that ENI returns do not follow a normal distribution, since to the curve described by them is sharper than the Gaussian one, i.e., ENI yields are leptokurtic. This is because its kurtosis is higher than normal (kurtosis = 3.98 > 3). We could analyze the abnormality of ENI in another way, paying more attention to the distribution tails, since ENI tails are wider than normal ones.

5.1.1 Jarque-Bera normality test

The Jarque-Bera test is a goodness of fit test that examines whether a data sample has the skewness and kurtosis of a normal distribution. The above measure is a parametric test, which is defined as:

\[ JB = \frac{n - k + 1}{6} \left( S^2 + \frac{1}{4} (K - 3)^2 \right) \]  

(29)
where $S$ represents the sample skewness and $K$ is the sample Kurtosis of the time series. Otherwise, $n$ are the degrees of freedom and $k$ is the number of regressors.

The Jarque-Bera statistic is asymptotically distributed as chi-squared distribution, $\chi^2$, with two degrees of freedom. In this case, we use it to test the null hypothesis that ENI yields follow a normal distribution. The null hypothesis is a joint hypothesis that skewness and kurtosis are nil.

Then, the JB-statistic and the P-value are used to analyze whether we reject the null hypothesis or not. The p-value is the probability of observing statistical evidence equal to or more extreme than the observed value under the null hypothesis (this hypothesis is assumed true at the beginning).

Given the above, we reject the null hypothesis at a significance level of 95%, since the p-value (1.40E-13) is less than $\alpha$ (0.05). Thus, it is not possible to affirm that ENI returns follow a normal distribution.

$$H_0 = \text{Normality in returns}$$

(30)

Linking to the above, we must mention that despite knowing that the assets’ returns have the problem of heavy tails, we continue this research with the assumption of normality in returns.

5.1.2 Simple and partial autocorrelation functions (FAS and FAP)

The analysis of persistence in performance is an interesting line of research because of the controversy over whether this phenomenon occurs, and if so, whether that persistence exists only in short-term time periods or in longer time horizons. Moreover, the existence of this phenomenon can be considered a useful information tool for participants in financial markets when making their investment decisions.

Correlation refers to the persistence in returns or correlation between current and past values of the studied variable. The autocorrelation of a stochastic process is measured by simple and partial autocorrelation functions. The $\rho$ parameter is generally known as the persistence of the process. An increase or decrease in the actual return of assets takes effect in their own future performance, although the influence of the current value gradually grows over time, in accordance with the decrease of the $\rho$ coefficient. A $\rho$ value closes to 1 indicates high
persistence in the process, and conversely, a value of $\rho$ close to 0 does not indicate persistence in the process.

\[ \text{INSERT FIGURE 2 HERE} \]

Figure 2 shown both simple and partial autocorrelation functions, they appear to be among the bands in the case of the studied equity, suggesting the absence of autocorrelation or autocorrelation that is weak. On this basis, we can conclude that a clear persistence in returns is not realized, although statistical tests we be used to confirm it.

Moreover, regarding the serial correlation of squared returns, we find that ENI correlations are out of bounds, which indicates heteroscedasticity or autocorrelation in second moments, i.e., it is changing over time. Therefore, it seems advisable to model volatilities and correlations over time by the applying several GARCH schemes.

However, the autocorrelation functions are merely qualitative tools for analyzing the presence of autocorrelation in yield lags. Therefore, to evaluate the combined autocorrelation of some lags in a more quantitative way, we will use the Box-Pierce statistical test.

### 5.1.3 Box-Pierce statistical test

Box and Pierce (1970) developed a statistic based on the squares of the first autocorrelation coefficients of residual yields, to analyze whether there is autocorrelation. The statistic is defined as a cumulative sum of squares of the correlation coefficients, that is,

\[ Q_p = n \sum_{j=1}^{p} \hat{\rho}_j^2 \]  

where $\hat{\rho}_j = \frac{\sum_{t=j+1}^{n} \epsilon_t \epsilon_{t-j}}{\sum_{t=1}^{n} \epsilon_t^2}$ and $\epsilon_t$ is the residual.

Under the null hypothesis of no autocorrelation, the $Q$ statistic is asymptotically distributed as a chi-squared, $\chi^2$, with degrees of freedom equal to the difference between the accumulated number of coefficients ($p$) and the number of parameters estimated by adjusting the considered process.

Note that, as in the JB-test, we analyze this test according to the associated p-value. In addition, we must mention that we have used 5 lags in the calculus of this test.
Thus, for a significance level of 95%, we could affirm that there is no autocorrelation between returns, since the p-value (0.8642) is greater than 0.05 for both studied cases, we accept the null hypothesis in such cases.

\[ H_0 = \text{No Autocorrelation} \]  

(32)

Following the above, this equity has no persistence in returns. Given that, this equity follows a white noise structure.

We must mention that despite knowing that there are a few cases in which a clear persistence in returns is found (see Appendix. A), we will continue this research without adjusting the equities’ returns by autocorrelation.

6. EMPIRICAL EVIDENCE. THE USE OF TIME-VARYING OPTIMAL PORTFOLIOS

In this section, we provide an overview of the main results and findings obtained by the application of the different studied models. In particular, we analyze the estimated parameters, the monthly evolution of the different portfolio’s weights and the conditional evolution of the risk aversion parameter over the two selected periods (calm and stress).

6.1 Parameter estimation

First, we want to make it clear that in this section, we only analyze in detail the estimated parameters for each one of the risk aversion models described in section 3., while the estimated parameters for the construction of conditional optimal portfolios (GARCH and DCC GARCH schemes), are shown in Appendix C.

Then, we show the main results and parameters obtained for Model A. In addition, we must remember that the estimations are based on the EuroStoxx-50 returns at monthly frequency. Note that, in this case, we exclude Model B because we do not have to estimate any parameters in these schemes. Since we have assigned a constant risk aversion parameter (8.51) obtained as a derivation of the Sharpe Ratio, so we do not have to show any results.

[INSERT TABLE 2 HERE]
From Table 2, we find the $\lambda$ parameter, showing the effect of the variance in the mean model. In this model and for both studied periods, the previous variance has a low impact on the current variance, while most of the effects come from the returns of the previous period. Even though an extra parameter was added in the equation of the mean, it has no impact on the variance in the mean model, i.e., it is not significantly different from 0.

[INSERT FIGURE 3 HERE]

According to Figure 3, the risk aversion parameter is greater, on average, in the stress period, which makes sense, since the common investor is more risk averse when the market is having a negative trend (bearish market). In addition, according to this model, we can assess that the risk-aversion time series presents a random variability, so we can observe groupings or “clusters”.

[INSERT TABLE 3 HERE]

Moreover, from the previous table (Table 3), we find that the maximum values are greater, and the minimum values are lower during the stress period (the values range between 0.5 and 10). In addition, the above is also reflected in the conditional parameters’ standard deviation, since it is higher in the second case.

6.2 Monthly evolution of the optimal portfolio weights

Then, we determined the optimal portfolio weights of each of the mentioned models (A, B), i.e., those weights that maximize the investor’s expected utility. As in the last section, we analyze the portfolio evolution for the two selected periods. Moreover, we only assess in detail the evolution of the constant risk aversion attitude, Model B. The above is because the conditional evolution trend is similar for all the studied cases.

[INSERT FIGURE 4 HERE]

According to Figure 4, we can assess the evolution of the optimal weights that an investor has to assign to each of the previously selected equities to maximize his expected utility function at each time of the market. In particular, top graph analyze the evolution of the optimal portfolio in the case of the calm period. On the other hand, bottom graph shows the evolution of the portfolio over the stress period. From here, we find that the assets’ trend has several differences regarding the calm period analysis. First, the optimal weights are higher, on average. In addition, we can observe that the weights distribution is slightly different.

[INSERT TABLE 4 HERE]
Then, we show a summary table of the analyzed weights according to each of the mentioned models and for the two studied periods. Following Table 4, which shows the average of the conditional portfolio weights expressed as a percentage, we can evaluate more in detail how each of the assessed portfolios behaves over the selected period. Thus, we can observe the conditional evolution of IBERDROLA (23.80%) and ESSILOR (22.90%), which are the firms that have captured the greatest weights on average over the considered period. Moreover, the lowest weights have been assigned to LVMH (-0.40%) and SAFRAN (-0.77%), which are the only two companies for which we introduce the short selling strategy.

Otherwise, Table 5 shows the weights of the 10 firms expressed as a percentage. In this case, we must highlight the conditional evolution of ESSILOR (47.28%), because this firm captured the highest average weights over the considered period, standing well above the rest of companies. Moreover, the lowest weights have been assigned to INTESA (-14.83%), LVMH (-5.62%) and BASF (-3.94%) which are the base of the short selling strategy.

7. ANALYSIS OF PORTFOLIO MANAGEMENT. SOME PERFORMANCE RATIOS

In this section, we show a number of ratios and risk measures to analyze the performance of our portfolio. In addition, we assess the exposures of the different studied funds. Note that we spend the majority of this section on the analysis of the Sharpe Ratio and, as a consequence, it is the one that we explain in greater detail.

7.1 Unconditional distribution moments. Individual assets versus composed portfolios

From Table 6, we can observe that the means and standard deviations of individual assets and composed portfolios are expressed in annual terms, because this is the preferred approach in finance. In addition, regarding the assets’ moments, we need to annualize them because this is a requirement to ascertain the optimal portfolio weights. Otherwise, we must make it
clear that skewness and kurtosis of the proposed portfolios are assumed to be 0 and 3, respectively, as we have assumed normality when calculating such portfolios\(^9\).

Then, regarding the calm period, we can assess that the firm, SAFRAN, has a negative return, on average, and a rather high standard deviation. However, with portfolio construction, through the diversification effect, we can greatly reduce this deviation, as seen in Table 6. Therefore, we have finally achieved a well-diversified portfolio for each of the studied models. The above means that we have obtained some portfolios with fairly good results, on average (Mean=17\%, Stdv=10\%).

Moreover, analyzing the stress period in more detail, we can observe that in general terms, the set of assets has a lower mean and a greater deviation, regarding the calm period. In fact, as seen at the bottom of Table 6, we find that four firms have a negative performance in terms of returns (INTESA, IBERDROLA, ENI and DEUSTCHE POST). In addition, as in the last period, we have finally reached a well-diversified portfolio for each of the studied models. However, due to the above reasons, the results of the stress portfolios are worse, on average (Mean=5\%, Stdv=16\%).

7.2 Sharpe Ratio

7.2.1 How to ascertain the ratio

The Sharpe ratio measures the excess return per unit of risk\(^{10}\). This ratio allows the different investment options to be prioritized based on return and risk. In addition, the Sharpe ratio should only be used when normality is assumed, since the standard deviation of the portfolio only makes sense if we have a stable probability distribution over the sample period. The higher the value of this ratio is, the better the performance of our portfolio, i.e., it indicates that we are receiving higher returns relative to their associated risk.

We can ascertain it as follows:

\[
SR = \frac{E(R_p) - R_f}{\sigma(R_p)} \tag{33}
\]

\(^9\) Although we know that the probability distribution of assets' returns has the problem of heavy tails (leptokurtosis), we assumed normality in returns

\(^{10}\) To review this section, we rely on Sharpe (1994)
where $E(R_p)$ is the expected portfolio return, $R_f$ is the risk-free rate and $\sigma(R_p)$ is the volatility approximated as the standard deviation of the portfolio. For the allocation period of our portfolios, we show how this ratio evolves for each of the studied portfolio models. We calculate it at a monthly frequency and we use the 3-month German Treasury Bills as the risk-free asset return.

Despite calculating this ratio monthly, we expressed it in annual terms for all the mentioned portfolios. To carry out this process, we annualized the risk-free return because we have previously ascertained the expected return and the variance of each of the portfolios in annual terms. Further, as in the previous sections, we analyze this ratio evolution for the two selected periods.

### 7.2.2 Conditional evolution of the Sharpe Ratio

First, we begin with the study of the calm period. According to Figure 5, we can observe the evolution of the Sharpe Ratio over the calm timeframe for both studied models. Note that for clarity, we have divided the graph into three clearly-defined parts. The first part of the graph compares the selected models against the benchmark (market performance). The second part follows the same line as the first one, but in this case, we are plotting both models in excess of the market index. The key of the above, is to provide more detail regarding how well we are performing by actively managing our equities portfolios versus the passive management (be invested in the EuroStoxx-50 index over the timeframe). The bottom of the graph shows model A in excess of the naïve one (Model B), to visualize how well the dynamic models are performing versus to the constant one.

![Insert Figure 5 Here](image)

Following the previous graphs (Figure 5), first, we find that the performance of our portfolio over the calm period is acceptable for the analyzed models, as it ranges between -6 and 7, and the constant risk aversion model (Model B) performs the best on average. According to the above, as seen in the graphs, we are outperforming the benchmark (EuroStoxx-50) for much of the timeframe in both cases. Further, a greater frequency of negative Sharpe values is observed at the end of the period, i.e., when we are really close to the beginning of the Economic Crisis. Note that the negative values are not too high in this calm period. The above seems reasonable, because we are talking about a calm period in which the equity investment offered more attractive returns than fixed income, so it is logical to assume that the Sharpe Ratio is going to almost always be positive.
Then, we continue our study with analyzing the stress period. According to Figure 6, as in the last case, we can observe the evolution of the Sharpe Ratio over the stress period for the mentioned models. In this case, we follow the same structure as for the calm period. Given that, the first part of the graph compares the selected models against the benchmark (market performance). The second part follows the same line as the first one, but, in this case, we are plotting both models in excess of the market index. The bottom of the graph shows the dynamic model in excess of the naïve one (Model B), to visualize how well the dynamic models perform versus the constant one.

Otherwise, regarding the stress timeframe, our portfolios continue performing well in comparison to the benchmark (in fact, the EuroStoxx-50 index has a negative stress ratio on average). However, in this case, the Sharpe Ratio shows negative values with a higher frequency than those observed in the previous period, ranging between -8 and 7. According to the previous discussion, in those periods in which we observe negative ratio values, we can predict that investing our money in fixed income would be more profitable than keeping it invested in our portfolio models since negative Sharpe Ratio values are caused by high risk free rate values, i.e., the risk free asset (fixed income) offers a greater return regarding our portfolios.

Then, according to Table 7, we continue the analysis of the Sharpe Ratio, and assess the differences in performance terms between the different selected models. First, regarding the calm period, the fund which exhibits the highest ratio over this timeframe is Model B, although it is followed closely by Model A. However, regarding the stress period, the best fund in Sharpe terms is Model A. Thus, we can conclude that Model A is the best model according to the Sharpe Ratio, because this fund has the best performance, on average, for the two considered periods.

### 7.2.3 Hypothesis testing for the ratio average

In this case, we implement a mean-difference test for two independent samples, that is to say, we compare whether the differences between the averaged ratios of the dynamic and constant models are significant or not. The above is well-known as a parametric test.
We can decide whether we reject the null hypothesis or not in two different ways, i.e., the \(t\)-statistic and the associated \(p\)-value. Given that, we can determine the \(t\) statistic, which is defined as:

\[
t = \frac{\mu_1 - \mu_2}{\sigma_{1,2} \sqrt{2/n}}
\]

where \(\mu_1 - \mu_2\) represents the mean difference between the dynamic risk aversion model (Model A) and the constant risk aversion model (Model B). Otherwise, \(\sigma_{1,2}\) is the joint deviation and \(n\) represents the data size. Then, the null hypothesis of this parametric test is as follows:

\[
H_0: \mu_1 - \mu_2 = 0
\]

Then, we accept the null hypothesis at a significance level of 95% for both studied models and for the two mentioned periods. This is because the \(p\) value is not less than \(\alpha\) (0.05). Thus, it is not possible to affirm that dynamic models are better than static ones, i.e., there is not significant evidence.

Therefore, if we had to invest some money in a risky portfolio, we would choose the one associated with Model A. However, if we observe the results obtained in the last test, we find that there are few differences between the different selected models. In fact, if we prefer a less complex method (by calculating the conditional risk-aversion attitude through different mathematical equations), we can select the constant risk-aversion scheme (Model B). As we have mentioned before, this is because the differences between the best model (Model A, based on time-varying risk aversion) and the worst one (Model B, based on a constant risk attitude parameter) are not significant.

**7.3 Certainty Equivalent**

In addition, we discuss another measure to assess our portfolio management, the certainty equivalent. This analysis tool can be calculated as follows:
\[
CE = E(R_p) - \frac{1}{2} \alpha \sigma_{R_p}^2 + \frac{\tau(R_p)}{6} \alpha^2 \sigma_{R_p}^3 - \frac{k(R_p) - 3}{24} \alpha^3 \sigma_{R_p}^4
\]  

(36)

Since we are assuming normality in returns, with expected return \( E(R_p) \), variance \( \sigma_{R_p}^2 \), skewness \( \tau(R) = 0 \) and kurtosis \( k(R_p) = 3 \), we can obtain a new expression, as follows:

\[
CE = E(R_p) - \frac{1}{2} \alpha \sigma_{R_p}^2
\]  

(37)

This expression refers to the amount of money necessary for an investor to give up keeping his portfolio invested under uncertainty. Obviously, as in the case of the Sharpe Ratio, the higher the Certainty Equivalent is, the better the performance of our portfolio model. Note that we ascertain it in annual terms, as in the previous section.

[INSERT FIGURE 7 HERE]

First, we begin with the analysis of the calm period. From Figure 7, we find that both models show the same trend along the calm timeframe, but have many differences in terms of magnitudes. Thus, the best performing fund in this period is Model A. Note that in this period, the Certainty Equivalent is almost always positive.

Analyzing the stress period, we can observe that the trend is not the same as in the previous timeframe. In this period, regarding the Sharpe Ratio, the best performing fund is Model A again. Furthermore, the Certainty Equivalent is not always positive. In fact, for the case of Model B, it is almost always negative, which means that it is the worst model in this context.

In summary, according to the last figures, we can conclude that the highest risk premium offered, on average, to exchange our portfolio, is the one shown by Model A, since it is the one that usually offers us the greatest relationship over time\(^{11}\). According to the above, the portfolios with better performances based on the Certainty-Equivalent ratio are those associated with the time-varying risk aversion attitude, while those with a constant risk aversion parameter (Model B) have a negative risk-return relationship and a very unstable trend throughout the whole studied period.

\(^{11}\) In this context, the average means, considering the performance of the models over the whole period, i.e., the sum of the calm and the stress periods.
7.4 Lower and Upper Partial Moments Family

Kaplan and Knowles (2004) introduced the kappa indices which risk measure is estimated by using the Lower Partial Moments to evaluate the properties of returns probability distribution in the left tail. The kappa ratio of order \( m \) is defined as:

\[
K(h, m) = \frac{E(R) - h}{LPM_{m,h}(R)^{1/m}}
\]

Then, we assess some performance measures based on partial moments, such as the Kappa of order 1 and its associated Omega statistic. In addition, we analyze the Sortino Ratio (kappa of order 2). The three mentioned ratios are determined as follows:

\[
K(R_f, 1) = \frac{E(R) - R_f}{E[Max(R_f - R, 0)]}
\]

\[
\Omega(R_f, 1,1) = \frac{E[Max(R - R_f, 0)]}{E[Max(R_f - R, 0)]} = K(R_f, 1) + 1
\]

\[
K(R_f, 2) = \frac{E(R) - R_f}{E[Max(R_f - R, 0)^2]^{1/2}}
\]

Note that we have fixed the value of the threshold as the risk-free rate for all the studied ratios.

[INSERT TABLE 8 HERE]

As seen in Table 8, the results follow the same trend as in the previous performance measures. The highest average ratio is the one shown by Model A, since it is the one that almost always
provides the greatest relationship over time, i.e., the greatest ratio, on average, for the two considered periods. In addition, we can appreciate that those portfolios with the lowest associated risk, also have the lowest associated performance ratios.
CONCLUSIONS

When an individual decides to invest money in a risky portfolio, he always chooses an efficient one with a composition that depends on his subjective preferences. Analyzing the market more in depth, we find that investor preferences are heterogeneous, i.e., there are some individuals who prefer to take some risks, but there are others who are more cautious. However, it is assumed in the financial literature that investors are traditionally risk-averse individuals. Given that, at first, we planned to build a well-diversified portfolio from the set of assets listed in the EuroStoxx-50 index. Following the approach proposed by Tapon and Vitali (2013), we can obtain a well-diversified portfolio including ten stocks that belong to different sectors or branches of business. The key of this proposal is to compensate the adverse movements in some equities with the earnings obtained in others. It is well known as “diversification effect”.

Otherwise, an investor is more or less risk-averse according to the economic and political circumstances, i.e., their investment attitude depends on the market trend. As an example of the last statement, currently, even the most adventurous investor has had to reduce his optimistic expectations since we are in an economic recession period. Given that, in this study, we aim to find the optimal portfolio that best meets with the customer’s behavior, considering the variability of the market. Therefore, we introduced time-varying risk aversion models and compared them against the constant risk aversion models. In addition, we analyzed data in two different periods (calm and stress), because we aimed to observe whether is better to fix a single parameter to build the optimal portfolio over the whole period, or to make it change over the timeframe.

Given that, we have immersed ourselves in the theory of utility and choosing the optimal portfolio for risk averse individuals. We began with the study of the unconditional Markowitz approach to analyze how we can build the optimal portfolio in a constant context. After that, we have studied more in depth how this portfolio changes over time, through conditional schemes, such as GARCH (1, 1) and DCC-GARCH. Then, we spent the majority of this paper focusing our study in the modeling of the risk aversion parameter so that it changes over time.

Despite the influence of risk aversion in the optimal portfolio context, there are few studies that have explicitly estimated the risk aversion of an investor. Instead, they choose random values to reflect the common levels of risk aversion. However, common
sense tells us that the use of arbitrary values for this parameter could yield optimal portfolios that do not reflect the actual investor’s attitude towards risk. Given that, one of the key elements of this paper has been to propose the modeling of the risk aversion parameter to make it varying over time. To carry out this proposal, we focused our attention on the review of the quadratic utility functions framework, analyzing the changes in relative risk aversion as a derivation of the market risk premium (Cotter and Hanly, 2010). Thus, the key of this paper was to assess whether is better to work with a constant or a time-varying risk aversion parameter. Then, analyzing the performance results for the proposed models and for both studied periods, we have tested that in general, the models related to time-varying risk aversion showed a better performance on average. This is so, both from the point of view of the Sharpe Ratio and the Certainty Equivalent. Furthermore, more specifically, the best way to model the risk aversion parameter in performance terms is the one associated with Model A.

In addition, we implement a mean-difference test (parametric test) for two independent samples, that is to say, we compare whether the differences between the averaged ratios of the dynamic and constant models are significant or not. Then, if we had to invest some money in a risky portfolio, we would choose the one associated with Model A. However, according to the results obtained in the mentioned test, we find that there are few differences between the different selected models. In fact, if we prefer a less complex method (by calculating the conditional risk aversion attitude through different mathematical equations), we can select the constant risk aversion scheme. As we have mentioned before, this is because the differences between the best model (Model A, based on time-varying risk aversion) and the worst one (Model B, based on a constant risk attitude parameter) are not significant.

As a possible extension of this research, note that we have made several assumptions about assets and portfolios’ returns. On the one hand, we have worked with returns unadjusted by autocorrelation, so the obtained results may be biased. In this case, we could correct the results by autocorrelation. In addition, the above allows us to estimate the conditional volatility and correlation models using the residual instead of the assets’ returns. On the other hand, despite knowing that the probability distribution of assets' returns has the problem of heavy tails, we assumed normality in returns. As a solution, we could model the returns according to other distributions, such as the student’s t distribution.
REFERENCES


## APPENDIX.A: NORMALITY AND AUTOCORRELATION TESTS

<table>
<thead>
<tr>
<th>Firms</th>
<th>JARQUE BERA TEST</th>
<th>BOX PIERCE AUTOCORRELATION TEST</th>
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</tr>
<tr>
<td>ENI</td>
<td>59.19</td>
<td>1.4011E-13</td>
</tr>
</tbody>
</table>
APPENDIX.B: THE LOG LIKELIHOOD. UNIVARIATE AND MULTIVARIATE MODELS

The parameters have been estimated by the application of the maximum likelihood method. Further, we have assumed normality in returns. Thus, the procedure for obtaining the covariance matrices through the previous calculus of dynamic correlations requires consideration of univariate and multivariate models. Univariate models are used in modelling the volatility of each of the ten selected assets. Otherwise, Multivariate models are used to model the conditional correlations between the mentioned assets. Note, we have also used the univariate schemes to model the volatility of the Eurostoxx-50 index.

B.1. Univariate Normal Distribution

First, we consider that asset returns follow a univariate normal distribution. Given that, its associated density function is as follows:

\[ f(x/\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{1}{2} \frac{(x - \mu)^2}{\sigma^2} \right) \]

where the likelihood function is obtained as the multiplication of the density function form 1 to n:

\[ L(\mu, \sigma^2/x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{1}{2} \frac{(x_i - \mu)^2}{2\sigma^2} \right) \]

\[ L(\mu, \sigma^2/x_i) = 2\pi^{-\frac{n}{2}} \sigma^{-\frac{2n}{2}} \exp \left( -\frac{1}{2} \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{\sigma^2} \right) \]

Then, the log likelihood is defined as:

\[ \log L(\mu, \sigma^2/x_i) = -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2} \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{\sigma^2} \]

B.2. Multivariate Normal Distribution

Moreover, we assume multivariate probability distributions for estimating the conditional correlations models, i.e., it is assumed that asset returns are distributed by a given multivariate distribution function. Then, we show the distribution and likelihood functions:

\[ f(x_1 \ldots x_n/\mu, V) = \frac{1}{(2\pi)^{\frac{n}{2}} |V|^{\frac{1}{2}}} \exp \left( -\frac{1}{2} (x - \mu)'V^{-1}(x - \mu) \right) \]
where \( x = [x_1 \ldots x_n] \) are the standardized returns and \( V^{-1} \) is the 10x10 inverse covariance matrix.

The likelihood function is defined as:

\[
L(\mu, V|x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{(2\pi)^n|V|}} \exp\left(-\frac{1}{2} (x_i - \mu)'V^{-1}(x_i - \mu)\right)
\]

\[
L(\mu, V|x_i) = 2\pi^{-\frac{n}{2}} |V|^{-\frac{2n}{2}} \exp\left(-\frac{1}{2} (x_i - \mu)'V^{-1}(x_i - \mu)\right)
\]

Then, we show the log likelihood:

\[
\log L(\mu, V|x_i) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log|V| - \frac{1}{2} \sum_{i=1}^{n} (x_i - \mu)'V^{-1}(x_i - \mu)
\]
### APPENDIX.C: ESTIMATED PARAMETERS

#### C.1 Conditional volatility model/ GARCH (1, 1)

<table>
<thead>
<tr>
<th></th>
<th>Calm period</th>
<th>Stress period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\omega$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>LVMH</td>
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<td>0.09914</td>
</tr>
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<td>AIRBUS</td>
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<td>0.17594</td>
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<tr>
<td>INTESA</td>
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<td>0.05455</td>
</tr>
<tr>
<td>IBER.</td>
<td>0.00000</td>
<td>0.19523</td>
</tr>
<tr>
<td>SAFRAN</td>
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<tr>
<td>UNIBAIL</td>
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</tr>
<tr>
<td>ESSILOR</td>
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</tr>
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<td>D.POST</td>
<td>0.00002</td>
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<td>ENI</td>
<td>0.00001</td>
<td>0.10002</td>
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<tr>
<td>BASF</td>
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<td>0.06768</td>
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</tbody>
</table>

#### C.2 Conditional correlations model/ DCC-GARCH

<table>
<thead>
<tr>
<th></th>
<th>Calm period</th>
<th>Stress period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\omega$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>LVMH</td>
<td>0.00001</td>
<td>0.01068</td>
</tr>
</tbody>
</table>
**FIGURES**

Figure 1. Financial time series analysis. ENI

Source: Compiled by the author based on ENI stock prices. Top graphs show the stock prices and returns of ENI firm at a daily frequency. Bottom graphs compare ENI returns against the normal distribution function. We plot the kernel graph.

Figure 2. Simple and partial autocorrelation functions

Source: Compiled by the author. To ascertain these functions, we have used 20 lags in Matlab. Top graphs show the simple and partial autocorrelation function of asset returns, while the graphs located at the bottom represent the same functions, but for squared returns. The possible existence of serial correlation and heteroscedasticity can be observed.
Source: Compiled by the author based on the market risk premium ascertained at monthly frequency. We use the GARCH-M schemes to model the conditional mean and variance simultaneously. We use the EuroStoxx-50 data, available from the DataStream database. Top graph describes the monthly evolution of the risk aversion parameter over the calm period. Bottom graph shows the evolution of the risk parameter over the stress period.
Figure 4. Monthly evolution of the optimal portfolio. Model B

Source: Compiled by the author. We use the EuroStoxx-50 data (ten equities traded on the index), available from the DataStream database. Top graph describes the monthly evolution of optimal portfolio weights over the calm period. Bottom graph shows the evolution of the optimal portfolio weights over the stress period. We use GARCH and DCC GARCH schemes to estimate the conditional parameters of the optimal portfolio problem.
Figure 5. The Sharpe Ratio. Calm period review

Source: Compiled by the author. Expressed in annual terms. Whole graphs show the evolution of the Sharpe Ratio over the calm timeframe for both studied models. Note that for clarity, we have divided the graph into three clearly-defined parts. The first part of the graph compares the selected models against the benchmark (market performance). The second part follows the same line as the first one, but in this case, we are plotting both models in excess of the market index. The key of the above, is to provide more detail regarding how well we are performing by actively managing our equities portfolios versus the passive management (be invested in the EuroStoxx-50 index over the timeframe). The bottom of the graph shows model A in excess of the naïve one (Model B), to visualize how well the dynamic models are performing versus to the constant one.
Figure 6. The Sharpe Ratio. Stress period review

Source: Compiled by the author. Expressed in annual terms. Whole graphs show the evolution of the Sharpe Ratio over the stress timeframe for both studied models. Note that for clarity, we have divided the graph into three clearly-defined parts. The first part of the graph compares the selected models against the benchmark (market performance). The second part follows the same line as the first one, but in this case, we are plotting both models in excess of the market index. The key of the above, is to provide more detail regarding how well we are performing by actively managing our equities portfolios versus the passive management (be invested in the EuroStoxx-50 index over the timeframe). The bottom of the graph shows model A in excess of the naïve one (Model B), to visualize how well the dynamic models are performing versus to the constant one.
Figure 7. The Certainty Equivalent. Calm period analysis.

Source: Compiled by the author. Expressed in annual terms. Both graphs show the evolution of the Certainty Equivalent over timeframe for both studied models (dynamic and constant risk aversion). Note that for clarity, we have divided the graph in two clearly-defined parts. The first part of the graph compares the selected models over the calm period. The second part follows the same line as the first one, but for the case of the stressed timeframe.
Table 1. Optimal set of assets

<table>
<thead>
<tr>
<th>FIRMS</th>
<th>INDUSTRY</th>
</tr>
</thead>
<tbody>
<tr>
<td>SAFRAN</td>
<td>Industrial Goods</td>
</tr>
<tr>
<td>UNIBAIL-RODAMCO</td>
<td>Real Estate</td>
</tr>
<tr>
<td>AIRBUS GROUP</td>
<td>Industrial Services</td>
</tr>
<tr>
<td>LVMH</td>
<td>Diversified</td>
</tr>
<tr>
<td>ESSILOR INTL.</td>
<td>Medical Equipment</td>
</tr>
<tr>
<td>DEUTSCHE POST</td>
<td>Industrial Goods</td>
</tr>
<tr>
<td>INTESA SANPAOLO</td>
<td>Banks</td>
</tr>
<tr>
<td>IBERDROLA</td>
<td>Utilities</td>
</tr>
<tr>
<td>BASF</td>
<td>Chemicals</td>
</tr>
<tr>
<td>ENI</td>
<td>Oil and Gas</td>
</tr>
</tbody>
</table>

Source: Compiled by the author based on the EuroStoxx-50 Index. This index adds the 50 largest companies by market capitalization. It includes companies from Austria, Belgium, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, Netherlands, Portugal and Spain. This index is calculated by weighting the floating capital of each of the 50 mentioned companies. That is, not all companies have the same weight, but their representation is based on their capitalization.

Table 2. Estimated parameters. Model A

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>MLE</th>
<th>Parameter</th>
<th>Value</th>
<th>MLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega )</td>
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<td>3019.5999</td>
<td>( \omega )</td>
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<td>2414.1845</td>
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<td>0.7266</td>
<td></td>
<td>( \alpha )</td>
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</tr>
<tr>
<td>( \beta )</td>
<td>0.2724</td>
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<td>( \beta )</td>
<td>0.0755</td>
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<tr>
<td>( \lambda )</td>
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<td></td>
<td>( \lambda )</td>
<td>0.9000</td>
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<tr>
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<td></td>
<td>( \delta )</td>
<td>0.000001</td>
<td></td>
</tr>
</tbody>
</table>

Source: Compiled by the author. We show the estimated parameters for the Garch-M. In addition, we show the maximum likelihood at the optimum. We show the parameters and the maximum likelihood method for both studies periods (calm and stress).
Table 3. Key elements of Model A

<table>
<thead>
<tr>
<th>Time frame</th>
<th>Mean</th>
<th>St. deviation</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calm period</td>
<td>1,9328</td>
<td>1,1420</td>
<td>0,8116</td>
<td>6,8949</td>
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<tr>
<td>Stress period</td>
<td>2,4284</td>
<td>2,2914</td>
<td>0,4657</td>
<td>9,9864</td>
</tr>
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</table>

Source: Compiled by the author based on the market risk premium ascertained at monthly frequency. We use the GARCH-M schemes to model the conditional mean and variance simultaneously. We use the EuroStoxx-50 data, available from the DataStream database. First and second columns describe the sample average and standard deviation of the conditional risk aversion parameter. The last columns show the minimum and maximum values reached by the parameter.

Table 4. Summary of the optimum percentage weights/ calm period

<table>
<thead>
<tr>
<th>Portfolio Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model/Firm</td>
</tr>
<tr>
<td>Model A</td>
</tr>
<tr>
<td>Model B</td>
</tr>
</tbody>
</table>

Source: Compiled by the author. Each column shows the average of the conditional portfolio weights expressed as a percentage. We can evaluate more in detail how each of the assessed portfolios behaves over the calm period.

Table 5. Summary of the optimum percentage weights/ stress period

<table>
<thead>
<tr>
<th>Portfolio Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>LVMH</td>
</tr>
<tr>
<td>Model A</td>
</tr>
<tr>
<td>Model B</td>
</tr>
</tbody>
</table>

Source: Compiled by the author. Each column shows the average of the conditional portfolio weights expressed as a percentage. We can evaluate more in detail how each of the assessed portfolios behaves over the stressed timeframe.
Table 6. Unconditional moments. Individual assets

<table>
<thead>
<tr>
<th>Firms</th>
<th>Calm period</th>
<th>Stress period</th>
<th>Calm period</th>
<th>Stress period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std</td>
<td>Kurt</td>
<td>Skew</td>
</tr>
<tr>
<td>LVMH</td>
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<td>-0.0164</td>
</tr>
<tr>
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<td>0.3163</td>
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<tr>
<td>INTESA</td>
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<td>IBERD.</td>
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<td>0.1784</td>
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<td>1.7457</td>
</tr>
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<td>SAFRAN</td>
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<td>UNIBAIL</td>
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<td>ESSILOR</td>
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<td>0.1801</td>
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<td>0.4027</td>
</tr>
<tr>
<td>D.POST</td>
<td>0.0855</td>
<td>0.2070</td>
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</tr>
<tr>
<td>ENI</td>
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</tr>
<tr>
<td>BASF</td>
<td>0.1972</td>
<td>0.1787</td>
<td>0.7119</td>
<td>-0.0168</td>
</tr>
<tr>
<td>Model A</td>
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<td>0.1068</td>
<td>3.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>Model B</td>
<td>0.1690</td>
<td>0.1067</td>
<td>3.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Source: Compiled by the author. Note that the expected returns and standard deviations are expressed in annual terms, which is calculated by multiplying the monthly returns by 12 and their associated standard deviations by $\sqrt{12}$.

Table 7. Sharpe Ratio average. Calm and Stress period

<table>
<thead>
<tr>
<th>Average Ratios</th>
<th>Calm Period</th>
<th>Stress Period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mod. A</td>
<td>Mod. B</td>
</tr>
<tr>
<td>Sharpe Standard</td>
<td>1.6535</td>
<td>1.6769</td>
</tr>
<tr>
<td>Standard-EStoxx</td>
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<td>0.7569</td>
</tr>
<tr>
<td>Dynamic-Const</td>
<td>-0.0082</td>
<td>-</td>
</tr>
</tbody>
</table>

Source: Compiled by the author. Note that both periods are expressed in annual terms. The first row describes the standard Sharpe Ratio reached for each one of the mentioned periods, on average. Moreover, the second row shows the mean of the Sharpe Ratio on excess of the EuroStoxx-50 index, reached for the models. The third row shows the average difference between the dynamic and the constant risk aversion portfolios.
Table 8. Performance measures based on partial moments

<table>
<thead>
<tr>
<th>Ratios</th>
<th>Calm period</th>
<th>Stress period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Model A</td>
<td>Model B</td>
</tr>
<tr>
<td>Kappa 1</td>
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<tr>
<td>Omega</td>
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<td>1.0739</td>
</tr>
<tr>
<td>Sortino</td>
<td>0.3347</td>
<td>0.3277</td>
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</tbody>
</table>

Source: Compiled by the author. Expressed in annual terms. We fix the following values for the order of the LPM to consider different risk attitudes: $m = 2$ (moderate investors-Sortino Ratio) and $m = 1$ (aggressive investors-Kappa(1)). Moreover, we set the value $q = 1$ for the case of Omega index.