An Efficient American Option Approximation Formula in Markets with Daily Price Limits

Jia-Hau Guo[*] Yi-He Chang

ABSTRACT

This paper proposes solutions for pricing American options on stocks in markets with daily price limits. We first extend the intraday density function of Guo and Chang (2020) to a multi-day one. Next, we adopt the fast Fourier transform (FFT) to derive accurate and efficient formulae for American options in the framework of Kim (1990) and Chang et al. (2016) and, further, employ the three-point Richardson extrapolation to accelerate the computation. Finally, the accuracy of our proposed methods is verified by simulations. We also note that more restrictive daily price limits could force put options to be exercised earlier.

Keywords: Daily Price Limit, Early Exercise, Fast Fourier Transform, Multi-day Density Function, Richardson Extrapolation

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[*] Guo and Chang are from the Department of Information Management and Finance, College of Management, National Yang Ming Chiao Tung University, Hsinchu, Taiwan. Correspondence to: Professor Jia-Hau Guo, Department of Information Management and Finance, College of Management, National Yang Ming Chiao Tung University, No. 1001 Ta-Hsueh Road, Hsinchu, Taiwan. Tel:+886-3-5712121 #57078 Fax:+886-3-5733260 E-mail:gjiahau@nycu.edu.tw

I. Introduction

This paper considers the problem of pricing American options on stocks in markets with daily price limits. We follow Black and Scholes (1973) to derive American option pricing formulae for non-dividend-paying stocks but extend the framework to markets with daily price limits. As per Guo and Chang's (2020) assertion, knowledge of pricing options in markets with daily price limits is quite limited, and our understanding of price limit mechanisms primarily comes from empirical studies. This illustrates the importance and contribution of this paper: it derives solutions for these types of pricing situations especially that Kim and Park (2010) point out that 23 out of 43 of the most important world markets use daily price limits.

We believe that the discussion of the impacts of early exercise on options could be further generalized to the bounded geometric Brownian motion. We extend the method of Kim (1990) and Chang et al. (2016) by studying daily price limits. Such an extension is important because most stock markets around the world use price limits. Price limits are believed to mitigate excessive price volatility, lower panic behavior, and/or minimize price manipulation.[1] Despite their significant presence, however, impacts of these price limit mechanisms on options are not well understood, and there remain many unanswered questions about how to make early-exercise decisions regarding market regulation because of the lack of appropriate study tools. In this

[1] See Kim and Rhee (1997), Kim (2001), Kim and Yang (2004), and Kim and Park (2010).

paper, we first extend the intraday density function of Guo and Chang (2020) to a multi-day density function. Then, we use the framework of Kim (1990) and Chang et al. (2016) to value American options on stocks in markets with daily price limits. We derive an efficient formula for the early exercise premium and employ the three-point Richardson extrapolation to accelerate the computation. In addition, we also explore the influence of price limits and interest rates on the decision of early exercise.

The rest of this paper is organized as follows: in Section 2, the model and methodology are briefly described. Section 3 provides a comparison of our proposed solution with simulations and illustrates our findings. Section 4 presents the conclusion.

II. Model and Methodology

2.1 Framework of Kim (1990) and Chang et al. (2016)

For American put options, it can be optimal to exercise at any time prior to expiration, even in the absence of dividends. So, in this case, we are generally forced to a numerical solution; this is a well-known backward iteration. The Richardson extrapolation technique is one possible solution to obtain an efficient scheme for American options on a stock without dividends. For example, according to Kim (1990) and Chang et al. (2016):

$$P_A(S_t, t)$$
$$= P_E(S_t, t) + rK \int_t^T e^{-r(s-t)} \left\{ \int_0^{S_s^*} \phi(S_t, S_s, s-t) dS_s \right\} ds \text{ for } t \geq 0, \quad (1)$$

where $P_A(\cdot)$ denotes the time t value of an American put on the underlying stock price S_t with the strike price K and the maturity T, $P_E(\cdot)$ denotes the corresponding value of its European put, r denotes the risk-free interest rate, and $\phi(\cdot)$ denotes the density function of the future stock price S_s at future time s given the time t filtration. The critical exercise boundary solves the following integral equation for S_t^*:

$$K - S_t^*$$
$$= P_E(S_t^*, t) + rK \int_t^T e^{-r(s-t)} \left\{ \int_0^{S_s^*} \phi(S_t^*, S_s, s-t) dS_s \right\} ds$$
$$\text{for } T \geq t \geq 0. \quad (2)$$

Once S_t^* is obtained, the price of the American put option can be calculated based on Eq. (1). Solving for S_t^* needs to be conducted recursively. We need to solve for S_s^* for $s \in (t, T]$. To rapidly evaluate American options without approximating the whole early exercise boundary between t and T, we follow Huang et al. (1996) and Chang et al. (2016) to utilize a three-point Richardson extrapolation to accelerate the recursive integration method. The Richardson extrapolation scheme gains efficiency without sacrificing much accuracy. Our proposed model is implemented in a similar way. Assuming that the option can be respectively exercised only once, twice, or three times between t and T, and denoting the corresponding option prices as P_1, P_2, and P_3, the three-point Richardson extrapolation for the American put option could be expressed as follows:

$$\hat{P}_A = \frac{1}{2}(P_1 - 8P_2 + 9P_3) \tag{3}$$

where $\hat{P}_A$ denotes the approximated American put option value. Our approach may not be limited to the GBM price process, and S_t could follow a very general continuous-time stochastic process whose transition density is known.

2.2 Intraday characteristic function

We next extend this new efficient scheme to pricing European options on stocks which pay discrete dividends in markets with daily price limits. We first extend the intraday density function of Guo and Chang (2020) to a multi-day density function for stocks in markets with daily price limits. Consider an example of a European option with maturity T on stocks with daily price limits defined as follows: (A.1) price limits are determined by stock prices at date t_i, where $i = 0, \cdots, N$ and $t_0 = 0 < t_1 < t_2 < t_3 < \cdots < t_N = T$. The time interval between t_i and t_{i+1} is often one day. (A.2) In each time interval, the pricing process is a function of a geometric Brownian motion until price limits are reached. (A.3) After reaching a boundary, the stock price may remain on the boundary for a time or rebound away from the boundary. Hence, as

Ban et al. (2000) claimed, the least complicated natural process in each time interval is given by the following stochastic differential equation:

$$\begin{cases} dS_t = \sigma S_t I_{(a,b)}(S_t)dW_t + \theta S_t I_{(a,b)}(S_t)dt + \delta_1 d\emptyset_t - \delta_2 d\varphi_t \\ I_{\{a\}}dt = \rho_1 d\emptyset_t \\ I_{\{b\}}dt = \rho_2 d\varphi_t, \end{cases} \tag{4}$$

where W_t denotes a standard Brownian motion, θ denotes the drift term, and ϕ and φ are, respectively, local times at a (the lower bound) and b (the upper bound) under the physical measure. ρ is the viscosity of the boundary with $\rho \geq 0$; larger values of ρ could inhibit the change in the stock price. $\delta(\geq 0)$ denotes the elasticity of the boundary; as δ increases, the stock price rebounds more violently. This is Ban et al.'s (2000) intraday model of daily price limit markets. With the vanishing transaction cost technique, Ban et al. (2000) showed that the transaction cost vanishes sufficiently fast and the hedging error vanishes as the size of the discretization interval shrinks to zero. Therefore, they derived the following intraday partial differential equation (PDE) for the value of the contingent claim C with maturity T under the price-limit process described by Eq. (16):

$$\begin{cases} \frac{\partial C}{\partial t}(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}(S,t) + rS\frac{\partial C}{\partial S}(S,t) - rC(S,t) = 0 \\ C(S,T) = Y(S) \\ \frac{\partial C}{\partial t}(a,t) + ra\frac{\partial C}{\partial S}(a,t) - rC(a,t) = 0 \\ \frac{\partial C}{\partial t}(b,t) + rb\frac{\partial C}{\partial S}(b,t) - rC(b,t) = 0 \end{cases} \tag{5}$$

where $(S,t) \in [a,b] \times [0,T]$, r denotes the risk-free rate, and $Y(S)$ is the value of the contingent claim expired at the end of the day. Guo and Chang (2020) show that the intraday transition density $p(t, Z_0, Z_t)$ with $L < Z_t < U$ is given by

$$p(t, x, y)$$

$$= \frac{2}{U-L} exp\left\{\frac{\mu(y-x)}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right\} \sum_{m=1}^{\infty} exp\left\{-\frac{m^2\pi^2\sigma^2 t}{2(U-L)^2}\right\} \times sin\left(\frac{m\pi(x-L)}{U-L}\right) sin\left(\frac{m\pi(y-L)}{U-L}\right) \quad (6)$$

where $t > 0$, $Z_t = \ln S_t$, $x = Z_0$, $y = Z_t$, $L\,\{= ln(a) = ln\,[(1-\alpha)S_0]\} < x,\ y < U\,\{= \ln(b) = \ln\,[(1+\beta)S_0]\}$, and $\mu = \vartheta - \sigma^2/2$. The drift parameter ϑ can be determined with the requirement of retaining the Martingale property.[2] In addition,

$$p(t, x, \{L\}) = \Pi_x(\tau_L < \tau_U) - \int_L^U \Pi_y(\tau_L < \tau_U) p(t, x, y) dy, \quad (7)$$

and

$$p(t, x, \{U\}) = \Pi_x(\tau_U < \tau_L) - \int_L^U \Pi_y(\tau_U < \tau_L) p(t, x, y) dy, \quad (8)$$

where τ_L and τ_U denote the stopping time at *L* and *U*, respectively. Given the initial position $Z_0 = x$, the expression $\Pi_x(\tau_L < \tau_U)$ and $\Pi_x(\tau_U < \tau_L)$ can be defined and given by[3]

$$\begin{cases} \Pi_x(\tau_L < \tau_U) = \frac{1-\exp(2\mu(U-x)/\sigma^2)}{1-\exp(2\mu(U-L)/\sigma^2)} \\ \Pi_x(\tau_U < \tau_L) = \frac{1-\exp(-2\mu(x-L)/\sigma^2)}{1-\exp(-2\mu(U-L)/\sigma^2)}. \end{cases} \quad (9)$$

Given the intraday transition density under the chosen measure, the intraday characteristic function can be further deduced. The characteristic function is defined by

$$J_1(\phi, Z_0, t_1) \equiv \tilde{E}\left[exp\left(i\phi Z_{t_1}\right)|Z_0\right]$$

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[2] The measure that meets the requirement of $\tilde{E}[e^{-rT} S_T | S_0] = S_0$ is called the risk-neutral measure (see Kou and Wang, 2004).

[3] Please refer to Bhattacharya and Waymire (1990) for details of the proof.

$$= \int_L^U e^{i\phi y} p(t_1, x, y) dy + e^{i\phi L} p(t_1, x, \{L\}) + e^{i\phi U} p(t_1, x, \{U\}). \quad (10)$$

Note that $p(t_1, x, \{L\})$ and $p(t_1, x, \{U\})$ are constants because they depend only on ϑ, σ, α, and β. The characteristic function of the closing price is given by

$$J_1(\phi, Z_0, t_1)$$

$$= e^{i\phi Z_0}\left\{C \sum_{m=1}^{\infty} F_m G_m(\phi) + (1-\alpha)^{i\phi} p(t_1, x, \{L\}) + (1+\beta)^{i\phi} p(t_1, x, \{U\})\right\}, \quad (11)$$

where $C = \frac{2}{U-L} exp\left(\frac{-\mu^2 t_1}{2\sigma^2}\right) exp\left(\frac{\mu ln\,(1-\alpha)}{\sigma^2}\right)$, $F_m = exp\frac{-m^2\pi^2\sigma^2 t_1}{2d^2} sin\left(\frac{-m\pi\, ln(1-\alpha)}{U-L}\right)$, and $G_m(\phi) = \exp(i\phi \ln(1-\alpha)) \int_0^{U-L} exp\left(\frac{(i\phi\sigma^2+\mu)y}{\sigma^2}\right) \sin\left(\frac{m\pi y}{U-L}\right) dy$. Note that $J_1(\phi, Z_0, t_1)$ contains two parts, which are $e^{i\phi Z_0}$ and

$$H(\phi, t_1, \alpha, \beta)$$

$$= C \sum_{m=1}^{\infty} F_m G_m(\phi) + (1-\alpha)^{i\phi} p(t_1, x, \{L\}) + (1+\beta)^{i\phi} p(t_1, x, \{U\}), \quad (12)$$

where $H(\phi, t_1, \alpha, \beta)$ is a function of ϕ without Z_0. Therefore, under the chosen measure, the multiday characteristic function of the logarithm price at the end of the N^{th} day is

$$J_N(\phi, Z_0, t_N) \equiv \tilde{E}\left[exp\left(i\phi Z_{t_N}\right)|Z_0\right] = e^{i\phi Z_0} H(\phi, t_1, \alpha, \beta)^N, \quad (13)$$

where t_1 is the period of one day.

2.3 Pricing option using the fast Fourier transform (FFT)

Given the characteristic function of the logarithm price, Carr and Madan (1999) show that the call price can be obtained numerically using the inverse transform

$$C_T(k) = \frac{exp(-\hat{\alpha}k)}{\pi} \int_0^{\infty} e^{-ivk} \psi_T(v) dv \tag{14}$$

for a range of positive values of $\hat{\alpha}$, where $k = log\ (K)$, and

$$\psi_T(v) = \frac{e^{-rT} J_{N+\eta}(v-(\hat{\alpha}+1)i, Z_0, T)}{\hat{\alpha}^2+\hat{\alpha}-v^2+i(2\hat{\alpha}+1)v}. \tag{15}$$

To avoid a highly oscillatory integrand in the Fourier inversion for out-of-the money options with very short maturities, Carr and Madan (1999) further suggest using

$$C_T(k) = \frac{1}{sinh(\hat{\alpha}k)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \gamma_T(v) dv, \tag{16}$$

where $\gamma_T(v) = (\zeta_T(v - i\hat{\alpha}) - \zeta_T(v + i\hat{\alpha}))/2$ and

$$\zeta_T(v) = e^{-rT} \left(\frac{1}{1+iv} - \frac{e^{rT}}{iv} - \frac{J_{N+\eta}(v-i, Z_0, T)}{v^2-iv} \right). \tag{17}$$

Hence, the approximation for $C_T(k)$ in Eq. (14) using the fast Fourier transform (FTT) is given by

$$C_T(k_u) = \frac{exp(-\hat{\alpha}k_u)}{\pi} \sum_{j=1}^{M} e^{-i\frac{2\pi}{M}(j-1)(u-1)} e^{ibv_j} \psi(v_j) \frac{\xi}{3} \left[3 + (-1)^j - \varrho_{j-1}\right] \tag{18}$$

where $v_j = \xi(j-1)$, $k_u = -b + \lambda(u-1)$ for $u = 1,2,\cdots,M$, $b = M\lambda/2$, $\lambda\xi = 2\pi/M$, and ϱ_n is the Kronecker delta function that is unity for $n = 0$ and zero otherwise. The use of the FFT for calculating out-of-the-money option prices is given by

$$C_T(k_u) = \frac{1}{\sinh\ (\hat{\alpha}k_u)} \frac{1}{\pi} \sum_{j=1}^{M} e^{-i\frac{2\pi}{M}(j-1)(u-1)} e^{ibv_j} \gamma(v_j) \frac{\xi}{3} \left[3 + (-1)^j - \varrho_{j-1}\right]. \tag{19}$$

2.4 Multi-day density function

To derive the multi-day density function, we consider the following transformation

$$p_N(t_N, x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi y} J_N(\phi, Z_0, t_N)\, d\phi \tag{20}$$

where $x = Z_0$ and $y = Z_{t_N}$.

2.5 Early exercise premium

For American put options, it is difficult to find an analytical solution to the boundary and we focus on numerical solutions. With the Richardson three-point extrapolation, the numerical put option value could be solved quickly as long as the boundary is known. Eq. (1) shows the early exercise premium (EEP) of an American put option:

$$EEP_{t\_T} = rK \int_t^T e^{-r(s-t)} \left\{ \int_0^{S_s^*} \phi(S_t, S_s, s-t) dS_s \right\} ds\,. \tag{21}$$

Let $Z = \log(S)$, $Z_t = \log(S_t)$ and $k_s^* = \log(S_s^*)$, Eq. (21) can be rewritten as:

$$rK \int_t^T e^{-r(s-t)} \left\{ \int_{-\infty}^{k_s^*} \phi_Z(Z_t, Z_s, s-t) dZ_s \right\} ds, \tag{22}$$

where $\phi_Z(\cdot)$ denotes the density function of Z. Eq. (13) gives the multiday characteristic function of the logarithm price at the end of the N^{th} day. $J_N(v, Z_t)$ is $e^{ivZ_t}H^N$, given Z_t, so we can imply the N_s days characteristic function as follows:

$$J_{N_s}(v, Z_t) = e^{ivZ_t} H^{N_s} \tag{23}$$

with $N_s = (s-t)/one\ day\ time$. The logarithm N_s-day price density function could be calculated by the Fourier transformation:

$$\phi_Z(Z_t, Z_s, s-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivZ_s} J_{N_s}(v, Z_t,) dv = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivZ_s} e^{ivZ_t} H^{N_s} dv. \tag{24}$$

Therefore, Eq. (21) can be rewritten into

$$EEP_{t\_T} = rK \int_t^T e^{-r(s-t)} \left\{ \int_{-\infty}^{k_s^*} \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} e^{-ivZ_s} e^{ivZ_t} H^{N_s} dv \right) dZ_s \right\} ds. \tag{25}$$

After setting $k_s^* = k_{t\_T}^*$ (a constant number), $Z_s = M$, and changing the integral order in Eq. (25), we have:

$$rK\int_{-\infty}^{k_{t\_T}^{*}}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ivM}e^{ivZ_t}\left\{\int_t^T e^{-r(s-t)}H^{N_s}ds\right\}dv\,dM. \tag{26}$$

After defining $J'(v)$ as:

$$J'(v,t,T)\equiv e^{ivZ_0}\left\{\int_t^T e^{-r(s-t)}H^{N_s}ds\right\}$$

$$=e^{ivZ_0}\frac{1-e^{-r(T-t)}H^{N_S}}{r-\log(H)}=\frac{J_0(v,Z_0,t_0)-e^{-r(T-t)}J_{N_S}(v,Z_0,t_{N_S})}{r-\log(H)}, \tag{27}$$

we could simplify Eq. (26) as follow

$$EEP_{t\_T}(k_{t\_T}^{*})=rK\int_{-\infty}^{k_{t\_T}^{*}}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ivM}J'(v,t,T)dv\,dM. \tag{28}$$

It is clear that the inner integral of Eq. (28) is also a Fourier transform, which means there exists $q'(M,t,T)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ivM}J'(v,t,T)dv$ such that $J'(v,t,T)=\int_{-\infty}^{\infty}e^{ivM}q'(M,t,T)dM.$

Then Eq. (28) could be abbreviated as:

$$EEP_{t\_T}(k_{t\_T}^{*})=rK\int_{-\infty}^{k_{t\_T}^{*}}q'(M,t,T)dM. \tag{29}$$

Finally, let

$$G'\left(k_{t\_T}^{*}\right)\equiv\int_{-\infty}^{k_{t\_T}^{*}}q'(M,t,T)dM=\int_{-\infty}^{k_{t\_T}^{*}}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ivM}J'(v,t,T)dv\,dM. \tag{30}$$

We define $g'\left(k_{t\_T}^{*}\right)\equiv e^{-\alpha k_{t\_T}^{*}}G'\left(k_{t\_T}^{*}\right)$ and $\psi'(v,t,T)\equiv\int_{-\infty}^{\infty}e^{ivk_{t\_T}^{*}}g'\left(k_{t\_T}^{*}\right)dk_{t\_T}^{*}$. After applying an inverse Fourier transformation, we have

$$G'\left(k_{t\_T}^{*}\right)=\frac{e^{\alpha k_{t\_T}^{*}}}{2\pi}\int_{-\infty}^{\infty}e^{-ivk_{t\_T}^{*}}\psi'(v,t,T)dv. \tag{31}$$

After changing the integral order, we have

$$\begin{aligned}\psi'(v,t,T)&\\ &=\int_{-\infty}^{\infty}e^{ivk_{t\_T}^{*}}\int_{-\infty}^{k_{t\_T}^{*}}e^{-\alpha k_{t\_T}^{*}}q'(M,t,T)dM\,dk_{t\_T}^{*}\\ &=\int_{-\infty}^{\infty}\int_{M}^{\infty}e^{(iv-\alpha)k_{t\_T}^{*}}dk_{t_T}^{*}\,q'^{(M,t,T)}dM\\ &=\int_{-\infty}^{\infty}\frac{e^{i(v+i\alpha)M}}{\alpha-iv}q'(M,t,T)dM\\ &=\frac{J'(v+i\alpha,t,T)}{\alpha-iv}.\end{aligned} \tag{32}$$

With $\alpha > 1$, we have the EEP

$$EEP_{t\_T}(k_{t\_T}^*) = rK \frac{e^{\alpha k_{t\_T}^*}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk_{t\_T}^*} \psi'(v,t,T) dv, \tag{33}$$

With $\alpha > 1$, the approximation for EEP in Eq. (33) using FFT is given by

$$EEP_{t\_T}(k_u)$$

$$= rK \frac{exp(\hat{\alpha} k_u)}{\pi} \sum_{j=1}^{M} e^{-i\frac{2\pi}{M}(j-1)(u-1)} e^{ibv_j} \psi'(v_j,t,T) \frac{\xi}{3}\left[3 + (-1)^j - \varrho_{j-1}\right], \tag{34}$$

where $v_j = \xi(j-1)$, $k_u = -b + \lambda(u-1)$ for $u = 1, 2, \cdots, M$, $b = M\lambda/2$, $\lambda\xi = 2\pi/M$, and ϱ_n is the Kronecker delta function that is unity for $n = 0$ and zero otherwise.

For instance, in the case of two early exercise time points, Eq. (21) can be rewritten into

$$\begin{aligned} EEP_{0\_T}^2 &= rK \int_0^{T/2} e^{-r(s-0)} \left\{ \int_{-\infty}^{k_s^*} \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} e^{-ivZ_s} e^{ivZ_0} H^{N_s} dv \right) dZ_s \right\} ds \\ &\quad + rK \int_{T/2}^{T} e^{-r(s-0)} \left\{ \int_{-\infty}^{k_s^*} \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} e^{-ivZ_s} e^{ivZ_0} H^{N_s} dv \right) dZ_s \right\} ds \\ &= rK \int_0^{T/2} e^{-r(s_1-0)} \left\{ \int_{-\infty}^{k_{s_1}^*} \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} e^{-ivZ_{s_1}} e^{ivZ_0} H^{N_{s_1}} dv \right) dZ_{s_1} \right\} ds_1 \\ &\quad + rK \int_{T/2}^{T} e^{-r(s_2-0)} \left\{ \int_{-\infty}^{k_{s_2}^*} \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} e^{-ivZ_{s_2}} e^{ivZ_0} H^{N_{s_2}} dv \right) dZ_{s_2} \right\} ds_2. \end{aligned} \tag{35}$$

After setting $k_{s_1}^* = k_{0\_T/2}^*$ (a constant number), $k_{s_2}^* = k_{T/2\_T}^*$ (a constant number), $Z_{s_1} = M_1$, $Z_{s_2} = M_2$, and changing the integral order in Eq. (37), we have:

$$\begin{aligned} EEP_{0\_T}^2 = &\ rK \int_{-\infty}^{k_{0\_T/2}^*} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivM_1} e^{ivZ_0} \left\{ \int_0^{T/2} e^{-r(s_1-0)} H^{N_{s_1}} ds_1 \right\} dv\, dM_1 \\ &+ rK \int_{-\infty}^{k_{T/2\_T}^*} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivM_2} e^{ivZ_0} \left\{ \int_{T/2}^{T} e^{-r(s_2-0)} H^{N_{s_2}} ds_2 \right\} dv\, dM_2, \end{aligned} \tag{36}$$

where $k_{T/2\_T}^* = K$ and $k_{0\_T/2}^*$ can be pinned down by

$$K - e^{k_{0\_T/2}^*} = P_E\left(e^{k_{0\_T/2}^*}, T/2\right)$$

$$+rK\int_{T/2}^{T} e^{-r(s-T/2)}\left\{\int_0^{e^{k^*_{T/2\_T}}} \phi(e^{k^*_{0\_T/2}}, S_s, s-T/2)dS_s\right\}ds. \tag{37}$$

Having $k^*_{0\_T/2}$ and $k^*_{T/2\_T}$ makes EEP in Eq. (36) can calculated by Eq. (34) with $(k^*_{t\_T}, t, T) = (k^*_{0\_T/2}, 0, T/2)$ and $(k^*_{t\_T}, t, T) = (k^*_{T/2\_T}, T/2, T)$. In the case of three early exercise time points, Eq. (21) can be rewritten into a similar formula

$$EEP^3_{0\_T} = rK\int_{-\infty}^{k^*_{0\_T/3}} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivM_1}e^{ivZ_0}\left\{\int_0^{T/3} e^{-r(s_1-0)}H^{N_{s_1}}ds_1\right\}dv\,dM_1$$

$$+\ rK\int_{-\infty}^{k^*_{T/3\_2T/3}} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivM_2}e^{ivZ_0}\left\{\int_{T/3}^{2T/3} e^{-r(s_2-0)}H^{N_{s_2}}ds_2\right\}dv\,dM_2$$

$$+\ rK\int_{-\infty}^{k^*_{2T/3\_T}} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivM_3}e^{ivZ_0}\left\{\int_{2T/3}^{T} e^{-r(s_2-0)}H^{N_{s_3}}ds_3\right\}dv\,dM_3, \tag{38}$$

where $k^*_{2T/3\_T} = K$. In addition, $k^*_{T/3\_2T/3}$ can be pinned down at time $2T/3$ by

$$K - e^{k^*_{T/3\_2T/3}} = P_E\left(e^{k^*_{T/3\_2T/3}}, 2T/3\right)$$

$$+rK\int_{2T/3}^{T} e^{-r(s-2T/3)}\left\{\int_0^{e^{k^*_{2T/3\_T}}} \phi(e^{k^*_{T/3\_2T/3}}, S_s, s-2T/3)dS_s\right\}ds, \tag{39}$$

and then $k^*_{0\_T/3}$ can be pinned down at time $T/3$ by

$$K - e^{k^*_{0\_T/3}} = P_E\left(e^{k^*_{0\_T/3}}, T/3\right)$$

$$+rK\int_{T/3}^{2T/3} e^{-r(s-T/3)}\left\{\int_0^{e^{k^*_{T/3\_2T/3}}} \phi(e^{k^*_{0\_T/3}}, S_s, s-T/3)dS_s\right\}ds$$

$$+rK\int_{2T/3}^{T} e^{-r(s-T/3)}\left\{\int_0^{e^{k^*_{2T/3\_T}}} \phi(e^{k^*_{0\_T/3}}, S_s, s-T/3)dS_s\right\}ds. \tag{40}$$

Having $k^*_{0\_T/3}$, $k^*_{T/3\_2T/3}$, and $k^*_{2T/3\_T}$ makes EEP in Eq. (38) can calculated by Eq. (34) with $(k^*_{t\_T}, t, T) = (k^*_{0\_T/3}, 0, T/3)$, $(k^*_{t\_T}, t, T) = (k^*_{T/3\_2T/3}, T/3, 2T/3)$ and $(k^*_{t\_T}, t, T) = (k^*_{2T/3\_T}, 2T/3, T)$.

Given $EEP_{0\_T}^{2}$ and $EEP_{0\_T}^{3}$, we have $P_1 = P_E(S_0, 0)$, $P_2 = P_E(S_0, 0) + EEP_{0\_T}^{2}$, $P_3 = P_E(S_0, 0) + EEP_{0\_T}^{3}$ and the three-point Richardson extrapolation for the American put option could be expressed as Eq. (3).

III. Numerical Results and Findings

3.1 Numerical results

In this section, we discuss the influence of early exercise in daily price limit markets by comparing the results of the proposed numerical solutions with simulations. Table 1 shows the solutions of Guo and Chang (2020) (denoted by GC) and our proposed three-point Richardson extrapolation solutions of the Chang et al. (2016) framework (denoted by RE) are consistent with the results of Monte Carlo simulations (denoted by MC) for European options and those of the least square Monte Carlo simulations (denoted by LSMC) for American options on stocks without dividends in daily price limit markets.[4] The differences between the analytic solutions and MC are quite small. As for the computation time in the framework of Chang et al. (2016), our extended solution seems not to increase with the time to maturity. Our method has a great advantage in time consumption. For example, Table 1 shows that the computation time of our numerical solution is much less than the MC and LSMC. The computation

[4] According to Hull (2000), American calls on stocks without dividends have no reason to be early exercised and could be treated as European ones.

time of our solution may consume more time for American put options, but it seems not to increase with the time to maturity and is apparently between six and seven seconds. However, the computation time of LSMC often quickly increases as the time to maturity increases. The comparison of the computation time of these two methods shows that the Richardson extrapolation is quite accurate and effective for the EEP in markets with daily price limits.

3.2 Sensitive Analysis and Findings

Figure 1 shows the relationship between daily price limits (γ) and early exercise boundaries of put options on stocks without dividends. A more restrictive daily price limit seems to incur an earlier exercise boundary. However, when the daily price limit is 10% or greater, there seems to be little difference between early exercise boundaries.

IV. Conclusion

In the valuation of American options, the derivation of the early exercise boundary often involves a recursive and numerical computation and poses practical problems. We find that the three-point Richardson extrapolation improves the computation efficiency of the EEP and extends this new efficient scheme to pricing options on stocks in markets with daily price limits. To the best of our knowledge, no study has yet applied this methodology for equity options on stocks in markets with daily price limits.

We first extend the intraday density function of Guo and Chang (2020) to a multi-day density function for stocks in markets with daily price limits. Then, we apply our multi-day density function in the framework of Kim (1990) and Chang et al. (2016) to value American options on stocks without dividends. Moreover, we build an efficient formula and take advantage of FFT to quickly calculate the EEP in markets with daily price limits. We also adopt the three-point Richardson extrapolation to accelerate the computation of American options. The accuracy of our proposed method is further verified by simulations. We also note that more restrictive daily price limits could force put options to be exercised earlier. The lower limit could be the primary factor affecting the early exercise boundary for American puts.

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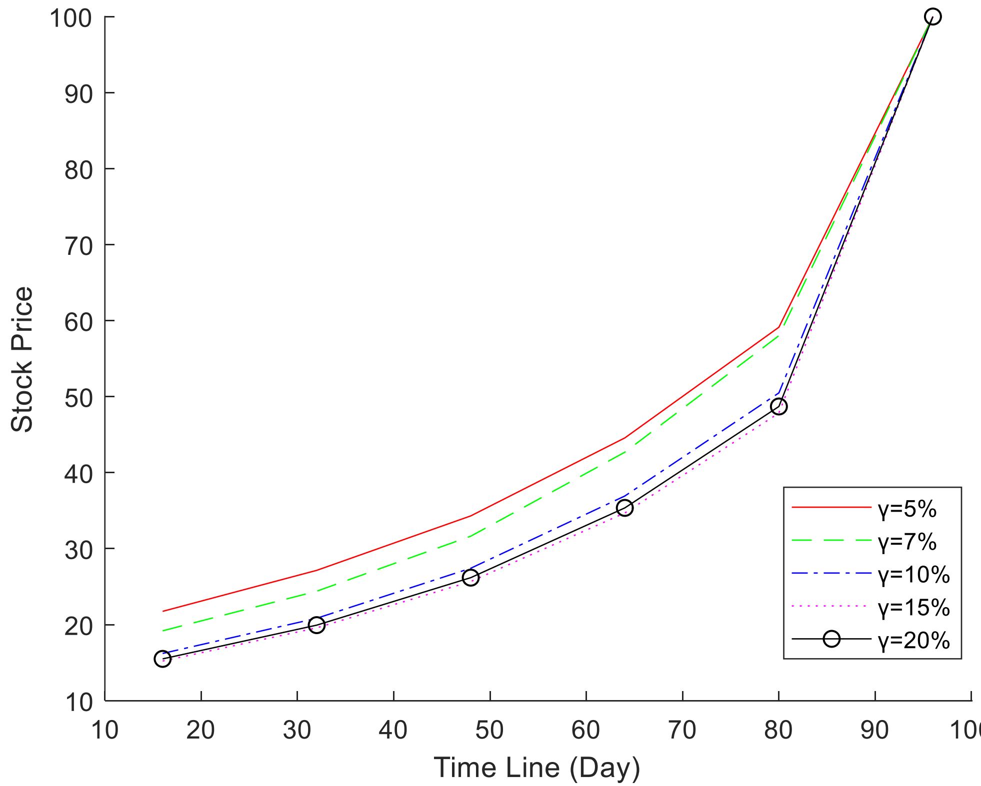
Table 1. Options on Stocks without Dividends in Markets with Daily Price Limits

		European Call					European Put					American Put				
S_0	T	GC	Time	MC	Time	Diff	GC	Time	MC	Time	Diff	RE	Time	LSMC	Time	Diff
90	6	0.88	0.06	0.88(0.007)	3.10	0.52%	10.86	0.06	10.88(0.005)	3.10	0.15%	10.88	7.14	10.86(0.013)	195.24	0.23%
	12	2.09	0.06	2.09(0.018)	6.63	0.08%	12.04	0.06	12.08(0.011)	6.63	0.31%	12.06	6.36	12.05(0.017)	684.57	0.07%
	24	4.11	0.06	4.10(0.015)	16.73	0.30%	14.02	0.06	14.08(0.007)	16.72	0.43%	14.10	7.45	14.04(0.026)	2945.67	0.46%
100	6	4.30	0.04	4.29(0.012)	3.03	0.23%	4.28	0.04	4.29(0.011)	3.03	0.32%	4.30	6.99	4.28(0.008)	189.60	0.45%
	12	6.08	0.04	6.06(0.021)	6.61	0.34%	6.04	0.04	6.06(0.015)	6.61	0.43%	6.05	6.34	6.05(0.011)	686.61	0.01%
	24	8.60	0.04	8.56(0.026)	16.32	0.47%	8.51	0.04	8.56(0.017)	16.31	0.63%	8.59	7.45	8.54(0.014)	2934.92	0.64%
110	6	11.18	0.03	11.17(0.012)	3.04	0.08%	1.16	0.04	1.17(0.008)	3.04	1.21%	1.18	6.98	1.17(0.007)	177.87	1.14%
	12	12.61	0.03	12.58(0.024)	6.63	0.21%	2.56	0.04	2.58(0.018)	6.63	0.91%	2.57	6.36	2.58(0.015)	643.20	0.32%
	24	14.92	0.03	14.84(0.052)	16.84	0.49%	4.82	0.04	4.85(0.031)	16.84	0.63%	4.90	7.43	4.84(0.024)	2821.80	1.27%

Model parameter specifications: r=1%, K=100, σ =70%, daily price limit γ=10%, T measured in days, and computation time measured in seconds. GC denotes the solution of Guo and Chang (2020). We set ξ=0.1702, $\hat{\alpha}$ =1.1 and use 4096 points in the quadrature. MC denotes the Monte Carlo simulation, which has 100,000 paths and 100 time steps in one day. Numbers in brackets denote standard deviations of MC. The absolute value of the different between GC and MC divided by MC is denoted by Diff. RE denotes our proposed three-point Richardson extrapolation solutions of the Chang et al. (2016) framework. LSMC denotes the least square Monte Carlo simulation, which has 100,000 paths and 100 time steps in one day. Numbers in brackets denote the standard deviations of LSMC. The absolute value of the difference between RE and LSMC divided by LSMC is denoted by Diff.

Figure 1. A Sensitive Analysis of Early Exercise Boundaries for Puts on Stocks without Dividends to Daily Price Limits



Model parameter specifications: S_0=K=100, r =10%, σ=70%, time to maturity N=96 days, and daily-price limit denoted by γ.