Investment, Uncertainty, and U-Shaped Return Volatilities*

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Abstract

I develop a real options model to explain means and variances of stock returns of portfolios sorted on book-to-market ratios. While average returns increase monotonically across portfolios, return volatilities are U-shaped. My model combines business cycle variations with countercyclical economic uncertainty. Operating leverage and procyclical growth options make both value stocks and growth stocks risky, generating U-shaped return volatilities. Growth stocks additionally load on the negative variance risk premium which reduces their expected return. Using structural estimation, my model *jointly* fits average returns and return volatilities, thereby solving a long-standing problem in investment-based asset pricing. Further reduced-form evidence supports the model channels.

KEYWORDS: Asset pricing, value premium, return volatility, real options, economic uncertainty, variance risk premium, simulated method of moments.

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1 Introduction

Economic uncertainty varies with the business cycle and is critical for policy makers, firms, and investors (see, e.g., Bloom (2009) and Jurado et al. (2015)). In particular, time-varying economic uncertainty is a major driver of asset prices (see, e.g., Bansal et al. (2014) and Campbell et al. (2018)), shapes corporate policies (see, e.g., Campello and Kankanhalli (2023)), and is itself priced in the cross-section of expected stock returns (see, e.g., Bali et al. (2017)). In this paper, I show that incorporating time-varying economic uncertainty in a dynamic investment model resolves a long standing problem in the investment-based asset pricing literature: jointly explaining average returns and return volatilities (see, e.g., Liu et al. (2009) and Gonçalves et al. (2020)).

The structural estimations of Carlson et al. (2004), Liu et al. (2009), Liu and Zhang (2014), Gonçalves et al. (2020), Li et al. (2023), and Kogan et al. (2023) confirm that models based on real options and q-theory indeed match average stock returns well. However, the same estimations consistently reject that investment-based theories fit return variances with Gonçalves et al. (2020) summarizing that this poor fit "*leaves much to be desired*" as return volatilities are crucial for Sharpe ratios, portfolio allocations, and hedging. Figure 1 illustrates why jointly fitting means and volatilities is difficult. I plot the mean (Panel A) and standard deviation (Panel B) of monthly excess returns of value-weighted quintile portfolios sorted on book-to-market ratios. While average returns increase monotonically and generate a positive value premium, return volatilities are U-shaped across portfolios. Volatility thus differs markedly from systematic risk, and challenges existing theories which make similar predictions for both quantities. My contribution lies in developing and structurally estimating an investment-based asset pricing model which *jointly* matches monotone average returns and U-shaped return volatilities, alongside further stylized facts, such as the value premium being countercylical and persistent.

My model features heterogeneous firms facing stochastic productivity that tracks the business cycle. Firms invest subject to real frictions such as irreversibility and fixed adjustment costs. Drawing from the recent macro-finance literature that stresses how economic uncertainty varies with the business cycle, I incorporate stochastic volatility (see, e.g., Bloom et al. (2018) and Alfaro et al. (2023)).

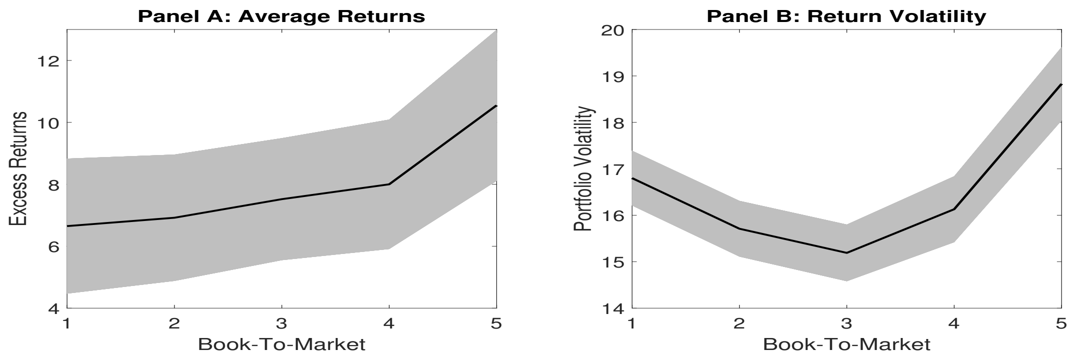

Figure 1: This figures shows the means (Panel A) and standard deviations (Panel B) of monthly excess returns of value-weighted quintile portfolios sorted on book-to-market ratios. The gray areas are standard errors. All numbers are annualized and reported as percentages. The data is from Ken French's website covering the period 1963 until 2022.

As a result, firms invest cautiously during periods of high economic uncertainty because capital adjustments involve giving up the volatility-sensitive option to wait.

Firms own assets-in-place and growth options which endogenously determine firms' sensitivity to the business cycle and to time-varying economic uncertainty. The sensitivity to the business cycle is U-shaped as a function of book-to-market ratios. Value firms are procyclical due to operating leverage resulting from fixed maintenance costs and investment irreversibility. The procyclicality of growth firms stems from their growth options being inherently levered claims which are very valuable during booms but of particularly low value during recessions. However, as growth stocks own volatility-sensitive growth options, they are additionally exposed to economic uncertainty. Value stocks with their abundance of fixed production assets are less responsive to time-varying economic uncertainty, generating monotonically declining volatility exposures as a function of book-to-market ratios. Because economic uncertainty peaks during recessions, exposure to it offers a partial hedge to economic downturns, thereby lowering the expected return of growth stocks and generating a positive value premium. A firm's return volatility depends on the squared sensitivities to the business cycle and economic uncertainty, and on their cross-product. Quantitatively, however, the first component dwarfs the other two such that return volatility is entirely driven by firms' U-shaped sensitivity to the business cycle. Thus, intriguingly, allowing for stochastic volatility helps matching expected returns via the variance risk premium while barely impacting return volatilities.

To quantitatively asses these model predictions, I next take the model to the data using the simulated method of moments (SMM). I choose value-weighted averages of stock returns and firm-level return volatilities within quintile portfolios sorted on book-to-market ratios as joint moment conditions. I focus on the value premium which has long been the central focus of investment-based asset pricing research (see, e.g., Carlson et al. (2004), Zhang (2005), Cooper (2006), Liu et al. (2009), Kogan and Papanikolaou (2012, 2013, 2014), Favilukis and Lin (2016), and Belo et al. (2022)). Using quintile portfolios ensures that the U-shaped volatilities survive averaging across a large part of the cross-section and do not result from extreme firms. Instead of aggregated portfolio volatilities, I target average firm-level return volatilities in the estimation after documenting that they are already U-shaped. This suggests that the U-shapes in portfolio volatility do not arise from correlations between stocks but are a characteristic of the average value and growth stock, which also aligns with the predictions of my model. The resulting ten moments are targeted by five model parameters which are key to the model channels: firms' fixed maintenance costs generating operating leverage, the risk premiums associated to the business cycle and economic uncertainty, the level of idiosyncratic volatility generating firm heterogeneity, and the volatility of uncertainty shocks.

The structural estimation confirms that the model fits the data well. The targeted moments are matched closely with an overall mean absolute error of only about 66 basis points (56 basis points for mean returns and 77 basis points for return volatilities). Importantly, the parameters are well-identified. For example, the average return and return volatility of value stocks largely pin down the fixed maintenance cost parameter responsible for the operating leverage channel. Conversely, the price of risk for economic uncertainty greatly impacts the expected return of growth stocks while having little impact on return volatilities. All parameters are estimated accurately with small standard errors.

The model reproduces further stylized facts beyond the ten imposed moment conditions. In particular, the model generates a value premium that is countercyclical, persistent, and not explained by the CAPM. Intuitively, the value premium in my model is largely attributable to volatility-sensitive growth options partially hedging recessions. The return difference between value and growth stocks is thus particularly large during bad states of the economy when economic uncertainty is high. Second, as the book-to-market ratio is fairly sticky and persistent in both the real data and my model, the resulting return premium is also persistent. Finally, as my model captures compensations for exposure to both the business cycle and economic uncertainty, a simple one-factor conditional CAPM cannot fully capture expected returns. Serving as a test for external validation, the model also fits a series of further untargeted moments, including average investment rates, the equity premium, and further factor premiums related to market size, long-term reversals, and expected growth. This fit ensures that the model gives a good overall description of the behavior of firms and the distribution of stock returns.

I conclude with documenting additional reduced-form evidence supporting the main channels of my model. First, I use market beta as a proxy for firms' sensitivity to the business cycle and find it indeed to be U-shaped across portfolios sorted on book-to-market ratios. Second, loadings on the VIX are monotone with growth stocks being more sensitive to economic uncertainty than value stocks. Average firm characteristics further confirm that growth firms are indeed characterized by many growth options, value firms are highly operationally levered, and that financial leverage, equity duration, and institutional ownership, which increase monotonically across book-to-market sorted portfolios, are an unlikely explanations for these U-shaped patterns. I also document U-shapes in implied volatilities, within all different size quintiles, in different subsamples, after removing highly levered or distressed firms, when sorting on alternative proxies for book-to-market ratio, and that these U-shapes are not mechanically the result of the price information contained in book-to-market ratios.

A few real options asset pricing studies posit that investment reversibility may weaken the operating leverage channel (see, e.g., Hackbarth and Johnson (2015) and Gu et al. (2018)). Following the empirical evidence of Bai et al. (2022) that merely 5.5% of firm-level investment rates are actually negative, I follow the literature standard and assume irreversible investment in my model. Still, the shutdown options in my model are very akin to the contraction options that may weaken the operating leverage channel, yet are found in my estimation to be of negligible value, thus aligning my evidence with a large literature supporting the operating leverage channel (see, e.g., Carlson et al. (2004), Zhang (2005), Cooper (2006), Garcia-Feijóo and Jorgensen (2010), Novy-Marx (2011), Obreja (2013), Favilukis and Lin (2016), Lambrecht et al. (2016), and Chen et al. (2022)).

I add to a growing literature of structural estimations of investment-based asset pricing models (see, e.g., Carlson et al. (2006), Gomes et al. (2006), Belo et al. (2013), Alti and Tetlock (2014), Vitorino (2014), Hackbarth and Johnson (2015), Zhu (2022), Kim et al. (2022), and Belo et al. (2023)). By deriving a model which makes distinct predictions for systematic risk and return volatility, I resolve a long standing puzzle in the investment-based asset pricing literature since the structural estimations of Carlson et al. (2004), Liu et al. (2009), Liu and Zhang (2014), Gonçalves et al. (2020), Li et al. (2023), and Kogan et al. (2023) find that investment-based theories fit average stock returns but not return volatilities. Intuitively, these models make identical predictions for average returns and return volatilities, and hence fail to jointly match both sets of moments. In line with Belo et al. (2022) and Delikouras and Dittmar (2022), I thus stress the importance of moment conditions beyond average returns when estimating a dynamic investment model.

This paper adds to the real options asset pricing literature by studying the asset pricing implications of introducing stochastic volatility to a popular investment-based model. Recently, Ai and Kiku (2016), Dou (2017), Bhamra and Shim (2017), McQuade (2018), and Barinov and Chabakauri (2023) employ time-varying volatility in real options asset pricing models to explain average returns of individual risk factors. I not only differ in research question and methodology (focus on jointly fitting means and variances using SMM), but I also derive novel and fully analytical solutions for real option values in a stochastic volatility model that are not restricted to models with only two states and that avoid numerical approximations but instead offer economic intuition about the model solution.

Finally, I contribute to the study of economic uncertainty. My model endogenizes the investment-hampering effect of heightened economic uncertainty documented in the macro-finance literature (see, e.g., Bloom (2009), Bloom et al. (2018), and Alfaro et al. (2023)), and employs the cross-sectional pricing of economic uncertainty (see, e.g., Bali et al. (2017) and Campbell et al. (2018)) to solve the important problem of jointly fitting average returns and return volatilities. Indeed, return volatilities are a useful tool to compare competing theories by taking *all* of their predictions seriously and testing how they fare with the data. In this sense, my structural estimation lends strong empirical support for real options asset pricing models with multiple risk sources such as time-varying economic uncertainty.

2 Model

In this section, I study means and variances of stock returns in a real options models in which heterogeneous firms make irreversible investment decisions while facing uncertain productivity and time-varying economic uncertainty. The model predicts that firms with both high and low valuation ratios are procyclical and sensitive to the state of economy due to operating leverage and levered growth options. This two-sided riskiness generates U-shaped return volatilities across portfolios. The addition of the variance risk premium allows volatility-sensitive growth options to partially hedge recessions and generate monotone expected returns without impacting the U-shaped return volatilities.

2.1 Firm technology

Consider a production economy in which a continuum of all-equity financed, risk-averse firms operate with an infinite and continuous time horizon. Every firm uses their assets-in-place to produce a unique output good which is sold instantaneously. I denote the number of production units ("capital stock") of firm i at time t by $\bar{K}_{i,t} \in [0, \infty)$ which each produce one output unit per unit of time. Firms' production capacity is their key choice variable to maximize their (net) market value. Firms pay fixed operating costs $f\bar{K}_{i,t}$ with $f \geq 0$. These costs combine, among others, maintenance costs, basic wage costs, and linear production costs. A firm's gross operating profit per unit of time, $\Pi_{i,t}$, is

$$\Pi_{i,t} = X_t Z_{i,t} \bar{K}_{i,t}^{\psi} - f\bar{K}_{i,t}, \tag{1}$$

where X_t and $Z_{i,t}$ are aggregate and firm-specific disembodied productivity components representing the business cycle, and $\psi < 1$ captures decreasing returns-to-scale. A firm's sales revenue is $X_t Z_{i,t} \bar{K}_{i,t}^{\psi}$ per unit of time. The aggregate component of productivity captures long-run growth and time-varying economic uncertainty in the business cycle using Heston (1993) stochastic volatility,

$$\mathrm{d}X_t = \alpha X_t \mathrm{d}t + \sqrt{v_t} X_t \mathrm{d}B_t^X, \tag{2}$$

$$\mathrm{d}v_t = \kappa(\bar{v} - v_t)\mathrm{d}t + \xi\sqrt{v_t}\mathrm{d}B_t^v. \tag{3}$$

The idiosyncratic component is driftless and generates firm heterogeneity through firm-specific shocks

$$\mathrm{d}Z_{i,t} = \sigma Z_{i,t} \mathrm{d}B^Z_{i,t}. \tag{4}$$

Here, $\alpha > 0$ is the aggregate productivity growth rate, v_t is time-varying economic uncertainty, and $\sigma > 0$ is firm-specific volatility. Aggregate variance, v_t, is characterized by its speed of mean reversion, $\kappa > 0$, its long-term mean, $\bar{v} > 0$, and its own volatility, $\xi > 0$. The innovations common to all firms are driven by two Brownian motions, B^X_t and B^v_t, correlated via $\mathrm{d}B^X_t \mathrm{d}B^v_t = \rho \mathrm{d}t$. Intuitively, κ controls the persistence of the variance process, ρ the skewness of the productivity distribution, and ξ the kurtosis of that distribution. A negative correlation, $\rho < 0$, induces a leverage effect, linking states of high (low) productivity and low (high) volatility. I assume constant idiosyncratic productivity variance which is sufficient to generate firm heterogeneity while not further convoluting the model. In Appendix B, I confirm that the model intuition carries through when idiosyncratic productivity variance is stochastic. Setting productivity variance to be constant, $\kappa = \xi = 0$, recovers a standard geometric Brownian motion employed in many real options models (see, e.g., Carlson et al. (2004, 2006, 2010), Cooper (2006), Aguerrevere (2009), Aretz and Pope (2018), and Zhu (2022)).[1]

A key variable in the model is firms' total productivity, denoted by $\theta_{i,t} = X_t Z_{i,t}$, which is given by

$$\mathrm{d}\theta_{i,t} = \alpha \theta_{i,t} \mathrm{d}t + \sqrt{v_t} \theta_{i,t} \mathrm{d}B^X_t + \sigma \theta_{i,t} \mathrm{d}B^Z_{i,t}, \tag{5}$$

$$\mathrm{d}v_t = \kappa(\bar{v} - v_t)\mathrm{d}t + \xi \sqrt{v_t} \mathrm{d}B^v_t. \tag{6}$$

A firm's state space is fully characterized by its total productivity, $\theta_{i,t}$, its capital stock, $\bar{K}_{i,t}$, and current uncertainty about future aggregate productivity, v_t.

[1] Following Engle (1982), economists identified heteroscedasticty in a great many variables that might proxy firms' productivity and the business cycle, turning Heston's (1993) stochastic volatility model into a popular benchmark. Using Engle's (1982) ARCH test, I strongly reject the null hypothesis of homoscedasticity in the innovations to monthly industrial production or quarterly TFP and GDP, three common proxies for the business cycle. Bloom et al. (2018) stress the role of time-varying second moments, negatively correlated with first moment shocks, to model firm decisions accurately. Indeed, stochastic volatility is an important state variable in finance, from consumption-based models (Bansal and Yaron (2004)) over DSGE models (Alfaro et al. (2023)) to credit risk models (Du et al. (2019)).

2.2 Growth options and investment policy

Firms' main choice variable is their investment policy which comes with several frictions. Investment is irreversible and subject to fixed and quasi-fixed adjustment costs. Following Cooper (2006) and Hackbarth and Johnson (2015), firms pay fixed installation costs $k > 0$ per capital unit as well as a proportion of the sales revenue generated from those new capital units, $k_p \in (0, 1)$. Thus, the capital expenditure for raising a firm's capital stock from $\bar{K}_{i,t}$ to $\bar{K}_{i,t} + I_{i,t}$ is

$$kI_{i,t} + k_p\theta_{i,t}\left(\left(\bar{K}_{i,t} + I_{i,t}\right)^{\psi} - \bar{K}_{i,t}^{\psi}\right). \tag{7}$$

The fixed investment costs (k) are independent of total productivity $\theta_{i,t}$, make continuous capital adjustments prohibitively expensive, and instead turn investment decisions into valuable growth options which are costly to exercise. Intuitively, rather than investing upon every increase in $\theta_{i,t}$, firms wait until their productivity increases sufficiently high. The additional costs (k_p) are proportional to current sales, capture the forgone profits during disruptive capital adjustments, and ensure that the investment costs are neither overwhelming nor negligible compared to firm size. Finally, firms may choose to irreversibly shut down operations if they are sufficiently unproductive.

The resulting investment policies align qualitatively with empirical evidence. First, investing entails losing the volatility-sensitive option to wait which makes investment decisions a function of productivity *and* economic uncertainty. Thus, investments are less responsive to total productivity in states of high uncertainty, see Bernanke (1983), Leahy and Whited (1996), Guiso and Parigi (1999), Bloom (2009), Carlson et al. (2010), Bloom et al. (2018), Campello et al. (2021), and Alfaro et al. (2023), among others. Furthermore, as real options introduce fixed capital adjustment costs, continuous investments as in standard q-theory are infinitely expensive and investment occurs in infrequent spurts. Doms and Dunne (1998) and Bai et al. (2022) indeed document lumpy investment rates. The investment irreversibility in the model reflects the finding that negative firm-level investment rates are rare and that investment rates instead are strongly positively skewed (see, e.g., Bai et al. (2022)).

2.3 Firm value

I next outline how to determine a firm's value. In line with the intuition that firms own valuable real options, Proposition 1 interprets firms as a portfolio of their assets-in-place and options to invest and shut down. The solution method is novel, exact, and akin to the pricing of standard financial options.

Proposition 1. *A firm's market value, $W_{i,t}$, permits the decomposition*

$$W_{i,t} = SO_{i,t} + AiP_{i,t} + GO_{i,t}, \tag{8}$$

where $SO_{i,t}$ captures the value of shutdown options, $AiP_{i,t}$ adds the value of assets-in-place, and $GO_{i,t}$ contributes the accumulated growth option values.

The value of the assets-in-place is

$$AiP_{i,t} = \frac{\theta_{i,t}}{\delta}\bar{K}_{i,t}^{\psi} - \frac{f}{r}\bar{K}_{i,t}, \tag{9}$$

where the risk-adjusted discount rate δ is defined in Appendix A.1 and r is the risk-free rate of return.

The value of the option to shut down the installed assets-in-place is

$$SO_{i,t} = \frac{f}{r}\bar{K}_{i,t}\mathcal{S}_{i,t} - \frac{\theta_{i,t}}{\delta}\bar{K}_{i,t}^{\psi}\mathcal{S}'_{i,t}, \tag{10}$$

where $\mathcal{S}_{i,t}, \mathcal{S}'_{i,t} \in [0,1]$ are exercise probabilities defined in Appendix A.2.

The value of growth options is divided into increments such that

$$GO_{i,t} = \int_{\bar{K}_{i,t}}^{\infty} \Delta GO_{i,t}(K)\mathrm{d}K, \tag{11}$$

with the value of the K^{th} incremental option to grow being

$$\Delta GO_{i,t} = (1-\delta k_p)\psi\theta_{i,t}K^{\psi-1}\mathcal{G}'_{i,t} - \left(\frac{f}{r} + k\right)\mathcal{G}_{i,t}, \tag{12}$$

where $\mathcal{G}_{i,t}, \mathcal{G}'_{i,t} \in [0,1]$ are exercise probabilities defined in Appendix A.3.

Proof. See Appendix A. □

Valuing a firm reduces to pricing its assets-in-place and real options. Appendix A contains all details which I briefly summarize here. Starting wit the assets-in-place, the capital stock offers a perpetual flow of profits, whose present value can be calculated using the appropriate discount rate (Equation (9)). Because production and sales are instantaneous and no optionality is involved, $AiP_{i,t}$ is unaffected by uncertainty about the future business cycle. Fixed maintenance costs can generate prolonged negative profits, such that firms may choose to seize operations. The value of this flexibility is captured in Equation (10) and represents a perpetual put option written on the firm's assets-in-place. Like usual option pricing formulas, the option value trades-off the present value of the gains of the exercise, namely the saved fixed maintenance costs ($\frac{f}{r}\bar{K}_{i,t}$), with the losses of the exercise, namely the lost sales revenues ($\frac{\theta_{i,t}}{\delta}\bar{K}^{\psi}_{i,t}$). The terms $\mathcal{S}_{i,t}$ and $\mathcal{S}'_{i,t}$ capture the likelihood of the option exercise and resemble the probabilities "$N(-d_2)$" and "$N(-d_1)$" as they appear in the Black-Scholes (1973) formula. To calculate the value of the growth options, I consider investments into new incremental productive units, see Equation (11). The value of the opportunity to expand the current capital stock and to add the next capital increment in Equation (12) depends on the present values of the potential gains and losses upon exercise. If the investment takes place, the firm increases its sales revenue which is captured by the term $\psi\theta_{i,t}K^{\psi-1}$. However, a proportion of current sales is lost as quasi-fixed adjustment cost which explains the factor $1-\delta k_p$. Investment also incurs sunk cost (k) and perpetually paying fixed maintenance costs ($\frac{f}{r}$). As with the shutdown options, the terms $\mathcal{G}_{i,t}$ and $\mathcal{G}'_{i,t}$ capture the exercise probability, thereby resembling "$N(d_2)$" and "$N(d_1)$" from the Black-Scholes (1973) formula. Armed with usual value-matching conditions, I can then use these closed-form expressions to identify firms' investment and shutdown policies and determine the firm value.

My solution offers a novel method to solve real options models with stochastic volatility. The decomposition of the value function of perpetual real options into their probability-weighted payoff is not only novel but also inserts economic intuition instead of opaque solutions to differential equations.

Contemporaneous papers restrict volatility to only switch between two states (see, e.g., Ai and Kiku (2016), Bhamra and Shim (2017), Dou (2017), and Alfaro et al. (2023)) or employ asymptotic expansions, Taylor polynomials, and other approximations (e.g., McQuade (2018), Du et al. (2019), and Barinov and Chabakauri (2023)). My approach, instead, is free from such assumptions and approximations.

2.4 Expected return

I next compute firms' expected return and decompose it into rewards for positively priced productivity risk and negatively priced variance risk. Firm characteristics such as valuation ratios determine the sensitivity to these risk sources and thus generate cross-sectional variation in expected returns.

PROPOSITION 2. *The conditional expectation of firms' excess return per unit of time is*

$$\frac{\mathbb{E}_t[\mathrm{d}R_{i,t} - r\mathrm{d}t]}{\mathrm{d}t} = \Omega_{i,t}^{(\theta)}(\mu - r) + \Omega_{i,t}^{(v)}\lambda, \tag{13}$$

where $\Omega_{i,t}^{(\theta)}$ *and* $\Omega_{i,t}^{(v)}$ *are elasticities, and* $\mu - r$ *and* λ *denote risk premiums. Exposure to procyclical productivity shocks is awarded by* $\mu > r$*, whereas* λ *denotes the negative variance risk premium.*[2]

The risk premiums are weighted by the firm-level elasticities

$$\Omega_{i,t}^{(\theta)} = \frac{\partial W_{i,t}/W_{i,t}}{\partial \theta_{i,t}/\theta_{i,t}} = \frac{SO_{i,t}}{W_{i,t}}\Omega_{SO}^{(\theta)} + \frac{AiP_{i,t}}{W_{i,t}}\Omega_{AiP}^{(\theta)} + \frac{GO_{i,t}}{W_{i,t}}\Omega_{GO}^{(\theta)}, \tag{14}$$

$$\Omega_{i,t}^{(v)} = \frac{\partial W_{i,t}/W_{i,t}}{\partial v_t/v_t} = \frac{SO_{i,t}}{W_{i,t}}\Omega_{SO}^{(v)} + \frac{AiP_{i,t}}{W_{i,t}}\Omega_{AiP}^{(v)} + \frac{GO_{i,t}}{W_{i,t}}\Omega_{GO}^{(v)}, \tag{15}$$

where $\Omega_{SO}^{(\theta)}$, $\Omega_{AiP}^{(\theta)}$, *and* $\Omega_{GO}^{(\theta)}$ *measure the individual firm value components' tilt toward productivity risk while* $\Omega_{SO}^{(v)}$, $\Omega_{AiP}^{(v)}$, *and* $\Omega_{GO}^{(v)}$ *represent the firm value components' tilt to productivity variance risk.*

Proof. See Appendix A.4. □

[2]The variance risk premium is the return earned for holding an asset whose returns perfectly correlate with variance innovations and are independent of other factors. As high variance corresponds to economically bad states of nature, this investment acts as an insurance against market risk and agents accept $\lambda < 0$ in equilibrium. In an ICAPM sense, higher variance represents a deterioration of the investment opportunity set as it increases dispersion in consumption allocations across states of nature and thus creates hedging demand for agents (see, e.g., Campbell et al. (2018)). Among many others, Carr and Wu (2009) provide empirical evidence for the variance risk premium being negative.

The expected excess return of a firm thus has two components: a compensation for the risk associated to the diffusive shocks that drive total productivity, $\mu - r$, representing the business cycle, and second a premium for the tilt to altering economic uncertainty, λ. Equations (14) and (15) show that the firm-level elasticities, $\Omega^{(\theta)}_{i,t}$ and $\Omega^{(v)}_{i,t}$, are value-weighted averages of elasticities of a firm's assets-in-place, shutdown option, and growth options. Thus, the expected excess return is ultimately composed of six ingredients. Assets-in-place offer a unit exposure to productivity risk, augmented by operating leverage resulting from fixed maintenance costs, but are unresponsive to changes to future economic uncertainty. Thus, $\Omega^{(\theta)}_{AiP} \geq 1$ and $\Omega^{(v)}_{AiP} = 0$. Growth options are levered and convex claims on assets-in-place and thus respond positively to increases in productivity and productivity variance, implying $\Omega^{(\theta)}_{GO} \gg 1$ and $\Omega^{(v)}_{GO} > 0$. Conversely, a shutdown option benefits from declining productivity, $\Omega^{(\theta)}_{SO} < 0$, whereas its convexity generates positive volatility dependence, $\Omega^{(v)}_{SO} > 0$. To sum up, the expected return has a "delta channel" compensating sensitivity to the business cycle and a "vega channel" compensating sensitivity to economic uncertainty, with the loadings on these two risk factors depending on firms' asset decomposition.

2.5 Return volatility

In this section, I calculate the volatility of a firm's excess return. Continuing to interpret a firm as a portfolio of a "productivity component" and an "uncertainty component," a firm's return volatility is also impacted by the correlation between both risk sources.

PROPOSITION 3. *The conditional variance of firms' excess return per unit of time is*

$$\frac{\mathbb{V}\text{ar}_t[\text{d}R_{i,t} - r\text{d}t]}{\text{d}t} = \left(\Omega^{(\theta)}_{i,t}\right)^2 (v_t + \sigma^2) + \left(\Omega^{(v)}_{i,t}\right)^2 \frac{\xi^2}{v_t} + 2\Omega^{(\theta)}_{i,t}\Omega^{(v)}_{i,t}\, \rho\xi, \tag{16}$$

where the firm-level elasticities $\Omega^{(\theta)}_{i,t}$ and $\Omega^{(v)}_{i,t}$ are given in Equations (14) and (15).

Proof. See Appendix A.4. □

Proposition 3 confirms that a firm's return volatility has three components, one each for productivity risk, variance risk, and their interplay which is driven by the correlation between shocks to productivity and its variance. Indeed, Equation (16) resembles that of the return variance of a two-asset portfolio, where the firm-level elasticities $\Omega_{i,t}^{(\theta)}$ and $\Omega_{i,t}^{(v)}$ are portfolio weights and $\rho\xi$ the covariance term.

Propositions 2 and 3 illustrate the different channels at work in the model. Growth stocks are procyclical and very responsive to the business cycle (large $\Omega_{i,t}^{(\theta)}$). So are value stocks which derive their value from assets-in-place that create lots of operating leverage (large $\Omega_{i,t}^{(\theta)}$). However, while the productivity elasticity is U-shaped as a function of book-to-market ratios, the sensitivity to economic uncertainty is not. This is because assets-in-place are independent of volatility and only growth opportunities load on this second risk factor (positive $\Omega_{i,t}^{(v)}$). Because return volatility depends on squared elasticities, it is dominated in magnitude by the U-shaped sensitivity to the business cycle, while expected returns are monotone because growth stocks additionally load on economic uncertainty which lowers their expected return and generates a positive value premium.

If economic uncertainty is constant, $\Omega_{i,t}^{(v)} = 0$, the formulas for expected returns and return variances collapse to $\mathbb{E}_t[\mathrm{d}R_{i,t} - r\mathrm{d}t] = \Omega_{i,t}^{(\theta)}(\mu - r)\mathrm{d}t$ and $\mathbb{V}\mathrm{ar}_t[\mathrm{d}R_{i,t} - r\mathrm{d}t] = \left(\Omega_{i,t}^{(\theta)}\right)^2 \sigma^2\mathrm{d}t$ which resemble those from previously studied single-factor real options models, see, e.g., Carlson et al. (2004), Cooper (2006), Hackbarth and Johnson (2015), and Aretz and Pope (2018). In such a one-factor model with constant productivity variance, mean returns and return volatilities are thus both proportional to $\Omega_{i,t}^{(\theta)}$, generating identical predictions for means and variances of stock returns (see, e.g., Carlson et al. (2010)). This equality, however, contradicts the observation of monotone average returns and U-shaped return volatilities and plagues many investment-based asset pricing models.

3 Econometric design

In this section, I outline how I take the model to the data and conduct a structural estimation to illustrate that my model indeed jointly fits means and variances of stock returns. In the SMM estimation, I match observed stock returns with model-implied stock returns. To this end, I consider an "empirical panel" with data retrieved from CRSP and Computstat and a "simulated panel" containing artificial firms simulated from my model. I then seek model parameters to make the simulated panel resemble key aspects of the data ("moment conditions"). When choosing possible moments, I carefully ensure to repeat calculations of the empirical moments as closely as possible with the simulated moments. Thus, I first describe my data and variable definitions, and then turn to the calculation of moments in the empirical and simulated panel and conclude with discussing the estimation procedure.

3.1 Data and variables

I retrieve accounting data from Compustat and market data from CRSP. My sample ranges from July 1963 until June 2021 and includes common stocks traded on the NYSE, Amex, or NASDAQ. I correct the stock return for delisting events, remove firms with negative book value which is alien to my model, exclude the utility and financial sectors to which real options theory is unlikely to apply, and address backfilling biases by only including stocks of firms after they have appeared in Compustat for at least two years (see Fama and French (1993)). I remove small and illiquid stocks by imposing a price filter of at least $2 at the beginning of each month. Using a price filter of $1, $5, or the bottom decile/quintile of NYSE stocks yields similar results.

Following Fama and French (1992, 1993), I calculate *BookToMarket* as ratio of book value of equity to market capitalization, where book value is stockholder's equity plus potential deferred taxes and investment tax credits minus preferred stocks. Accounting data from the fiscal year ending in calendar year $y-1$ is matched with market data from December of year $y-1$ and used from July of calendar year y onwards. Accordingly, portfolios based on *BookToMarket* are rebalanced annually at the end of every June. I correct portfolio weights if stocks delist during the portfolio holding period.

Following Schwert (1989) and Carlson et al. (2010), I estimate a firm-level volatility by the annualized standard deviation of daily stock returns within the prior calendar month. For robustness, I also use the daily returns of the prior twelve months. To ensure reliable inference, I require at least 15 (or 200) daily returns to be non-missing over the rolling window, which is considerably more conservative than Campbell et al. (2008) who only require five observations over three months to be non-missing when calculating realized return volatilities. This popular realized variance estimator avoids the strict parametric assumptions of, for example, GARCH models. Stock-level volatility is winsorized at the 0.5^{th} and 99.5^{th} percentiles per month to mitigate outliers.

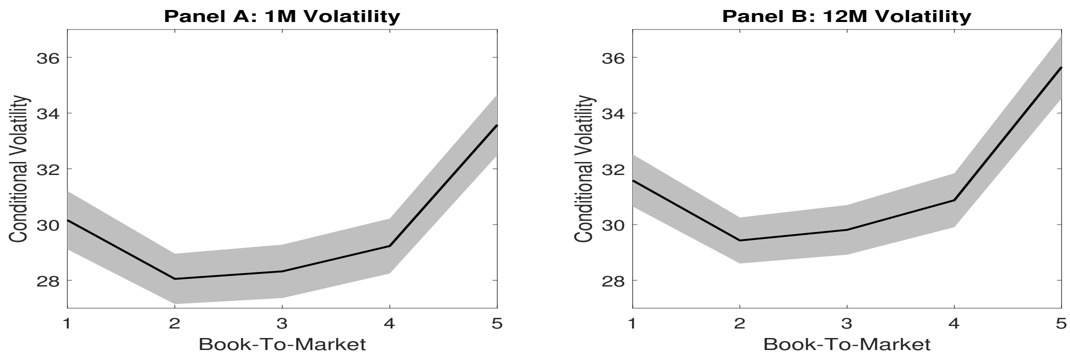

Figure 2: This figure plots the time series means of value-weighted averages within quintile portfolios sorted on book-to-market ratios of firm-level realized volatilities, estimated using rolling windows of either one month (Panel A) or twelve months (Panel B). All numbers are annualized and reported as percentages. The gray areas are standard errors corrected for heteroscedasticity and serial correlation following Newey and West (1987) with a lag length of six.

3.2 Choice of moment conditions

In this section, I explore what moment conditions best capture the positive value premium and its U-shaped return volatilities. To this end, I investigate whether portfolio volatilities are U-shaped as a result of covariances across firms or whether the high return volatility is already a firm characteristic of the average stock in the "high" and "low" portfolios. To disentangle the blending of average volatilities and covariances in the calculation of portfolio volatility, I group stocks into quintile portfolios based on book-to-market ratios and calculate the value-weighted average of single-stock volatilities within each quintile portfolio while ignoring correlations between stocks. Figure 2 plots the time series means of these average volatilities with Panel A showing return volatilities calculated from the prior month and Panel B showing return volatilities calculated over the prior year. Both panels confirm strong U-shaped volatilities and thereby confirm that the high volatility in the extreme portfolios drives from high firm-level volatility, and not from correlations between stocks. Put differently, high volatility is already a characteristic of the *average* value and growth firm.

I am now in the position to formulate my moment conditions. First, I employ the value-weighted average of excess returns of stocks in the quintile portfolios as first set of five moment conditions which define the value premium. I add the value-weighted average of firm-level volatilities within each portfolio as additional five moment conditions. These moment conditions are closely related to those from Liu et al. (2009), speak directly to the economic driver of U-shaped volatilities as seen in Figure 2, and, crucially, align with the outcomes variables from my real options model. I employ value-weighted averages within quintile portfolios of a data sample which excludes penny stocks to ensure that these patterns apply to a significant proportion of market capitalized equity and are economically important. Employing decile portfolios instead of quintiles yields similar estimation results.

I deliberately avoid employing Sharpe ratios, average returns divided by standard deviations, as alternative moment conditions because it is possible to fit these ratios perfectly while incorrectly estimating the level and shape of the first two moments themselves. Conversely, jointly matching means and variances ensures mechanically that Sharpe ratios are fitted as well.

3.3 Simulated moments

I continue with outlining how I simulate a panel of firms and conduct portfolio sorts on this data. Per simulation, I generate an economy with 1,500 firms for 720 months (60 years). The panel size resembles the average CRSP-Compustat universe after removing microcaps. I employ a Milstein discretization to simulate sample paths for total productivity and its variance and eliminate the impact of initial conditions by removing the first ten years for each firm as "burn-in" period.

Firms choose and adjust their capital stock by exercising their growth options. Firms may fully shutdown if they face low productivity and are unprofitable. To avoid that such firm shutdowns impact the panel size, I follow Carlson et al. (2004) and Cooper (2006) and permit new firms to enter whenever an existing firm exits. In this case, I add a new firm whose first ten years have been dropped.

In addition to a complete shutdown due to economic unprofitability, I allow firms to go private and leave the economy depending on an exogenous Poisson process $N_{i,t}$ whose arrival time follows an exponential distribution with parameter $\eta > 0$, see Carlson et al. (2004) and Hackbarth and Johnson (2015). Equivalently, this random lifetime can be interpreted as economic depreciation or stochastic obsolescence with cash flows being additionally discounted at rate η. I fix the leaving rate at $\eta = 0.11$ to match a median lifetime of 6.2 years of public firms, in line with Ai et al. (2022).

To conduct portfolio sorts on the panel of simulated firms, I measure a firm's market capitalization at the end of month t by its value, $W_{i,t}$. I proxy a firm's book value by the installation value of its productive capacity, $k\bar{K}_{i,t}$. For summary statistics in later parts of the paper, I calculate a firm's gross profits as sales revenue net production and maintenance costs, $\Pi_{i,t}$. Investment rates are $\frac{\bar{K}_{i,t+1}}{\bar{K}_{i,t}} - 1$. Finally, I use Equations (13) and (16) to calculate the expected return and return volatility for each firm-month observation. Firms report accounting variables (such as $\bar{K}$ and Π) at a quarterly and annual frequency. Monthly profits are compounded, or financed if negative, at the risk-fee rate of return. To best mimic the empirical analysis, investors use quarterly (annual) information with a three (six) month reporting delay. The estimation results are not sensitive to dropping this timing convention.

3.4 Choice of model parameters

I conclude this section by discussing the estimation procedure including parameter choice, objective function, and weighting matrix. I estimate five out of the thirteen model parameters which are key drivers of the model's economic channels and thus clearly identified and, crucially, to also avoid overfitting by using a large number of degrees of freedom. The remaining eight parameters do not drive the main channels of the model and are thus either estimated from the data or calibrated from prior studies. The estimation results can accommodate reasonable variations in these parameters. I fix the expected drift of aggregate productivity at the annualized growth rate of quarterly GDP growth, $\alpha = 0.031$, which represents the growth in the economy. I set the long-run average variance of aggregate productivity to the annualized unconditional volatility of GDP growth, $\bar{v} = 0.046^2$. These choices align with the calibrations of Cooper (2006) and Kogan and Papanikolaou (2013). Employing growth in quarterly TFP or monthly industrial production as alternative proxies yields similar results. I estimate the instantaneous correlation between first and second moment shocks from the correlation of daily changes in S&P 500 and VIX, suggesting $\rho = -0.79$. The persistence of the variance process has little impact on the asset pricing moments and I set $\kappa^* = 5.53$, which is the maximum likelihood estimate when fitting the square root diffusion in Equation (3) to the VIX. Following Cooper (2006), I normalize the installation costs to unity, $k = 1$, and I set the quasi-fixed adjustment costs to be $k_p = 0.12$ which aligns with Bloom (2009) and Cooper and Priestley (2016). I choose $\psi = 0.58$ to capture decreasing returns-to-scale which lies symmetrically between the estimates of Hennessy and

Whited (2005, 2007) and is of similar magnitude to the values used by Livdan et al. (2009) and Bai et al. (2019), among others. Finally, I fix the annualized real risk-free rate at $r = 0.01$.

Let $\Theta = (\mu, \lambda, \sigma, \xi, f)'$ denote the remaining five parameters. In addition to the risk premiums μ and λ, I include the volatilities σ and ξ because they largely determine the magnitude of the firm-level elasticities to productivity and uncertainty shocks. Finally, I include f in the estimation because operating leverage is a key mechanism in my model. The consistent SMM point estimator, $\hat{\Theta}$, minimizes a criterion function which is a quadratic form measuring the weighted sum of squared average "errors,"

$$\hat{\Theta} = \underset{\Theta}{\text{argmin}} \; Q(\Theta) = (M_T - m_T(\Theta))' W_T (M_T - m_T(\Theta)), \tag{17}$$

where M_T is a vector of empirical moments, $m_T(\Theta) = \frac{1}{S}\sum_{s=1}^{S} m_T^{(s)}(\Theta)$ is a vector of simulated model moments, and W_T is a positive definite weighting matrix. I choose $W_T = \Sigma_T^{-1}$ to be the inverted covariance matrix of the the empirical moments. I correct Σ_T for heteroscedasticity and autocorrelation using a Bartlett kernel with a lag of six. This choice for W_T is optimal, puts more weight on moments that are more accurately estimated and that are less correlated with other moments, takes the different units of mean returns and return variances into account, is independent of the parameters, and has favorable finite-sample properties, see Bazdresch et al. (2018). In line with Kogan et al. (2023), I average the model moments over $S = 100$ simulations and find $\hat{\Theta}$ using simulated annealing to avoid local minimums, see also Bloom (2009).

4 Estimation results

In this section, I report the results of the strucutural estimation when fitting my model to the means and volatilities of portfolios sorted on book-to-market ratios. I first report parameter estimates alongside standard errors. To this end, let $N_m = 10$ denote the number of imposed moment conditions, $N_p = 5$ the number of sought model parameters, and $D = \frac{\partial m}{\partial \Theta}$ the $N_m \times N_p$ Jacobian matrix, containing partial derivatives of the simulated moments. I calculate D using central differences, averaged over different step sizes as suggested by Bloom (2009). I estimate the $N_p \times N_p$ covariance matrix of the SMM point estimator $\hat{\Theta}$ from its asymptotic distribution,

$$\sqrt{T}\left(\hat{\Theta}-\Theta_0\right) \overset{a}{\sim} N\left(0, \left(1+\frac{1}{S}\right)\left(D'WD\right)^{-1} D'W\Sigma WD\left(D'WD\right)^{-1}\right), \tag{18}$$

where the factor $1+\frac{1}{S}$ corrects for simulation noise. Panel A of Table 1 lists the estimated parameters alongside their standard errors. I estimate the volatility of variance, ξ, to be 0.10 which equals about twice the long-run mean of aggregate volatility, $\sqrt{\bar{v}} = 0.046$. This relation echoes the dynamics of VIX whose volatility is about twice as large as its long-run average. For example, maximum likelihood estimates of the square-root diffusion (3) to VIX suggest $\bar{v} = 0.19^2$ and $\xi = 0.5$. The level of idiosyncratic volatility of productivity, σ, is estimated to be 0.15. Total volatility of θ is thus $\sqrt{\bar{v}+\sigma^2} = 0.16$ which is comparable to Carlson et al. (2004) and Cooper (2006). Furthermore, as with stock returns, most variability in total productivity is driven by idiosyncratic shocks. Firm fundamentals are less volatile than return volatility, as in the data. The fixed maintenance cost parameter, f, is estimated to be 0.085 which is close to Hackbarth and Johnson's (2015) SMM estimate. The estimates for risk premiums compensating shocks to the business cycle and economic uncertainty, μ and λ, are 0.08 and –79.24, respectively. All five parameters are estimated accurately with small standard errors.

The magnitude of μ and λ deserves further exploration. Unlike the parameter μ which can be interpreted as the expected return of a tradable portfolio mimicking productivity shocks, the variance risk premium λ scales with variances (squared percentages) and thus is of an entirely different magnitude. Indeed, the magnitude of μ and λ is best understood by the magnitude of the corresponding elasticities that take the differences in units into account, see Equation (13). The productivity elasticity $\Omega^{(\theta)}$ tends to take values larger than one while the uncertainty elasticity $\Omega^{(v)}$, measuring percentage changes in the firm value given percentage changes in economic uncertainty, is orders of magnitude smaller, hence requiring a large point estimate for λ to exert impact on expected stock returns. To illustrate this effect, I aggregate all stocks into one single portfolio and calculate the value-weighted expected return, alongside its two components, $\Omega^{(\theta)}(\mu - r)$ and $\Omega^{(v)}\lambda$. The time series mean of the annualized expected return of the resulting market portfolio is 6.74% which is close to the data. The

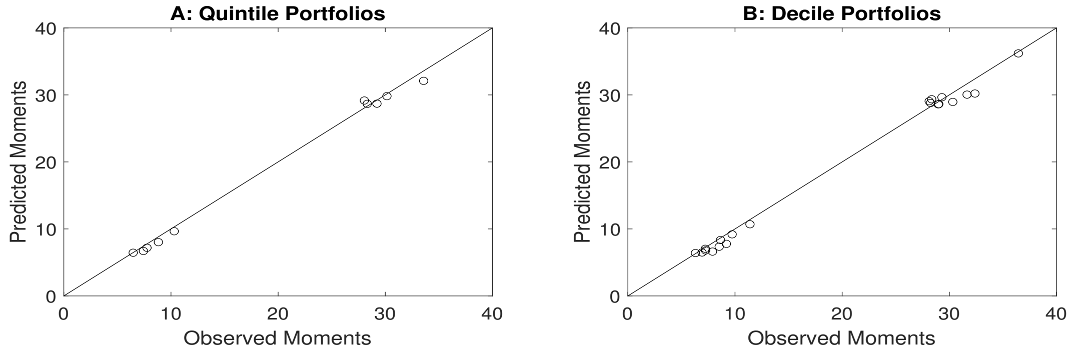

Figure 3: This figures illustrates the model fit by scattering data moments (x-axis) and simulated moments (y-axis). Panel A shows the means and volatilities of quintile portfolios targeted in the estimation. Panel B shows means and volatilities of decile portfolios. The solid black line has 45°. All numbers are annualized and reported as percentages.

average productivity component is 11.52% while the average uncertainty component amounts to –4.78%. Put in the right units, this variance risk premium now aligns with the literature and lies between the estimates of Ang et al. (2006) who estimate the variance risk premium to be about –2% p.a. using a VIX mimicking portfolio and Bali et al. (2017) who estimate the variance risk premium to be about –6% p.a. using cross-sectional regressions on loadings on Jurado et al.'s (2015) macroeconomic uncertainty index.

TABLE 1 ABOUT HERE.

I visualize the estimation results in Figure 3 by plotting both observed data moments and predicted simulated moments. The closer the data points align with the 45° line, the better the model fit. Panel A of Figure 3 shows ten moments: the means and volatilities of quintile portfolios targeted in the structural estimation while Panel B of Figure 3 shows the average means and volatilities of decile portfolios sorted on book-to-market ratios. The data points around 6% and 12% correspond to the portfolio mean returns while the data points around 28% to 36% represent the averages of conditional return volatilities. Overall, the model aligns well with the data and replicates monotone expected returns and U-shaped return volatilities. When fitting a regression line through the scatter plot, I estimate a slope coefficient of 1.01 (standard error: 0.02) with R^2=0.996, illustrating how closely both sets of moments mirror each other. Panel A of Table 2 lists all ten moments individually. To quantify the model fit, I next calculate the mean absolute error (MAE) of the estimation residuals.

The MAE for all ten moments is only about 66 basis points. The MAE achieved here amounts to only a sixth of that reported by Liu et al. (2009). Average returns are slightly better fitted than return volatilities, reflecting the difference in magnitude. Using mean relative errors, return volatilities are in fact fitted more accurately than expected returns (2.5 basis points versus 6.6 basis points). To sum up, my estimation not only matches the dynamics of the "high-minus-low" portfolio but fits the level and shape of the average means and volatilities of all anomaly portfolios.

TABLE 2 ABOUT HERE.

Because the moment conditions outnumber the model parameters, the structural estimation is overidentified, and I can test whether all pricing errors ("alphas") jointly differ statistically significantly from zero. The test statistic is $J = \left(1 + \frac{1}{S}\right) Q(\hat{\Theta})$ and follows asymptotically a χ^2 distribution with $N_m - N_p$ degrees of freedom. Intuitively, the test statistic is the weighted sum of squared pricing errors. The p-value for my estimation is small (<0.00), suggesting that this formal test is too high of a hurdle. Indeed, contemporaneous benchmark papers such as Gonçalves et al. (2020) and Kogan et al. (2023) are also rejected by the J-test, turning this test into a high bar to pass, in line with Newey's (1985) critique that the J-test is prone to misspecification. Overall, with pricing absolute errors averaging at only 66 basis points, the model successfully jointly matches average returns and return volatilities.

5 Economic mechanism

In this section, I explain the channels in the model that allow the model to jointly fit average returns and return volatilities. To do so, I illustrate the mechanics and channels of the model and how each parameter is identified by a clear set of moment conditions.

5.1 Model channels

The key variables in my model that drive expected returns and return volatilities of stocks is their responsiveness to the business cycle ($\Omega_{i,t}^{(\theta)}$) and economic uncertainty ($\Omega_{i,t}^{(v)}$), see Propositions 2 and 3. These sensitivities depend on firms' asset decomposition, namely how much of their value derives from assets-in-place and from growth opportunities. I first revisit these two elasticities which will then clarify their impact on expected returns and return volatilities.

Using simple algebra, the productivity elasticity from Equation (14) can be understood as follows

$$\Omega_{i,t}^{(\theta)} = 1 + \frac{f\bar{K}_{i,t}/r}{W_{i,t}} + \frac{GO_{i,t}}{W_{i,t}}\left(\Omega_{GO}^{(\theta)} - 1\right), \tag{19}$$

which combines operating leverage arising from fixed maintenance costs (first term) with risky growth options (second term). The "operating leverage ratio" $\frac{f\bar{K}_{i,t}/r}{W_{i,t}}$ is proportional to firms' book-to-market ratio, $\frac{k\bar{K}_{i,t}}{W_{i,t}}$ and is large for value firms burdened with much excess capital. Turning to the second term, $\Omega_{GO}^{(\theta)} \gg 1$ measures the riskiness of growth options and how much their value comoves with the economy.[3] This inherent option leverage is scaled by the proportion of growth options to the total firm value. All in all, the productivity elasticity $\Omega_{i,t}^{(\theta)}$ is high for value firms due to their operating leverage and high for growth firms tilting to risky growth options whose values strongly comove with the business cycle. This makes $\Omega_{i,t}^{(\theta)}$ U-shaped as a function of firms' book-to-market ratios.

Turning to the firm value's responsiveness to economic uncertainty, we can interpret the variance elasticity of a firm from Equation (15) as follows

$$\Omega_{i,t}^{(v)} = \frac{GO_{i,t}}{W_{i,t}}\Omega_{GO}^{(v)}, \tag{20}$$

which simply follows from assets-in-place being unaffected by changes in economic uncertainty. As growth options are convex claims, their values increase with heightened uncertainty ceteris paribus, $\Omega_{GO}^{(v)} > 0$. The sensitivity of the firm value to economic uncertainty is thus largely a function of how much the firm value is attributable to growth options. Growth firms are very responsive to economic uncertainty compared to value stocks.

Figure 4 visualizes these patterns. To do so, I simulate a panel of artificial firms using the parameters from Section 4 and sort those firms into quintile portfolios based on their book-to-market ratio. I

[3] While $\Omega_{GO}^{(\theta)}$ depends on state variables and all model parameters, I find the elasticity derived from a standard geometric Brownian motion to be good approximation, $\Omega_{GO}^{(\theta)} = \frac{1}{2} - \frac{r-\delta}{\nu^2} + \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\nu^2}\right)^2 + \frac{2r}{\nu^2}} \gg 1$, where $\nu^2 = \bar{v}^* + \sigma_I^2$. Using the parameter values from the SMM estimation, this rule of thumb suggests $\Omega_{GO}^{(\theta)} = 3.58$.

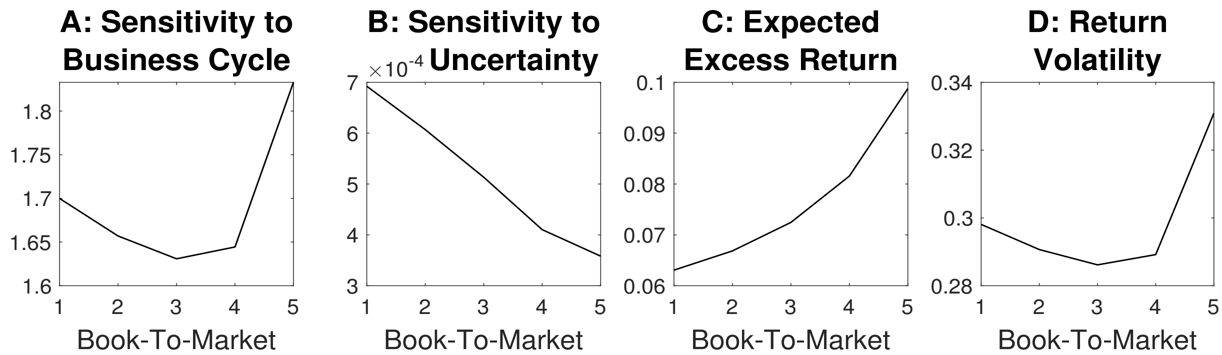

Figure 4: This figure plots the time series means of value-weighted averages of simulated firm-level productivity elasticities (Panel A), variance elasticities (Panel B), expected returns (Panel C), and return volatilities (Panel D). Numbers in Panels C and D are annualized and reported as percentages. The simulation procedure is as described in Section 3.3 with parameter values taken from the estimation results in Section 4.

then calculate the value-weighted average of productivity elasticities and variance elasticities within each portfolio and plot the corresponding time series means in Panels A and B of Figure 4. Panels C and D then report the resulting value-weighted expected return and return volatility in the average month which match the moments observed in the data. The productivity elasticity is U-shaped in Panel A of Figure 4, suggesting that both, growth firms and value firms are sensitive to the business cycle. Intuitively, growth firms derive their value from growth options which make their stocks very procyclical and aligned with the business cycle. Value stocks in portfolio five are burdened with operating leverage which makes those stocks very sensitive to the aggregate economy. Panel B of Figure 4 plots the average variance elasticity of each quintile portfolio. The relationship is monotone and growth stocks are more sensitive to economic uncertainty than value stocks. Intuitively, this is because assets-in-place are not responsive to economic uncertainty, unlike volatility-sensitive growth options.

Panel C of Figure 4 shows that expected excess returns increase monotonically across portfolios. As of Proposition 2, the expected return compensates for exposure to total productivity and economic uncertainty. When added up, the U-shaped risk premium earned for exposure to the business cycle and the monotonously declining risk premium earned for exposure to economic uncertainty result in increasing expected returns and a positive value premium. Intuitively, the loading on economic uncertainty removes that left wing of the "U," which generates the monotonicity in expected returns. Put differently, while both value stocks and growth stocks have high market betas and are exposed to the business cycle, growth stocks additionally lower their systematic risk by loading on a second factor, economic uncertainty, which lowers their expected returns.

Finally, return volatilities are U-shaped in Panel D of Figure 4. To understand how the model fitted the data quantitatively, recall Equation (16). Return volatility has three components which depend on the *squared* productivity elasticity, the *squared* variance elasticity, and the *cross-product* of productivity elasticity and variance elasticity. While productivity elasticities are numbers typically larger than one (see Equation (19)), variance elasticities are of completely different magnitude (multiplies of 10^{-4} in my estimations). This is because a one percent increase in variance (i.e., a 0.5% change in volatility) might raise 0.046^2 to 0.0462^2. The corresponding change in firm value will be small. This difference in magnitudes between $\Omega_{i,t}^{(\theta)}$ and $\Omega_{i,t}^{(v)}$ could easily be overcome for expected returns by estimating a variance risk premium that takes this difference of units into account. However, return volatility is almost entirely attributable to firms' responsiveness to total productivity due to the larger magnitude. Operating leverage and levered growth options then create high return volatilities of value and growth stocks. Quantitatively, sensitivity to economic uncertainty plays a negligible role at best.

5.2 Parameter identification

I further corroborate the model channels by illustrating what moment identifies which parameter. Economic models are often sufficiently complex such that most moments are impacted by various parameters which can make identification difficult and the model appear as a black box. Andrews et al. (2017) stress how the sensitivity of moments with respect to parameters sheds further light on the economic mechanisms of the model and what moments identify which parameters. I thus follow Hennessy and Whited (2007), Kim et al. (2022), and Kogan et al. (2023), I calculate sensitivities of simulated moments with respect to key model parameters. Using central differences, the sensitivity of moment m with respect to parameter a is

$$\delta_{m,a} = 100 \cdot \frac{\partial m}{\partial a} = 100 \cdot \frac{m(\frac{3}{2}a) - m(\frac{1}{2}a)}{\frac{3}{2}a - \frac{1}{2}a} = 100 \cdot \frac{m(\frac{3}{2}a) - m(\frac{1}{2}a)}{a}. \tag{21}$$

I multiply $\delta_{m,a}$ by 100 because all moments (average returns and return volatilities) percentages.

Table 3 displays the resulting sensitivity matrix for $\delta_{m,a}$ by showing in each column by what percentage each of the ten moments change if either fixed maintenance costs (f), idiosyncratic volatility (σ), productivity risk premium (μ) increase by 0.01, or if the variance risk premium (λ) increases by one. The different scaling takes the different units of the model parameters into account. The results are intuitive. An increase in fixed costs largely impacts the expected return and return volatility of value stocks as they are burdened with lots of excess capital and are highly operationally levered. For example, the return volatility of growth stocks increases by only 42 basis points while the return volatility of value stocks rises by 122 basis points if f changes from 8.5% to 9.5%. Changes to idiosyncratic volatility increases all return volatilities while moderately impacting expected returns, in line with Equation (16). Indeed, the sensitivities for return volatilities are 13–40 times larger than the impact on expected returns. Still, increases in idiosyncratic volatility lead to higher expected returns. To understand this, note that there are two channels at work: first, higher idiosyncratic volatility increases the value of growth options and thereby increases the relative role of growth opportunities in firms' asset mix. Second, higher idiosyncratic volatility makes firms less responsive to aggregate shocks and thereby pulls $\Omega_{GO}^{(\theta)}$ and $\Omega_{GO}^{(v)}$ closer to zero, see Babenko et al. (2016). Lower responsiveness to productivity shocks make growth options less risky while lower sensitivity to variance shocks makes them riskier. Overall, these channels add up to a moderate increase in expected returns following higher idiosyncratic volatility.

Table 3 also depicts how changes in risk premiums affect expected returns and return variances. If the risk premium associated to productivity shocks, μ, increases by 0.01, then the expected returns of all portfolios increase significantly, reflecting the higher discount rate, see Equation (13). Return volatilities decline if μ rises because the higher discount rate particularly lowers the value of growth options, which is the main driver of return volatilities, see Equation (19). Indeed, the return volatility of growth stocks declines particularly strongly. An increase in the negative variance risk premium λ (which is not a percentage number) from –79.24 to –78.24 (i.e., moving the price of variance risk closer to zero) reduces the hedging reward of growth stocks and thus increases their expected return but barely impacts return volatilities.

Table 3 about here.

6 Extensions and model validation

In this section, I show that my main estimation result – jointly targeting average returns and volatilities – also explains additional stylized facts that were not part of the estimation. This exercise serves as "cross-validation" (or "out-of-sample test") of the model. In particular, I reproduce the stylized facts that the value premium is countercyclical, persistent, and not explained by the CAPM. The model further matches untargeted moments such as the average book-to-market ratios of the quintile portfolios, average investment rates, the equity premium, and further factor premiums.

6.1 Countercyclical value premium

The value premium is inversely linked to the business cycle, typically being strong during recessions and weak during booms. Among others, Zhang (2005) and Petkova and Zhang (2005) document this pattern and conjecture a countercyclical driver for the value premium. In my model, the value premium mainly arises from growth stocks being sensitive to economic uncertainty which offers them a partial hedge from recessions. During poor states of the economy, economic uncertainty tends to increase which, in turn, raises the value of growth options and the responsiveness of firms with many growth options to economic uncertainty. Put differently, $\Omega_{i,t}^{(v)}$ increases with v_t which means that the variance risk premium reduces the systematic risk of growth options in particular during recessions characterized by high economic uncertainty. I illustrate this argument in Figure 5. I plot the expected excess returns of two firms, a value firm (solid line) and a growth stock (broken line) as a function of economic uncertainty ($\sqrt{v_t}$). The spread between both expected returns indicates a value premium. For clean identification of the impact of economic uncertainty on expected returns, I keep the productivity level constant and only vary $\sqrt{v_t}$. As economic uncertainty increases, the growth firm which derives most of its value from growth options has particularly low expected returns, reflecting the increased sensitivity to economic uncertainty (high $\Omega_{i,t}^{(v)}$). The value firm is much less affected by higher economic uncertainty, leading to a high value premium during recessions characterized by heightened economic uncertainty.

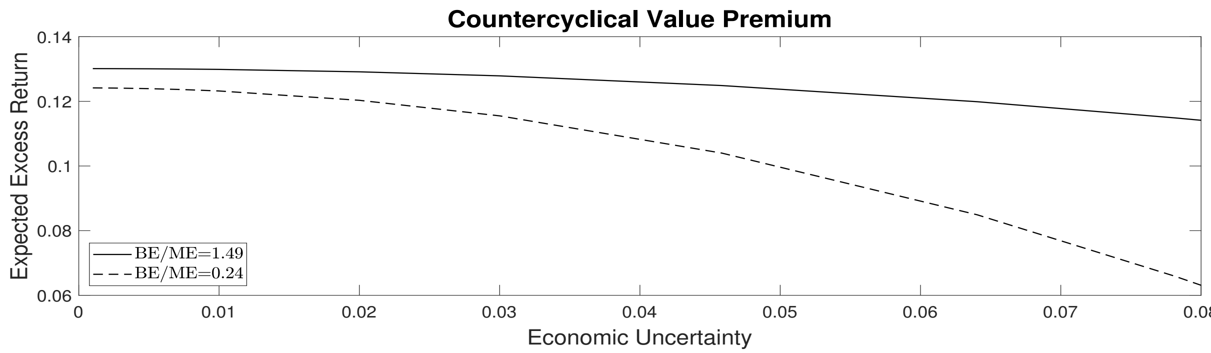

Figure 5: This figure plots the expected excess returns of two hypothetical firms as a function economic uncertainty ($\sqrt{v_t}$). The solid line corresponds to a value firm with high book-to-market ratio (BE/ME=1.49) while the broken line corresponds to a growth firm with low book-to-market ratio (BE/ME=0.24). Parameter values are taken from the estimation results in Section 4.

6.2 Persistent value premium

Another stylized fact about the value premium is its persistence as a long-lived return premium over extensive holding periods, as opposed to short-lived premiums like momentum. My model also reproduces this stylized fact. Intuitively, the book-to-market ratios is very sticky and does not vary much over time. This persistence is true in the data and in my model and generates the longevity in the value premium. Figure 6 visualizes this behavior by plotting the expected returns of quintile portfolios sorted on *lagged* book-to-market ratios. I use stale information to show that even book-to-market ratios from several years ago possess some power in explaining the cross-section of expected stock returns. For example, the black broken (solid) line represents the expected excess returns of the high (low) book-to-market ratio quintile when the sorting characteristic has been lagged by one year, two years, three years, etc. The lines in between correspond to quintile portfolios two, three, and four. If there is no lagging and the most recent book-to-market ratio is used for sorting, the value premium amounts to 3.5% per annum (left end of the x-axis). As I use more outdated information to form portfolios, the return spread between high and low book-to-market portfolios declines and the value premium becomes weaker. Nonetheless, there is still a sizeable value premium of about 2.0% when lagging book-to-market ratio by five years, which aligns well with the 2.3% value premium among large stocks five years after portfolio formation documented by Fama and French (1995).

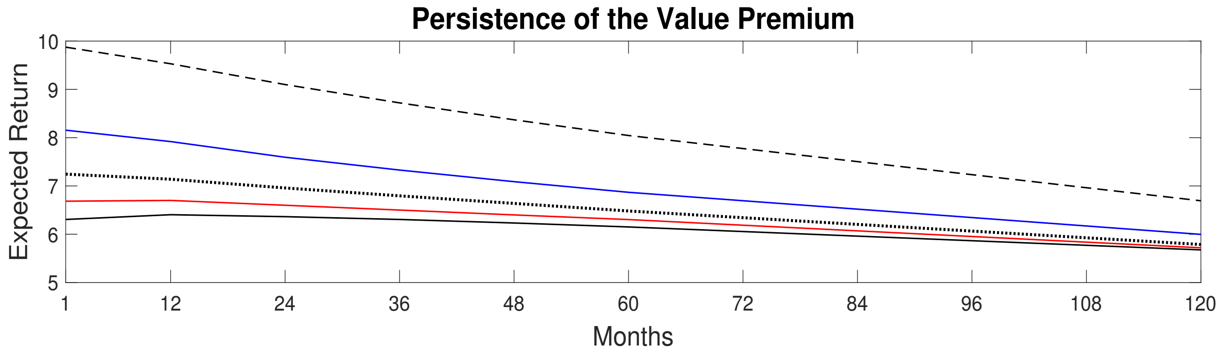

Figure 6: This figure plots the expected excess returns of simulated quintile portfolios sorted on *lagged* book-to-market ratios as function of the lags. The solid black line corresponds to growth firms (quintile 1), the red line to quintile 2, the dotted black line to quintile 3, the blue line to quintile 4, and the broken black line to value firms (quintile 5). Parameter values are taken from the estimation results in Section 4.

6.3 Failure of CAPM

One of the most well-known facts in asset pricing is that the conditional CAPM fails to accurately describe the cross-section of average stock returns, see Lewellen and Nagel (2006) among others. My model replicates this failure of the conditional CAPM. Intuitively, there are two sources of risk in my model: first, there is productivity which tracks the business cycle and thus resembles the market portfolio and second, there is time-varying economic uncertainty which earns its own independent risk premium. Because the conditional CAPM is a single-factor model, it cannot fully explain expected stock returns in my model. To further illustrate this failure, I calculate the ten moments (average returns and return volatilities of quintile portfolios) using the estimated parameters from Section 4 while shutting down the variance risk premium as second channel for expected returns, that is, I counterfactually set $\lambda = 0$. Table 4 compares the simulated moments with the observed moments in the data and calculates mean average errors. Without the variance risk premium channel, the model fit deteriorates and the MAE increases six-fold from 66 basis points to 384 basis points. This failure cannot be fixed by re-estimation either because without the second source of risk, the model makes identical predictions for means and variances, contradicting the different shapes observed in the data.

Table 4 about here.

6.4 External validity

I conclude this section by confirming that the parameters estimated to fit average returns and volatilities of value and growth stocks also generate a panel of simulated firms that matches further moments that were not part of the estimation objective function. While there is a plethora of potential moments to choose from, I focus on those that can be clearly identified in my model and are relevant to its channels and choice variables. To start with, I compare the average book-to-market ratio of the quintile portfolios in the data and the model. Because my model features investment decisions as main choice variable, I add average Tobin's q and average investment rates as additional moments. I also include average gross profitability. While my paper is firmly about the cross-section of stock returns, I next add the equity premium and average return volatility of firms as further benchmarks. Finally, I choose the average premium of various return factors that are known to explain asset prices. Table 5 compares all these untargeted moments from the data with their simulated counterparts.

TABLE 5 ABOUT HERE.

In Panel A of Table 5, I focus on the value-weighted average of book-to-market ratios across the quintile portfolios. The sorting variable is matched well on average with book-to-market ratios slightly lower in my model than in the data. A reason might be that book-to-market ratio is highly skewed in the cross-section and that my model implies less extreme cross-sectional dispersion.

I next turn to the firm policies in Panel B of Table 5 and compare Tobin's q, gross profitability, and investment rates between the data and my model. To calculate these numbers, I determine the value-weighted average firm characteristic in each month and then report the time series means of those averages. This panel is particularly important because Clementi and Palazzo (2019) stress that investment-based asset pricing models often fail to jointly explain equity returns and firm policies: estimating the model on firm policies does not generate enough variation in expected returns, while estimating the model asset pricing moments often results in unrealistically extreme firm policies. Panel B of Table 5 suggests that Clementi and Palazzo's (2019) critique does not apply to my model. Both gross profitability and investment rates are very closely matched, though not targeted by the estimation. The slightly lower book-to-market ratios in the simulated model from Panel A turn into an inflated Tobin's q that suggests higher relative valuation ratios in the simulated economy.

Panel C of Table 5 turns to the aggregation across all firms. To do so, I calculate the value-weighted average of excess returns and return volatilities for every month and then report the corresponding time series means in Panel C. My model closely matches the average equity premium of 7.30% in the data with 6.80% in the simulations. The average volatility is even closer matched (29.13% in the data versus 29.26% in the model). While none of these two moments were directly part of the estimation, the estimation did target the cross-sectional distribution of expected returns and return volatilities. Thus, it is not surprising that these quantities are also matched once aggregated.

Finally, I turn to other factors in the cross-section of average stock returns in Panel D of Table 5. For each sorting variable, I sort stocks into quintile portfolios, calculate value-weighted average returns, calculate the difference between the high and low portfolio, and report the corresponding time series means as factor premium. To start with, I study three alternative proxies for a firm's relative valuation: earnings-to-price ratio (profits divided by market capitalization), sales-to-price (sales divided by market capitalization), and long-term reversal (compounded returns over prior 60 months). The model generates a positive premium if sorting stocks on earnings-to-price and sales-to-price as well as a negative premium of stocks sorted on long-term reversals. The premiums for sales-to-price and earnings-to-price are higher yet in the data because these strategies also loads on the profitability premium which is beyond the scope of my model. Another variable related to the value premium is the negative TFP premium documented by İmrohoroğlu and Tüzel (2014), indicating that productive firms earn lower expected returns than unproductive firms. To replicate this, I sort stocks based on their total factor productivity ($\theta_{i,t}$ in the model), anticipating that productive firms are characterised by growth options which load strongly on the negative variance risk premium. Indeed, my model generates a corresponding premium of –3.00% p.a. which is close to the –2.55% reported by İmrohoroğlu and Tüzel (2014). The model furthermore generates a negative return premium when sorting stocks on idiosyncratic volatility, which aligns with Ang et al. (2006). Intuitively, firms with high idiosyncratic volatility are more sensitive to aggregate uncertainty which carries a negative risk premium. The larger tilt to economic uncertainty goes hand in hand with a larger proportion of growth options, echoing Bali et al. (2020) who also explain the IVOL discount using the convexity of the payoff of growth options. The model also prices market size: large stocks earn lower average returns than small stocks (–2.93% in the data, –2.51% in the model). Recently, Hou et al. (2021) propose a new expected growth factor which is based on expected future investment rates. A natural analogue in a real options setting is the distance of a firm's current productivity to the firm's next investment threshold. Sorting stocks on this distance to investment reveals indeed a sizeable positive return difference of 3.88% p.a. between the high and low quintile portfolio which aligns well with the positive pricing of the variable documented by Hou et al. (2021). All in all, these results confirm that the model does not only fit the targeted moments but also matches a series of additional "out-of-sample" moments.

7 Reduced-form evidence

In this section, I report several facts from non-parametric portfolio sorts that provide further reduced-form evidence for the main channels in my model. In particular, I document that market betas are U-shaped as a function of book-to-market ratios which indicates that growth and value stocks are indeed both sensitive to the business cycle. Turning to loadings on VIX, I find that growth stocks are more sensitive to economic uncertainty than value stocks. The two facts align directly with the two main channels of my model. As a final step, I also show U-shaped patterns exist in different measures of return volatility and within different size quintiles. Firm characteristics further underpin the main model channels and financial leverage seems an unlikely first-order explanation for U-shaped volatilities because these patterns also exist once I remove firms that are high levered or close to bankruptcy.

7.1 Market betas are U-shaped

In this first subsection, I document U-shapes in conditional market betas across portfolios sorted on book-to-market ratios. Using market betas to proxy a firm's sensitivity to the business cycle is common in the real options literature (see, e.g., Carlson et al. (2010), Hackbarth and Johnson (2015), and Lambrecht et al. (2016)). To do so, I estimate market betas each month using rolling windows by projecting daily excess stock returns on daily excess market returns over the last twelve

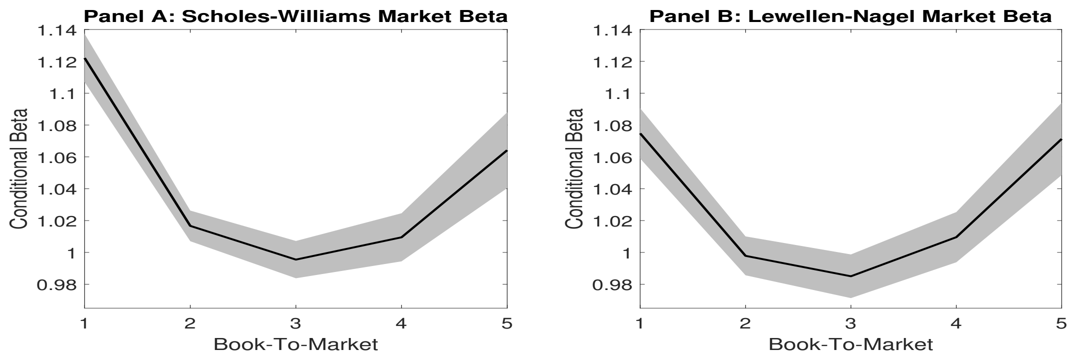

Figure 7: This figure plots the time series means of value-weighted averages within quintile portfolios sorted on book-to-market ratios of firm-level market betas estimated using a rolling window of twelve months, using the estimation method from Scholes and Williams (1977) (Panel A) or Lewellen and Nagel (2006) (Panel B). The gray areas are standard errors corrected for heteroscedasticity and serial correlation following Newey and West (1987) with a lag length of six.

months while requiring at least 200 daily return observations to be non-missing. I correct for potential asynchronous trading issues using the methods from Scholes and Williams (1977) and Lewellen and Nagel (2006) which incorporate lagged market returns. I winsorize market betas at the 0.5^{th} and 99.5^{th} percentiles per month to mitigate outliers. Given those market betas, I then group stocks into quintile portfolios based on book-to-market ratios as usual and calculate value-weighted averages of these conditional market betas for each portfolio. Figure 7 plots the time series means of those averaged betas and thereby displays the sensitivity of the average stock in each quintile portfolio during an average month.

Firms in the "high" and "low" portfolios do not only have volatile returns but also high loadings on market returns which translates into procyclicality and high sensitivity to the business cycle. These results apply to both ways of calculating market beta. Using the loadings from Lewellen and Nagel's (2006) method in Panel B as an example, growth stocks have on average a market beta of 1.075, while stocks in portfolio three have an average market beta of only 0.985 and value stocks have a market beta of 1.072 on average. These two differences are statistically significant. Given that the value-weighted average of market betas across all stocks is one by definition, these differences in value-weighted averages across quintile portfolios are economically very sizeable. The evidence of Figure 7 of U-shaped exposures to the business cycle as a function of book-to-market ratios thus supports the model channels described in Section 5.

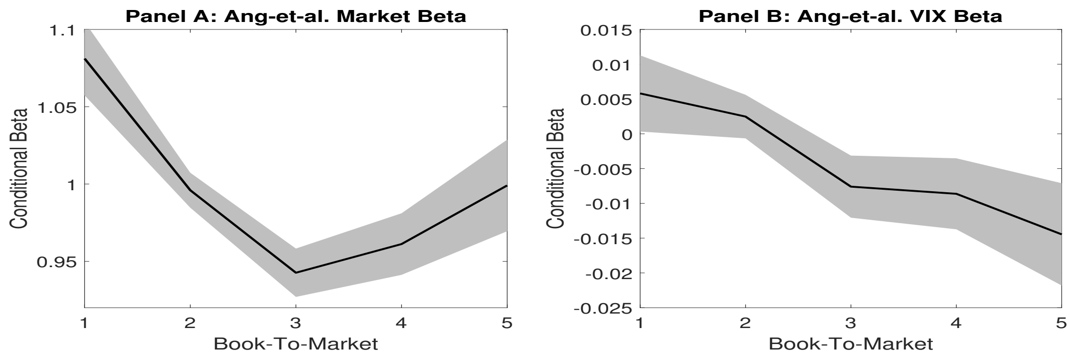

Figure 8: This figure plots the time series means of value-weighted averages within quintile portfolios sorted on book-to-market ratios of firm-level market betas (Panel A) and VIX betas (Panel B) estimated using a rolling window of twelve months following Ang et al. (2006). The gray areas are standard errors corrected for heteroscedasticity and serial correlation following Newey and West (1987) with a lag length of six.

7.2 Growth stocks are sensitive to economic uncertainty

I next report that growth stocks are more sensitive to aggregate uncertainty than value stocks. To do so, I repeat Ang et al.'s (2006) regression analysis and project daily single-stock excess returns on the span of daily market excess returns and daily changes in VIX. I estimate both factor loadings using a rolling window of twelve months if at least 200 daily return observations are available. The sample size for this exercise is January 1991 until June 2021. I winsorize both betas at the 0.5^{th} and 99.5^{th} percentiles per month to mitigate outliers. As usual, I then sort stocks into quintile portfolios based on book-to-market ratios and calculate the value-weighted average of these conditional betas within each portfolio. Figure 8 displays the corresponding time series means.

Echoing the prior subsection, both value stocks and growth stocks are strongly sensitive to the business cycle and market betas are U-shaped across portfolios (Panel A). The loadings on VIX innovations are not U-shaped but largely monotone suggesting that growth stocks are more sensitive to shocks to aggregate uncertainty than value stocks are (Panel B). This insight is indicative as VIX is only a noisy proxy for economic uncertainty and confounds expectations about return volatility with expected price jumps and risk premiums. The high volatility in VIX and the shorter available time series add further estimation imprecision to VIX betas. The sign and magnitude of the VIX betas are important. First, as with the elasticities in Figure 4, VIX betas are much smaller than market betas in magnitude, indicating that they contribute little to return volatilities. Second, VIX betas are negative for portfolios 3–5 but positive for growth stocks. Intuitively, a high VIX coincides with low stock returns. Panel B of Figure 8 shows that firms with lower book-to-market ratios posses a hedging feature and lose less (or even gain) value if VIX increases. This hedging property reflects the increase in the value of the firms' growth option in response to heightened economic uncertainty which lowers the expected return of growth stocks and results in a positive value premium.

7.3 Further channel validation

In this next subsection, I lend further support for the channels in my model using average firm characteristics. To do so, I sort stocks into quintile portfolios based on book-to-market ratios, calculate value-weighted averages of firm characteristics informative about my model channels within each portfolio, and report the time series means in columns one to five of Table 6. Columns six and seven report p-values for the null hypothesis that the characteristics do not differ across portfolios one and two and four and five. I follow Newey and West (1987) to correct for heteroscedasticity and serial correlation. All variables are winsorized at the 0.5^{th} and 99.5^{th} percentiles per month to mitigate outliers.

Table 6 about here.

My model first claims that growth firms are characterized by an abundance of growth opportunities. The primary measure for those is Tobin's q which is inversely related to book-to-market and thus mechanically high for firms in portfolio 1. To add further support for this channel, I follow Cao et al. (2008) and calculate the ratio of capital expenditure (capx) and a firm's fixed assets (net PPE) as one proxy for growth options as well as Peters and Taylor's (2017) total q measure which includes the value of intangible capital in the calculation of Tobin's q. Both measures confirm that growth firms indeed derive much value from growth options. My model next claims that value stocks are highly operationally levered. To test this, I follow Chen et al.'s (2022) and proxy fixed costs by selling, general, and administrative expenses (xsga) which I divide by firms' market value, closely aligned with my model's operating leverage channel identified in Equation (19). This variable increases monotonically across portfolios sorted on book-to-market, suggesting that value firms are indeed burdened with high operating leverage. To address market microstructure issues, Table 6 also shows that market size and Amihud's (2002) illiquidity measure are monotone across book-to-market portfolios, suggesting that neither is a likely explanation for the pervasive U-shaped patterns in return volatility. Alongside market size, institutional ownership is often used as a proxy for short sale constraints, see Nagel (2005). I document in Table 6 that institutional ownership, as reported in institutional holdings (13f forms) accessed via Refinitiv, is also largely monotone across portfolios such that short sale constraints seem unlikely to be able to explain U-shaped return volatilities. Finally, Lettau and Wachter (2007) point to equity duration as a potential explanation for the value premium. I calculate duration for each stock following Gonçalves (2021) which confirms that growth stocks are long duration assets while value stocks expect their cash flows to occur in the nearer future. The monotonicity of this variable again hints to the theory struggling to explain U-shaped return volatilities.

I next examine whether financing decisions may be an explanation for the high volatility in extreme portfolios. To do so, Table 6 also compares the leverage ratio and Ohlson's (1980) O-score for expected bankruptcy of the average firm in the five quintile portfolios. Following Fama and French (1992, 1993), I define leverage as ratio of total debt to the sum of total debt and market capitalization following the usual timing convention of book-to-market ratios. The results show a clear monotonic relationship across portfolios with value firms being more reliant on debt financing than growth stocks, echoing a large literature in corporate finance that documents a low debt capacity of risky growth options (see, e.g., Myers (1977)). Such a monotonic relationship is thus unlikely to generate U-shaped return volatility. Still, to move the analysis one step further, I calculate the cross-sectional median of either leverage ratio or O-score each month and remove firms strongly relying on debt or close to bankruptcy. As a result, the pooled average leverage ratio, for example, declines from 37.1% to only 18.2%. Doing so halves my sample and strikes a balance between the focus on stock returns less affected by financing decisions and retaining a large enough sample for reliable inference and statistical power. As before, I then group the remaining stocks into quintile portfolios based on book-to-market ratios, calculate the value-weighted average of firm-level volatilities estimated as annualized standard deviation of daily returns in the prior month, and visualize the corresponding time series means in Figure 9. Return volatilities are strongly U-shaped across portfolios, regardless whether we focus on firms with low leverage ratios (Panel A) or on firms far away from financial

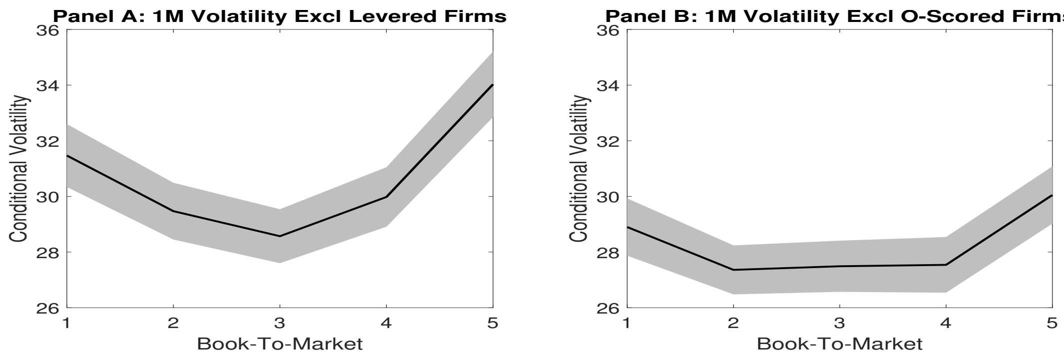

Figure 9: This figure plots the time series means of value-weighted averages within quintile portfolios sorted on book-to-market ratios of firm-level realized volatilities estimated using a rolling window of one month, after removing firms with high leverage ratios (Panel A) or high Ohlson (1980) O-score (Panel B). All numbers are annualized and reported as percentages. The gray areas are standard errors corrected for heteroscedasticity and serial correlation following Newey and West (1987) with a lag length of six.

distress (Panel B). The high volatility of the average value stock and growth stock thus seems unlikely to be primarily driven by financing decisions, motivating my modeling choice of all-equity financed firms. In unreported results, I document that the U-shapes in market beta remain unchanged when removing firms based on their leverage ratio or O-score. VIX loadings also do not change materially after the removal of strongly levered firms while the monotonic decline across book-to-market sorted portfolios becomes stronger once firms near bankruptcy are excluded.

7.4 Additional robustness

In this subsection, I show that U-shaped return volatilities are ubiquitous across different measures of return variance, do not mechanically arise from the price information included in book-to-market ratios, are robust to using alternative proxies for "value" than book-to-market ratios, and exist in different subsamples and in double sorts on market size and leverage ratio. To address the first point, I group stocks into quintile portfolios based on book-to-market ratios and calculate the value-weighted average of annualized firm-level volatilities calculated using a one month and twelve month rolling window, of annualized systematic and idiosyncratic volatility with respect to the market model calculated over the prior twelve months, and of implied volatilities from at-the-money single-stock options with 30 days to expiry (data available from January 1996). Panel A of Table 7 shows the corresponding time series means and p-values for testing whether these volatilities vary across portfolios. All five volatility measures show indeed strong U-shapes with volatilities in portfolios one and five being higher from those in portfolios two and four. Those differences are highly statistically significant.

Table 7 about here.

I next address Berk's (1995) critique of book-to-market ratios simply reflecting recent price information by sorting stocks into quintile portfolios based on one-year lagged book-to-market ratios. Fama and French (1992, 1993) match accounting data of the fiscal year ending in calendar year $t-1$ with market capitalization from the end of calender year $t-1$ to calculate book-to-market ratios and to sort stocks from July in calendar year t until June in calendar year $t+1$. This way, market data is between six and 18 months old during the portfolio holding period. In the following test, I lag the variable by an additional year to ensure that the portfolio sort is based on stale stock prices that are between 18 and 30 months old. Panel B of Table 7 displays the time series means of value-weighted averages of conditional return variances within quintile portfolios sorted on lagged book-to-market ratios. The table shows yet again that the average value and growth stock are both volatile during the portfolio holding period, regardless of how return volatility is measured. Lagging book-to-market by a second year gives very similar results. Because of the stale information in the sorting variable, these U-shaped return volatilities are unlikely to arise mechanically from the price information contained in book-to-market ratios.

As an additional robustness check, I document that return volatilities are also U-shaped when sorting stocks based on different proxies of being a cheap "value" stock. One such possible characteristic is the earnings-to-price ratio which is popular among practitioners. I calculate earnings-to-price ratios using the same timing convention as book-to-market ratio and employ income before extraordinary items (ib) as measure for earnings. As usual, Panel C of Table 7 shows the time series average of value-weighted averages of various measures of firm-level return volatility. Using realized variance in the prior calendar month as a proxy, average return volatilities for portfolios 1, 3, and 5 are 33.54%, 26.81%, and 30.40%, indicating a clear U-shape. In Panel D of Table 7 I repeat the above exercise using the inverse of Peters and Taylor's (2017) total q measure which divides market values by the combined value of physical book value and intangible value. This is important because Eisfeldt et al. (2022) argue that including intangibles can enhance value strategies. Using this new measure to identify value stocks, average return volatilities remain strongly U-shaped across portfolios with stocks in the first and fifth portfolio being more volatile than those in intermediate portfolios. Taken together, these two panels indicate that U-shaped volatilities are not sensitive to the way in which "being a value stock" is measured.

Return volatilities are also U-shaped across book-to-market sorted portfolio in different subsamples. In Panel E of Table 7, I split my sample from 1963 until 2021 into two blocks, one ranging from July 1963 until June 1991 and a second one ranging from July 1991 until June 2021. This choice aligns with the original data used in Fama and French (1992, 1993). Return volatilities are U-shaped in either period with the return volatilities in portfolios 1 and 5 being statistically significantly different from those in portfolios 2 and 4. Overall, return volatilities are larger in the second sample, which aligns with the findings of Fama and French (2021). Splitting the period in different subsamples (e.g., twenty year blocks) yields comparable results. All in all, U-shape return volatilities exist in different subsamples.

To gauge whether the U-shaped patterns in volatilities are just a size effect, I next perform dependent double sorts on market capitalization and book-to-market ratio. I first sort stocks into quintiles based on NYSE size breakpoints and then, within each size portfolio, I group stocks into five portfolios based on book-to-market ratios. For each of the 25 resulting portfolios, I calculate the value-weighted average of annualized firm-level return volatilities. Panel F of Table 7 shows the corresponding time series means and p-values for testing whether these volatilities vary across portfolios. Within each size quintile, return volatilities are U-shaped as a function of book-to-market ratio. The differences between portfolios one and two, and four and five are statistically significant. Thus, U-shaped volatilities are not just a size effect. Interestingly, return volatilities decrease across size quintiles as larger stocks are less volatile than small stocks. Crucially, this confirms that return volatilities are not always mechanically U-shaped in portfolio sorts but can be monotone across portfolios.

As a final robustness check, I conduct depend double sorts on leverage ratio and book-to-market ratios. As in the size double sorts, I first stocks into quintile portfolios based on NYSE breakpoints of leverage ratios and then, within each leverage portfolio, I sort the contained stocks into five portfolios based on their book-to-market ratio. Panel G of Table 7 displays the time series means of value-weighted cross-sectional averages of firm-level realized volatilities calculated over the last month, or the last twelve months. Return volatilities are U-shaped as a function of book-to-market ratios within each leverage portfolio. These U-shapes are almost always statistically highly significant.

The results in Figures 7–9 and Table 7 are robust to a battery of further methodological changes. Though unreported here for brevity, similar pictures emerge when using, for example, decile sorts instead of quintiles, weighting stocks equally, rebalancing monthly, or including penny stocks. Still, I deliberately choose to assign maximal weight to large and liquid stocks which drive market capitalization and matter most for real-world investors. If anything, return volatility is often more "U-shaped" when, for example, using equally-weighted averages within portfolios.

8 Conclusion

This paper studies average returns and return volatilities of portfolios sorted on book-to-market ratios. I document that while average returns increase monotonically across portfolios, return volatilities are U-shaped. Many leading behavioral and risk-based theories fail to account for these different shapes.

To explain these patterns in the data, I develop a dynamic real options model in which operating leverage and levered growth options make both value stocks and growth stocks sensitive to the business cycle. As growth options raise in value when economic uncertainty increases, they offer a partial hedge to recessions and lower the expected return of growth stocks. These two channels predict monotone expected returns but U-shaped return volatilities as a function of book-to-market ratios.

Using structural estimation, the model jointly matches average returns and return volatilities of firms grouped into portfolios based on their book-to-market ratio. This joint fit resolves a long standing open question in investment-based asset pricing. The model also reproduces further stylized facts such as the value premium being countercyclical, persistent, and unexplained by the CAPM. As a matter of external validity, the model fits moreover untargeted moments such as the equity premium, average investment rates, and further premiums in the cross-section of average stock returns. Additional reduced-form evidence about U-shaped market betas, monotone VIX betas, and monotonic firm characteristics further supports the main channels in the model.

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Table 1

Estimation results when fitting means and volatilities of B/M sorted portfolios

The table presents the SMM estimation results. Panel A shows the estimated parameter values, while Panel B shows calibrated parameter values. The plain numbers in Panels A and B are parameter values, while the numbers in round brackets are standard errors calculated from their asymptotic distribution, see Equation (18). Parameters are annualized where applicable.

Panel A: Estimated parameters	
Volatility of variance (ξ)	0.104
	(0.008)
Idiosyncratic volatility (σ)	0.153
	(0.000)
Fixed operating cost (f)	0.085
	(0.001)
Productivity risk premium (μ)	0.079
	(0.001)
Variance risk premium (λ)	-79.244
	(0.264)
Panel B: Calibrated parameters	
Productivity growth rate (α)	0.031
Average variance level ($\bar{v}$)	0.046^2
Speed of mean reversion (κ^*)	5.530
Correlation coefficient (ρ)	-0.790
Risk-free rate (r)	0.010
Returns-to-scale (ψ)	0.580
Investment cost (k)	1.000
Adjustment cost (k_p)	0.120

Table 2

Inference when fitting means and volatilities of B/M sorted portfolios

This table presents the goodness-of-fit of the SMM estimation. Panel A compares moments from the data with their fitted counterparts from the model. Panel B calculates the mean absolute error for all ten moments (MAE), for the five average returns (MAE(1)), and for the five return volatilities (MAE(2)). Panel C shows the p-value for Hansen's J test.

Panel A: Moments		
Moment	Data	Model
Return of growth stocks	6.47	6.45
Return of portfolio 2	7.43	6.71
Return of portfolio 3	7.76	7.19
Return of portfolio 4	8.83	8.02
Return of value stocks	10.30	9.65
Volatility of growth stocks	30.16	29.82
Volatility of portfolio 2	28.05	29.17
Volatility of portfolio 3	28.32	28.67
Volatility of portfolio 4	29.23	28.70
Volatility of value stocks	33.58	32.09
Panel B: Mean absolute error		
MAE	0.66	
MAE (1)	0.56	
MAE (2)	0.77	
Panel C: Overidentification test		
# moments	10	
# parameters	5	
p-value	0.00	

Table 3
Identification of parameters by moment conditions
This table presents sensitivities of the ten simulated moments with respect to four parameters: the level of fixed costs (f), idiosyncratic volatility (σ), risk premium associated to the business cycle (μ), and risk premium associated to economic uncertainty (λ). Using central differences, the sensitivity of moment m with respect to parameter a is

$$\delta_{m,a} = 100 \cdot \frac{\partial m}{\partial a} = 100 \cdot \frac{m(\frac{3}{2}a) - m(\frac{1}{2}a)}{\frac{3}{2}a - \frac{1}{2}a} = 100 \cdot \frac{m(\frac{3}{2}a) - m(\frac{1}{2}a)}{a}. \tag{22}$$

I multiply by 100 such that the difference in the moments (average returns and return volatilities) is measured as a percentage. For the parameters f, σ, and μ which are percentages, I divide $\delta_{m,a}$ by 100 such that $\delta_{m,f}$, $\delta_{m,\sigma}$, and $\delta_{m,\mu}$ indicate the absolute change in the moments if the three parameters each increase by 0.01. The variance risk premium is of a different unit and $\delta_{m,\lambda}$ indicates the change in moments if λ increases by one.

Moment	f	σ	μ	λ
Return of growth stocks	0.05	0.04	1.61	0.13
Return of portfolio 2	0.07	0.06	1.60	0.12
Return of portfolio 3	0.09	0.08	1.64	0.10
Return of portfolio 4	0.14	0.13	1.72	0.08
Return of value stocks	0.21	0.19	1.94	0.08
Volatility of growth stocks	0.42	1.57	−0.72	−0.07
Volatility of portfolio 2	0.48	1.48	−0.62	−0.07
Volatility of portfolio 3	0.55	1.48	−0.56	−0.07
Volatility of portfolio 4	0.71	1.63	−0.47	−0.07
Volatility of value stocks	1.22	2.49	−0.23	−0.08

Table 4

Counterfactual analysis under conditional CAPM

This table presents a counterfactual analysis of setting the variance risk premium equal to zero, $\lambda = 0$. All other parameter values are the same as in Table 1. Panel A compares moments from the data with their simulated counterparts from the model. Panel B calculates the mean absolute error for all ten moments (MAE), for the five average returns (MAE(1)), and for the five return volatilities (MAE(2)).

Panel A: Moments		
Moment	Data	Model
Return of growth stocks	6.47	10.54
Return of portfolio 2	7.43	10.42
Return of portfolio 3	7.76	10.30
Return of portfolio 4	8.83	10.46
Return of value stocks	10.30	11.96
Volatility of growth stocks	30.16	24.22
Volatility of portfolio 2	28.05	23.94
Volatility of portfolio 3	28.32	23.67
Volatility of portfolio 4	29.26	24.05
Volatility of value stocks	33.58	27.99
Panel B: Mean absolute error		
MAE	3.84	
MAE (1)	2.58	
MAE (2)	5.10	

Table 5
External validity
This table compares various untargeted moments in the data with simulated counterparts from the model. Numbers in Panels A, B and C are time series means of value-weighted cross-sectional averages, either within B/M sorted portfolios (Panel A) or of all firms in the economy (Panels B and C). Gross profitability and investment rate are reported as percentages. Equity premium and volatility are annualized and reported as percentages. Panel D shows the time series means of the high-minus-low quintile spread portfolios sorted on earnings-to-price ratio, sales-to-price ratio, long-term reversal, idiosyncratic volatility, market capitalization, firm-level TFP, and a firm's distance to investment.

	Data	Model
Panel A: Portfolio characteristics		
B/M of growth stocks	0.22	0.25
B/M of portfolio 2	0.45	0.33
B/M of portfolio 3	0.66	0.44
B/M of portfolio 4	0.92	0.64
B/M of value stocks	1.50	1.29
Panel B: Firm policies		
Tobin's q	1.97	3.12
Gross profitability	39.90	42.05
Investment rate	17.85	17.36
Panel C: Macro variables		
Equity premium	6.74	6.85
Stock volatility	29.29	29.26
Panel D: Cross-sectional premiums		
Earn/Price	4.57	2.07
Sales/Price	5.03	3.20
Reversal	−2.82	−1.99
Idio Vol	−1.96	−2.42
Size	−2.93	−2.58
Firm-level TFP	−2.55	−3.00
Dist to Invest		3.88

Table 6
Model channel validity
This table reports value-weighted average firm characteristics in quintile portfolios sorted on book-to-market ratios (columns one to five). Columns six (seven) also reports the p-value for the null hypothesis that there is no difference between portfolio one and two (four and five). The firm characteristics include two real options proxies: ratio of capital expenditures to fixed assets (capx/ppent, Cao et al. (2008)), and total q (Peters and Taylor (2017)). Operating leverage is selling, general and administrative expense (xsga) divided by market capitalization (Chen et al. (2022)), illiquidity is Amihud's (2002) absolute return-to-volume measure averaged over twelve months, size is log market capitalization, institutional ownership is from 13f forms following Nagel (2005), financial leverage is total debt divided by the sum of total debt and market capitalization (Fama and French (1992, 1993)), O-score is from Ohlson (1980), and equity duration is from Gonçalves (2021). All variables are winsorized at the 0.5^{th} and 99.5^{th} percentiles per month. p-values are corrected for heteroscedasticity and serial correlation following Newey and West (1987) with a lag length of six. The data sample ranges from July 1963 until June 2021.

	Growth (1)	BM2 (2)	BM3 (3)	BM4 (4)	Value (5)	$p(2-1)$ (6)	$p(5-4)$ (7)
CAPEX/PPE	0.29	0.24	0.20	0.19	0.18	0.00	0.00
Total q	3.49	1.39	0.81	0.60	0.40	0.00	0.00
Operating leverage	0.11	0.14	0.15	0.16	0.18	0.00	0.00
Illiquidity	0.03	0.05	0.06	0.10	0.26	0.00	0.00
Log market size	8.93	8.70	8.41	8.14	7.48	0.00	0.00
Institutional ownership	0.48	0.49	0.48	0.45	0.41	0.02	0.00
Financial leverage	0.17	0.30	0.38	0.47	0.59	0.00	0.00
O-Score	−4.99	−4.51	−4.26	−3.93	−3.44	0.00	0.00
Equity duration	44.12	38.89	34.31	31.76	27.90	0.00	0.00

Table 7
Robustness
This table reports value-weighted average of conditional firm-level return volatilities, calculated within quintile portfolios (columns one to five). Columns six (seven) also reports the p-value for the null hypothesis that there is no difference between portfolio one and two (four and five). Panel A reports average single-stock volatilities estimated from rolling windows of one month or twelve months, idiosyncratic and systematic volatilities according to the market model, and implied volatilities from 30-day ATM options (data available from January 1996). Panel B, C, and D report the same statistics for quintile portfolios sorted on one-year lagged book-to-market ratios, earnings-to-price ratios, and $1/q^{\text{total}}$ from Peters and Taylor (2017). Panels E report average firm-level return volatilities for different subsamples, split at June 1991. Finally, Panel F (G) reports return volatilities averaged across portfolios dependently double sorted on market size (leverage ratio) and book-to-market ratio. All volatilities are annualized, reported as percentage, and winsorized at the 0.5^{th} and 99.5^{th} percentiles per month. p-values are corrected for heteroscedasticity and serial correlation following Newey and West (1987) with a lag length of six. The data sample ranges from July 1963 until June 2021.

	Growth	2	3	4	Value	$p(2-1)$	$p(5-4)$
Panel A: Univariate sort on book-to-market							
Avg Vol (1M)	30.16	28.05	28.32	29.23	33.58	0.00	0.00
Avg Vol (12M)	31.59	29.43	29.81	30.87	35.66	0.00	0.00
Avg Sys Vol (12M)	17.23	15.62	15.38	15.68	16.56	0.00	0.00
Avg Idio Vol (12M)	25.31	23.92	24.50	25.51	30.35	0.00	0.00
Avg Impl Vol	32.32	30.07	31.23	31.76	35.32	0.00	0.00
Panel B: Univariate sort on 1-year lagged book-to-market							
Avg Vol (1M)	30.32	29.07	29.29	30.68	34.75	0.00	0.00
Avg Vol (12M)	31.46	30.11	30.52	32.14	36.70	0.00	0.00
Avg Idio Vol (12M)	25.20	24.18	24.74	26.41	30.89	0.00	0.00
Avg Sys Vol (12M)	17.09	16.43	16.12	16.53	17.63	0.00	0.00
Avg Impl Vol	30.41	30.09	31.01	32.42	35.38	0.00	0.34
Panel C: Univariate sort on earnings-to-price							
Avg Vol (1M)	33.54	28.42	26.81	27.27	30.40	0.00	0.00
Avg Vol (12M)	34.89	29.57	27.77	28.33	31.85	0.00	0.00
Avg Idio Vol (12M)	28.26	24.14	22.56	23.39	26.56	0.00	0.00
Avg Sys Vol (12M)	18.59	15.57	14.74	14.40	15.81	0.00	0.00
Avg Impl Vol	36.55	29.98	27.94	27.74	30.80	0.00	0.00
Panel D: Univariate sort on $1/q^{\text{total}}$ from Peters and Taylor (2017)							
Avg Vol (1M)	34.47	30.09	28.11	29.58	34.03	0.00	0.00
Avg Vol (12M)	36.18	31.27	29.33	30.88	35.94	0.00	0.00
Avg Idio Vol (12M)	29.96	25.48	23.96	25.38	29.94	0.00	0.00
Avg Sys Vol (12M)	18.57	16.46	15.19	15.79	17.89	0.00	0.00
Avg Impl Vol	35.46	29.95	28.95	31.20	36.13	0.00	0.00

(continued on next page)

Table 8
Robustness (cont.)

	Growth	2	3	4	Value	$p(2-1)$	$p(5-4)$
Panel E: Univariate sort on book-to-market over different subsamples							
Avg Vol (1M), 1963–1991	28.15	26.11	25.44	25.84	29.60	0.00	0.00
Avg Vol (1M), 1991–2021	32.03	29.89	30.98	32.41	37.26	0.00	0.00
Avg Vol (12M), 1963–1991	29.42	27.35	26.81	27.16	30.97	0.00	0.00
Avg Vol (12M), 1991–2021	33.61	31.37	32.61	34.33	40.02	0.00	0.00
Panel F: Double sort on book-to-market and size							
Avg Vol (1M) in Size1	58.71	51.39	47.44	46.52	49.14	0.00	0.00
Avg Vol (1M) in Size2	50.01	43.64	39.82	38.21	40.33	0.00	0.00
Avg Vol (1M) in Size3	44.60	38.13	36.00	34.77	36.32	0.00	0.00
Avg Vol (1M) in Size4	38.74	33.70	31.44	31.60	32.98	0.00	0.00
Avg Vol (1M) in Size5	28.10	26.50	26.18	25.57	26.57	0.00	0.00
Avg Vol (12M) in Size1	62.68	54.14	49.77	48.85	52.29	0.00	0.00
Avg Vol (12M) in Size2	53.37	46.37	42.11	40.17	42.47	0.00	0.00
Avg Vol (12M) in Size3	47.21	40.22	37.93	36.47	38.32	0.00	0.00
Avg Vol (12M) in Size4	40.67	35.36	33.02	33.18	34.62	0.00	0.00
Avg Vol (12M) in Size5	29.01	27.50	27.13	26.54	27.63	0.00	0.00
Panel G: Double sort on book-to-market and leverage							
Avg Vol (1M) in Leverage1	32.62	30.25	30.39	32.83	35.40	0.00	0.00
Avg Vol (1M) in Leverage2	28.68	27.38	27.35	30.17	34.37	0.00	0.00
Avg Vol (1M) in Leverage3	28.96	27.36	28.05	28.68	33.37	0.00	0.00
Avg Vol (1M) in Leverage4	30.89	29.37	28.65	29.49	33.16	0.00	0.00
Avg Vol (1M) in Leverage5	32.64	33.02	34.76	36.56	39.86	0.54	0.00
Avg Vol (12M) in Leverage1	34.18	31.81	31.88	34.90	37.61	0.00	0.00
Avg Vol (12M) in Leverage2	30.05	28.66	28.89	31.73	36.68	0.00	0.00
Avg Vol (12M) in Leverage3	30.26	28.58	29.46	30.31	35.55	0.00	0.00
Avg Vol (12M) in Leverage4	32.53	30.90	30.17	30.79	35.28	0.00	0.00
Avg Vol (12M) in Leverage5	34.58	35.29	37.16	38.73	43.41	0.28	0.00

A Theory appendix

In this appendix, I outline how I solve the model. I first discuss the pricing of productivity and productivity variance risk and then turn to the valuation of a firm. While production assets are valued as a perpetual stream of future profits, the values of capital adjustment options are decomposed into present values of gains and costs of their exercise. I derive analytical value functions by generalizing their early exercise premium. For notational convenience, I drop the subscript i in this appendix.

A.1 Market prices of risks

In this section, I lay the groundwork for the valuation of firms' assets and cash flow streams. The dynamics of θ_t and v_t in Equations (5) and (6) are under the real-world probability measure $\mathbb{P}$. By the first fundamental theorem of asset pricing, the absence of arbitrage implies the existence of a risk-neutral probability measure $\mathbb{Q} \sim \mathbb{P}$. The wedge between $\mathbb{P}$ and $\mathbb{Q}$ is driven by risk premiums. Following Heston (1993) among many others, I change between measures by parametrizing the risk premiums to be $\mathbb{E}_t[\mathrm{d}X_t] - \mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}X_t] = (\mu - r)X_t\mathrm{d}t$ and $\mathbb{E}_t[\mathrm{d}v_t] - \mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}v_t] = \lambda v_t\mathrm{d}t$. The resulting market prices of risks scale these risk premiums (differences in drifts) by the corresponding volatilities of $\mathrm{d}X_t$ and $\mathrm{d}v_t$, $\varphi_t^X = \frac{(\mu-r)X_t\mathrm{d}t}{\sqrt{v_t}X_t\mathrm{d}t} = \frac{\mu-r}{\sqrt{v_t}}$ and $\varphi_t^v = \frac{\lambda v_t\mathrm{d}t}{\xi\sqrt{v_t}\mathrm{d}t} = \frac{\lambda\sqrt{v_t}}{\xi}$. Under this parametrization, Girsanov's theorem implies that the unique stochastic discount factor, M_t, is given by

$$\mathrm{d}M_t = -rM_t\mathrm{d}t - \varphi_t^X M_t\mathrm{d}B_t^X - \varphi_t^v M_t\mathrm{d}B_t^v. \tag{A1}$$

The first term ensures that $\mathbb{E}[M_t] = e^{-rt}$ while the remaining two terms compensate exposures to the two aggregate risk sources (B_t^X and B_t^v) with the corresponding prices of risks (φ_t^X and φ_t^v). The resulting pricing kernel is not log-normally distributed but is driven by time-varying prices of risk. Under the new probability measure $\mathbb{Q}$, the dynamics of θ_t and v_t are

$$\mathrm{d}\theta_t = (r - \delta)\theta_t\mathrm{d}t + \sqrt{v_t}\theta_t\mathrm{d}\tilde{B}_t^X + \sigma\theta_t\mathrm{d}\tilde{B}_t^Z, \tag{A2}$$

$$\mathrm{d}v_t = \kappa^*(\bar{v}^* - v_t)\mathrm{d}t + \xi\sqrt{v_t}\mathrm{d}\tilde{B}_t^v, \tag{A3}$$

where the tildes denote Brownian motions under $\mathbb{Q}$. As before, $\tilde{B}_t^X$ and $\tilde{B}_t^v$ correlate via ρ. While the instantaneous volatilities of $\mathrm{d}\theta_t$ and $\mathrm{d}v_t$ are the same across both measures, the drifts are different. In Equation (2), productivity grows at rate $\alpha = \mu - \delta$, where the expected-return shortfall $\delta = \mu - \alpha$ mimics a dividend yield, see Pindyck (1988). In the risk-neutral world, productivity only grows at rate $r - \delta$. The drift of the variance process v_t also changes with the new parameters being $\kappa^* = \kappa + \lambda$ and $\bar{v}^* = \frac{\kappa\bar{v}}{\kappa+\lambda}$. A negative variance risk premium ($\lambda < 0$) implies slower mean reversion ($\kappa^* < \kappa$) and higher mean variance levels ($\bar{v}^* > \bar{v}$), consistent with the left part of the risk-neural distribution of the variance process being inflated. Idiosyncratic productivity volatility, σ, bears no risk premium and remains unaltered by the change of measure.

Given an arbitrary twice continuously differentiable value function $J(\theta_t, v_t)$, its expected change under $\mathbb{Q}$ can be calculated from the two-dimensional Itô formula via

$$\mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}J(\theta_t, v_t)] = \left(\frac{1}{2}v_t\theta_t^2\frac{\partial^2 J}{\partial\theta_t^2} + \rho\xi v_t\theta_t\frac{\partial^2 J}{\partial\theta_t\partial v_t} + \frac{1}{2}\xi^2 v_t\frac{\partial^2 J}{\partial v_t^2} + (r-\delta)\theta_t\frac{\partial J}{\partial\theta_t} + \kappa^*(\bar{v}^* - v_t)\frac{\partial J}{\partial v_t}\right)\mathrm{d}t. \tag{A4}$$

Consequently, the usual risk-neutral pricing rule $\mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}J(\theta_t, v_t)] = rJ(\theta_t, v_t)\mathrm{d}t$ enforces that value functions solve the linear partial differential equation (PDE)

$$\frac{1}{2}v_t\theta_t^2\frac{\partial^2 J}{\partial\theta_t^2} + \rho\xi v_t\theta_t\frac{\partial^2 J}{\partial\theta_t\partial v_t} + \frac{1}{2}\xi^2 v_t\frac{\partial^2 J}{\partial v_t^2} + (r-\delta)\theta_t\frac{\partial J}{\partial\theta_t} + \kappa^*(\bar{v}^* - v_t)\frac{\partial J}{\partial v_t} - rJ = 0. \tag{A5}$$

This PDE also underpins the corresponding Hamilton-Jacobi-Bellman equation if I were to solve the model using dynamic programming instead of the equivalent contingent claim analysis employed here.

If productivity variance is constant ($v_t = \sigma^2$), the model for total productivity θ_t collapses to a geometric Brownian motion. In this case, all partial derivatives with respect to v_t vanish and Equation (A5) reduces to the standard ordinary differential equation

$$\frac{1}{2}\sigma^2\theta_t^2\frac{\partial^2 J}{\partial\theta_t^2} + (r-\delta)\theta_t\frac{\partial J}{\partial\theta_t} - rJ = 0. \tag{A6}$$

This differential equation is featured in many one-factor real options asset pricing papers, see, for example, Pindyck (1988), Cooper (2006), Aguerrevere (2009), and Aretz and Pope (2018).

For future reference, I denote the expected drift from Equation (A4) by the shortcut $\mathcal{A}$, such that

$$\mathcal{A} = \frac{1}{2} v_t \theta_t^2 \frac{\partial^2}{\partial \theta_t^2} + \rho \xi v_t \theta_t \frac{\partial^2}{\partial \theta_t \partial v_t} + \frac{1}{2} \xi^2 v_t \frac{\partial^2}{\partial v_t^2} + (r - \delta) \theta_t \frac{\partial}{\partial \theta_t} + \kappa^* (\bar{v}^* - v_t) \frac{\partial}{\partial v_t}. \tag{A7}$$

Mathematically, $\mathcal{A}$ is an elliptic (but degenerate) differential operator and the infinitesimal generator of the two-dimensional Itô process (θ_t, v_t) under $\mathbb{Q}$. More intuitively, $\mathcal{A}$ defines the instantaneous drift of a value function such that $\mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}J(\theta_t, v_t)] = \mathcal{A}J(\theta_t, v_t)\mathrm{d}t$. The usual risk-neutral pricing rule $\mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}J(\theta_t, v_t)] = rJ(\theta_t, v_t)\mathrm{d}t$ then imposes that the value function solves the PDE $(\mathcal{A} - r)J(\theta_t, v_t) = 0$, which is a useful shorthand notation for Equation (A5).

A.2 Valuing assets-in-place

The value of owning $\bar{K}$ installed capital units is the present value of perpetually receiving a firm's entire operating profits, $\Pi_t = \theta_t \bar{K}^\psi - f\bar{K}$,

$$AiP(\theta_t; \bar{K}) = \int_t^\infty e^{-r(u-t)} \mathbb{E}_t^{\mathbb{Q}}[\Pi_u] \mathrm{d}u \tag{A8}$$

$$= \frac{\theta_t}{\delta} \bar{K}^\psi - \frac{f}{r} \bar{K}. \tag{A9}$$

Because production and sales are instantaneous and because no optionality is involved, the value of a firm's installed assets-in-place is independent of economic uncertainty. Instead, the assets-in-place award a unit exposure to total productivity, less some fixed maintenance costs.

If the firm is unprofitable and faces low productivity, it is possible for the value of the firm's assets-in-place to be negative due to the fixed costs. A rational firm would then consider seizing operations. I capture this irreversible shutdown decision via a perpetual put option. Upon exercise, this option pays $d\bar{K} - AiP$, that is, it removes all installed capital units and sets the firm's book value to zero. Furthermore, the firm might earn a recovery value (if $d > 0$) or face contractual penalties for early shutdown (if $d < 0$). For simplicity, I set $d = 0$ but I do state all formulas for an arbitrary d.

I derive formulas for the value of the shutdown option by considering the decomposition of finite-maturity American options into a European-style option and an early exercise premium, and letting time to maturity tend to infinity. I then decompose these real option values into the present values of accrued gains and losses upon exercise. Using the technique from Appendix A.5, I can derive the value and exercise policy of this abandonment option which is summarized in the following proposition.

PROPOSITION 4. *The value of the shutdown option is*

$$SO(\theta_t, v_t; \bar{K}) = \left(d + \frac{f}{r}\right) \bar{K}\mathcal{S} - \frac{\theta_t}{\delta}\bar{K}^{\psi}\mathcal{S}', \tag{A10}$$

where $\mathcal{S}, \mathcal{S}' \in [0,1]$ *are pseudo-probabilities defined in the proof.*

Proof. Let $\theta^S(v_t)$ the time-varying shutdown threshold for total productivity. In the continuation region ($\theta_t \geq \theta^S(v_t)$), the option value solves the pricing PDE $(\mathcal{A} - r)SO(\theta_t, v_t; K) = 0$. Turning to the exercise region ($\theta_t \leq \theta^S(v_t)$), I can substitute the payoff $SO(\theta_t, v_t; K) = d\bar{K} - AiP(\theta_t; K)$ into that same PDE, and, using $(\mathcal{A} - r)AiP(\theta_t; K) = -\Pi_t$, I get

$$(\mathcal{A} - r)SO(\theta_t, v_t, \bar{K}) = -\left(rd\bar{K} + f\bar{K} - \theta_t\bar{K}^{\psi}\right)\mathbb{1}_{\{\theta_t \leq \theta^S(v_t)\}}. \tag{A11}$$

Combining this equation with Equation (A43) yields

$$\begin{aligned} SO(\theta_t, v_t; \bar{K}) = (rd + f)\bar{K} &\int_t^{\infty} e^{-r(u-t)}\mathbb{Q}_t\left[\theta_u \leq \theta^S(v_u)\right] \mathrm{d}u \\ - \bar{K}^{\psi} &\int_t^{\infty} \theta_t e^{-\delta(u-t)}\mathbb{Q}'_t\left[\theta_u \leq \theta^S(v_u)\right] \mathrm{d}u, \end{aligned} \tag{A12}$$

where $\mathbb{Q}' \sim \mathbb{Q}$ is the equivalent martingale measure that employs $\theta_t e^{\delta t}$ as numéraire. To be precise, I apply the change of numéraire $\frac{\mathrm{d}\mathbb{Q}'}{\mathrm{d}\mathbb{Q}}\big|_t = \frac{\theta_u e^{\delta u}/\theta_t e^{\delta t}}{e^{ru}/e^{rt}}$ such that $\mathbb{E}_t^{\mathbb{Q}}\left[\theta_u \mathbb{1}_{\{\theta_u \geq x\}}\right] = \theta_t e^{(r-\delta)(u-t)}\mathbb{Q}'_t\left[\theta_u \geq x\right]$. The derivation of Equation (A10) is complete by defining

$$\mathcal{S} = r\int_t^{\infty} e^{-r(u-t)}\mathbb{Q}_t\left[\theta_u \leq \theta^S(v_u)\right] \mathrm{d}u, \tag{A13}$$

$$\mathcal{S}' = \delta\int_t^{\infty} e^{-\delta(u-t)}\mathbb{Q}'_t\left[\theta_u \leq \theta^S(v_u)\right] \mathrm{d}u. \tag{A14}$$

The convergence of the improper integrals follows from $r > 0$ and $\delta > 0$ because the probabilities are bounded between zero and one. The positivity of r and δ implies furthermore $\mathcal{S}, \mathcal{S}' \in [0, 1]$. $\square$

Having found the continuation values of the shutdown option, the final step is locating the exercise curve $\theta^S(v_t)$. Following Rouah (2013), a textbook solution is to set

$$\ln(\theta^S(v_t)) = \mathfrak{b}_0^S + \mathfrak{b}_1^S v_t, \tag{A15}$$

where $\mathfrak{b}_1^S < 0$. Log-linearity is based on strong empirical evidence for financial options, see, e.g., Broadie et al. (2000), and can easily be extended to incorporate higher order terms, which is, however, unlikely to add much economic insight because option values are insensitive to an exact location of the optimal exercise boundary.

I can now pick two values for v_t (say v_1 and v_2) and identify the two free parameters $\mathfrak{b}_0^S$ and $\mathfrak{b}_1^S$ by solving the following system of value-matching conditions

$$\begin{aligned} SO(\theta^S(v_1), v_1; \bar{K}) &= d\bar{K} - AiP(\theta^S(v_1); \bar{K}), \\ SO(\theta^S(v_2), v_2; \bar{K}) &= d\bar{K} - AiP(\theta^S(v_2); \bar{K}). \end{aligned} \tag{A16}$$

The value-matching condition states that a firm shuts down its production when its productivity is sufficiently low and the firm is unprofitable. Myneni (1992, Theorem 4.1) proves that valuing options and locating exercise curves by solving integral equations is equivalent to solving differential equations with smooth-pasting along the free boundary. Intuitively, similar to binomial trees and finite difference schemes, the payoff condition $\max\{\cdot, 0\}$ is already incorporated in the integral solutions, see Equation (A11), which therefore bypasses the need of imposing additional smooth-pasting conditions.

Intuitively, Equation (A10) represents the option value as present value of perpetual gains and costs accrued in the exercise regions. Conversely, the first term in Equation (A10) represents the gain from shutting down the $\bar{K}$ production units, namely the saved fixed maintenance costs and the earned recovery value, from which the loss in sales revenue from giving up those production assets is subtracted. Present values are calculated by integrating the values of Arrow-Debreu securities that pay one unit of the numéraire in the exercise regions under $\mathbb{Q}$ and $\mathbb{Q}'$, respectively. In analogy to the famous Black-Scholes (1973) formula, $\mathcal{S}'$ and $\mathcal{S}$ can be interpreted as integrated "$N(-d_1)$" and "$N(-d_2)$" exercise probabilities. When exercised, $\mathcal{S} = \mathcal{S}' = 1$, the value function in Equation (A10) still applies and reduces to the "option payoff" via the value-matching conditions in Equation (A16).

Equation (A10) applies to any such real options model in which θ_t is a continuous Itô process. If θ_t is a geometric Brownian motion, the formulas recover standard closed-form solutions as in Pindyck (1988), Cooper (2006), or Hackbarth and Johnson (2015). Exercise probabilities in my stochastic volatility model can be calculated fully analytically using Heston's (1993) characteristic function.

A.3 Valuing growth options

To determine the value of the firm's growth options, I follow Pindyck (1988), Aguerrevere (2009), and Aretz and Pope (2018), and assume that firms invest incrementally. To this end, I divide a firm's capital stock into a continuum of increments $\mathrm{d}K$. For example, the profit of producing and selling the K^{th} marginal output is $\Delta\Pi = \frac{\partial \Pi}{\partial K} = \psi\theta_t K^{\psi-1} - f$ per unit of time. Accordingly, the value of marginal unit of capital is $\Delta AiP(\theta_t; K) = \frac{\partial AiP(\theta_t;K)}{\partial K} = \psi\frac{\theta_t}{\delta}K^{\psi-1} - \frac{f}{r}$. Exercising incremental growth options raises a firm's capital stock by an infinitesimal amount. Denoting the value of the K^{th} incremental option to investment by $\Delta GO(\theta_t, v_t; K) = -\frac{\partial GO(\theta_t, v_t; K)}{\partial K}$, the total value of a firm's outstanding growth options is

$$GO_t = \int_{\bar{K}_t}^{\infty} \Delta GO(\theta_t, v_t; K)\mathrm{d}K, \tag{A17}$$

The firm value, W_t, can now be written as

$$W_t = SO_t + AiP_t + \int_{\bar{K}_t}^{\infty} \Delta GO(\theta_t, v_t; K)\mathrm{d}K. \tag{A18}$$

The valuation of the incremental growth options is similar to that of the shutdown option: I first determine the continuation value and then identify the free investment boundary using value-matching conditions. Upon exercise, the growth option adds to the value of the existing productive assets and their shutdown option. On the other hand, investment also incurs fixed and quasi-fixed adjustment costs. To continue, I thus first state the incremental value of assets-in-place and sales revenue.

The value of the K^{th} marginal capital unit is

$$\Delta AiP(\theta_t; K) = \psi \frac{\theta_t}{\delta} K^{\psi-1} - \frac{f}{r} \tag{A19}$$

which solves the inhomogeneous PDE

$$(\mathcal{A} - r)\Delta AiP(\theta_t; K) + \Delta\Pi = 0. \tag{A20}$$

Finally, when investing, firms pay quasi-fixed adjustment costs in the form of a proportion of the current sales revenue of the corresponding incremental productive unit which is given by

$$\Delta Sales(\theta_t; K) = \psi\theta_t K^{\psi-1}, \tag{A21}$$

which solves the inhomogeneous PDE

$$(\mathcal{A} - r)\Delta Sales(\theta_t; K) + \delta\Delta\Pi + \delta f = 0. \tag{A22}$$

With all of that notation established, I am in the position to next apply Equation (A43) to determine the continuation values of a firm's growth options.

PROPOSITION 5. *The value of the K^{th} incremental option to grow is*

$$\Delta GO(\theta_t, v_t; K) = (1 - \delta k_p)\psi \frac{\theta_t}{\delta} K^{\psi-1}\mathcal{G}' - \left(\frac{f}{r} + k\right)\mathcal{G}, \tag{A23}$$

where $\mathcal{G}, \mathcal{G}' \in [0,1]$ are pseudo-probabilities defined in the proof.

Proof. Let $\theta^S(v_t)$ the time-varying investment threshold for total productivity. In the continuation region ($\theta_t \leq \theta^I(v_t)$), the option value solves the pricing PDE $(\mathcal{A} - r)\Delta GO(\theta_t, v_t; K) = 0$. Turning to the exercise region ($\theta_t \geq \theta^I(v_t)$), I can substitute the payoff $\Delta GO(\theta_t, v_t; K) = \Delta SO(\theta_t, v_t; K) + \Delta AiP(\theta_t; K) - k_p \Delta Sales(\theta_t; K) - k$ into that same PDE, and, using Equations (A20) and (A22), get

$$(\mathcal{A} - r)\Delta GO(\theta_t, v_t; K) = -\Big((1 - \delta k_p)\psi\theta_t K^{\psi-1} - f - rk\Big)\,\mathbb{1}_{\{\theta_t \geq \theta^I(v_t)\}}. \tag{A24}$$

Combining this equation with Equation (A43) yields

$$\begin{aligned} \Delta GO(\theta_t, v_t; K) = (1 - \delta k_p)\psi K^{\psi-1} \int_t^\infty \theta_t e^{-\delta(u-t)} \mathbb{Q}'_t \left[\theta_u \geq \theta^I(v_u)\right] \mathrm{d}u \\ - (f + rk) \int_t^\infty e^{-r(u-t)} \mathbb{Q}_t \left[\theta_u \geq \theta^I(v_u)\right] \mathrm{d}u. \end{aligned} \tag{A25}$$

The derivation of Equation (A23) is complete by defining

$$\mathcal{G} = r \int_t^\infty e^{-r(u-t)} \mathbb{Q}_t \left[\theta_u \geq \theta^I(v_u)\right] \mathrm{d}u, \tag{A26}$$

$$\mathcal{G}' = \delta \int_t^\infty e^{-\delta(u-t)} \mathbb{Q}'_t \left[\theta_u \geq \theta^I(v_u)\right] \mathrm{d}u. \tag{A27}$$

The convergence of the improper integrals follows from $r > 0$ and $\delta > 0$ because the probabilities are bounded between zero and one. The positivity of r and δ implies furthermore $\mathcal{G}, \mathcal{G}' \in [0, 1]$. □

Indeed, the first term in Equation (A23) captures the additional sales revenue gained from investing into the K^{th} marginal production asset while the second term in that equation captures the associated losses in terms of fixed maintenance costs and investment costs.

Given the continuation values of the growth options, I locate the exercise curves $\theta^I(v_t)$ by setting

$$\ln(\theta^I(v_t)) = \mathfrak{b}_0^I + \mathfrak{b}_1^I v_t, \tag{A28}$$

where $\mathfrak{b}_1^I > 0$. I can now pick two values for v_t (say v_1 and v_2) and identify the two free parameters

$\mathfrak{b}_0^I$ and $\mathfrak{b}_1^I$ by solving the following system of value-matching conditions

$$\begin{aligned}\Delta GO(\theta^I(v_1), v_1; K) + k + k_p \Delta Sales(\theta^I(v_1); K) &= \Delta SO(\theta^I(v_1), v_1; K) + \Delta AiP(\theta^I(v_1); K),\\ \Delta GO(\theta^I(v_2), v_2; K) + k + k_p \Delta Sales(\theta^I(v_2); K) &= \Delta SO(\theta^I(v_2), v_2; K) + \Delta AiP(\theta^I(v_2); K).\end{aligned} \tag{A29}$$

The value-matching conditions state that a firm invests installed capital until the gain of investing into the next marginal unit of capital equals the cost of doing so.

A.4 Expected return and return variance

In this section, I compute the first two moments of a firm's stock return. Let $\mathrm{d}R_t = \frac{\mathrm{d}W_t + \Pi_t \mathrm{d}t}{W_t}$ denote a firm's instantaneous return where Π_t denotes its profit per unit of time. Using Itô's Lemma and recalling the parametrization of the risk premiums from Section A.1, I obtain

$$\mathbb{E}_t\left[\mathrm{d}R_t - r\mathrm{d}t\right] = \mathbb{E}_t\left[\mathrm{d}R_t\right] - \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{d}R_t\right] \tag{A30}$$

$$= \frac{(\mu-\delta)\theta_t \frac{\partial W_t}{\partial \theta_t} + \kappa(\bar{v} - v_t)\frac{\partial W_t}{\partial v_t} - (r-\delta)\theta_t \frac{\partial W_t}{\partial \theta_t} - \kappa^*(\bar{v}^* - v_t)\frac{\partial W_t}{\partial v_t}}{W_t}\mathrm{d}t \tag{A31}$$

$$= \Omega_t^{(\theta)}(\mu - r)\mathrm{d}t + \Omega_t^{(v)}\lambda \mathrm{d}t, \tag{A32}$$

where $\Omega_t^{(\theta)} = \frac{\partial W_t / W_t}{\partial \theta_t / \theta_t}$ and $\Omega_t^{(v)} = \frac{\partial W_t / W_t}{\partial v_t / v_t}$. From Equation (8), the firm-level elasticities are value-weighted sums of the elasticities of the firm's assets-in-place and real options to shut down and grow,

$$\Omega_t^{(\theta)} = \frac{SO_t}{W_t}\frac{\partial SO_t/SO_t}{\partial \theta_t/\theta_t} + \frac{AiP_t}{W_t}\frac{\partial AiP_t/AiP_t}{\partial \theta_t/\theta_t} + \frac{GO_t}{W_t}\frac{\partial GO_t/GO_t}{\partial \theta_t/\theta_t}, \tag{A33}$$

$$\Omega_t^{(v)} = \frac{SO_t}{W_t}\frac{\partial SO_t/SO_t}{\partial v_t/v_t} + \frac{AiP_t}{W_t}\frac{\partial AiP_t/AiP_t}{\partial v_t/v_t} + \frac{GO_t}{W_t}\frac{\partial GO_t/GO_t}{\partial v_t/v_t}. \tag{A34}$$

These elasticities can thus be calculated fully analytically by simply partially differentiating Equations (A9), (A10), and (A23) with respect to θ_t and v_t. These derivatives are available upon request.

Following Cooper (2006) and Hackbarth and Johnson (2015), the productivity elasticity from Equation (14) can be decomposed as follows

$$\Omega_t^{(\theta)} = \left(1 + \frac{f\bar{K}_t/r}{W_t}\right)\left(\frac{SO_t}{W_t^+}\Omega_{SO}^{(\theta)} + \frac{AiP_t^+}{W_t^+}\cdot 1 + \frac{GO_t}{W_t^+}\Omega_{GO}^{(\theta)}\right), \tag{A35}$$

where $W_t^+ = W_t + \frac{f\bar{K}_t/r}{W_t}$ denotes the "unlevered" firm value free of fixed maintenance costs. Freed from fixed costs, the productivity elasticity of the assets-in-place is normalized to one, representing the unit exposure to total productivity. In the model, the shutdown options make up only a small proportion of firms at best because firms are typically far away from their terminal shutdown threshold. Thus, using $\frac{SO_t}{W_t^+} \approx 0$ and $\frac{AiP_t^+}{W_t^+} \approx 1 - \frac{GO_t}{W_t^+}$, I arrive at

$$\begin{aligned}
\Omega_t^{(\theta)} &= \left(1 + \frac{f\bar{K}_t/r}{W_t}\right)\left(1 + \frac{GO_t}{W_t^+}\left(\Omega_{GO}^{(\theta)} - 1\right)\right) && \text{(A36)}\\
&= 1 + \frac{f\bar{K}_t/r}{W_t} + \frac{GO_t}{W_t}\left(\Omega_{GO}^{(\theta)} - 1\right). && \text{(A37)}
\end{aligned}$$

I next turn to the elasticity of the firm value with respect to productivity variance. Assets-in-place are unresponsive to changes in economic uncertainty and thus do not impact $\Omega_t^{(v)}$. Using again the insight that shutdown options barely contribute to firm value, a firm's variance elasticity reduces to

$$\Omega_t^{(v)} = \frac{GO_t}{W_t}\Omega_{GO}^{(v)}. \tag{A38}$$

I compute the conditional variance of a firm's instantaneous excess return using Itô's Lemma,

$$\begin{aligned}
\mathbb{V}\mathrm{ar}_t\left[\mathrm{d}R_t - r\mathrm{d}t\right] &= \frac{\mathbb{E}_t[(\mathrm{d}W_t)^2]}{W_t^2} && \text{(A39)}\\
&= \frac{(v_t + \sigma^2)\theta_t^2\left(\frac{\partial W_t}{\partial \theta_t}\right)^2 + \xi^2 v_t\left(\frac{\partial W_t}{\partial v_t}\right)^2 + 2\rho\xi v_t\theta_t\frac{\partial W_t}{\partial \theta_t}\frac{\partial W_t}{\partial v_t}}{W_t^2}\mathrm{d}t && \text{(A40)}\\
&= \left(\Omega_t^{(\theta)}\right)^2(v_t + \sigma^2)\mathrm{d}t + \left(\Omega_t^{(v)}\right)^2\frac{\xi^2}{v_t}\mathrm{d}t + 2\Omega_t^{(\theta)}\Omega_t^{(v)}\rho\xi\mathrm{d}t. && \text{(A41)}
\end{aligned}$$

In the same vain, the conditional covariance between the returns of stocks i and j is

$$\mathbb{C}\mathrm{ov}_t\left(\mathrm{d}R_{i,t},\mathrm{d}R_{j,t}\right)=v_t\Omega_{i,t}^{(\theta)}\Omega_{j,t}^{(\theta)}\mathrm{d}t+\rho\xi\Omega_{i,t}^{(\theta)}\Omega_{j,t}^{(v)}\mathrm{d}t+\rho\xi\Omega_{j,t}^{(\theta)}\Omega_{i,t}^{(v)}\mathrm{d}t+\frac{\xi^2}{v_t}\Omega_{i,t}^{(v)}\Omega_{j,t}^{(v)}\mathrm{d}t. \tag{A42}$$

A.5 Valuing perpetual options

In this section, I state and prove a lemma which allows me to calculate the value of a perpetual claim by solving a single integral.

LEMMA 1. *The value of a perpetual option is*

$$h(\theta_t,v_t)=-\int_t^{\infty}e^{-r(u-t)}\mathbb{E}_t^{\mathbb{Q}}\left[(\mathcal{A}-r)h(\theta_u,v_u)\right]\mathrm{d}u. \tag{A43}$$

Proof. Let $g(t,\theta_t,v_t;T)$ be the time-t value of a standard American option with maturity T and $h(\theta_t,v_t)$ the value of an equivalent perpetual option such that $\lim_{T\to\infty}g(t,\theta_t,v_t;T)=h(\theta_t,v_t)$. Applied to the convex value function of a discounted American option, $e^{-rt}g(t,\theta_t,v_t;T)$, the generalized Itô formula gives

$$g(t,\theta_t,v_t;T)=e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[g(T,\theta_T,v_T;T)\right]-\int_t^{T}e^{-r(u-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\left(\frac{\partial}{\partial u}+\mathcal{A}-r\right)g\left(u,\theta_u,v_u;T\right)\right]\mathrm{d}u. \tag{A44}$$

The value of the American option, $g(t,\theta_t,v_t;T)$, is thus decomposed into the value of a European-style option, $e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[g(T,\theta_T,v_T;T)\right]$, and an early exercise premium, the integral term. This decomposition resembles Dynkin's formula and, among others, Myneni (1992) employs this result to price American options analytically. Letting the maturity T of the option in Equation (A44) tend to infinity yields Equation (A43). The positivity of the early exercise premium follows from the no-arbitrage condition $\mathbb{E}_t^{\mathbb{Q}}[\mathrm{d}h(\theta_u,v_u)]\leq rh(\theta_u,v_u)\mathrm{d}u$ which implies $(\mathcal{A}-r)h(\theta_u,v_u)\leq 0$. □

B Time-varying idiosyncratic volatility

In this appendix, I show that my main result – U-shaped sensitivities to the business cycle and volatility-sensitive growth options can jointly fit the first two moments of stock returns – also holds true when generalizing my model and adding stochastic firm-specific volatility. Intuitively, because idiosyncratic risk does not contribute to systematic risk, expected return are still composed of two components, while return volatility now has six components, resembling the variance of a three-asset portfolio. As before, because of the small magnitude of firms' variance elasticities, the return volatility of stocks is quantitatively entirely driven by the U-shaped productivity elasticity. Time-varying idiosyncratic variance has been studied by Ai and Kiku (2016), Bhamra and Shim (2017), and Barinov and Chabakauri (2023), among others.

The model set-up largely follows the setting of the main papers with firms facing the production technology and investment frictions. In this extension, however, total productivity ($\theta_{i,t}$) is modeled via

$$\mathrm{d}\theta_{i,t} = \alpha\theta_{i,t}\mathrm{d}t + \sqrt{v_t^X}\theta_{i,t}\mathrm{d}B_t^X + \sqrt{v_{i,t}^Z}\theta_{i,t}\mathrm{d}B_{i,t}^Z, \tag{B1}$$

$$\mathrm{d}v_t^X = \kappa^X(\bar{v}^X - v_t^X)\mathrm{d}t + \xi^X\sqrt{v_t^X}\mathrm{d}B_t^{X,v}, \tag{B2}$$

$$\mathrm{d}v_{i,t}^Z = \kappa^Z(\bar{v}^Z - v_{i,t}^Z)\mathrm{d}t + \xi^Z\sqrt{v_{i,t}^Z}\mathrm{d}B_{i,t}^{Z,v}. \tag{B3}$$

Here, α, κ^X, κ^Z, $\bar{v}^X$, $\bar{v}^Z$, ξ^X, and ξ^Z are positive constants, similar to the ones in Equation (3). Productivity and productivity variance shocks correlate via $\mathrm{d}B_t^X\mathrm{d}B_t^{X,v} = \rho^X\mathrm{d}t$ and $\mathrm{d}B_{i,t}^Z\mathrm{d}B_{i,t}^{Z,v} = \rho^Z\mathrm{d}t$, while the two variance processes are correlated via $\mathrm{d}B_t^{X,v}\mathrm{d}B_{i,t}^{Z,v} = \rho^v\mathrm{d}t$. The model now incorporates 19 parameters.

In this "double Heston model," the conditional correlation between changes in productivity and total productivity variance, $\mathbb{C}\mathrm{orr}\left(\mathrm{d}\theta_{i,t}, \mathrm{d}(v_t^X + v_{i,t}^Z)\right)$, is state-dependent. If $\rho^X, \rho^Z < 0$, then there is a leverage effect between productivity and total productivity variance. The choice $\rho^v > 0$ captures the stylized fact that states of high aggregate uncertainty tend to coincide with periods of high firm-specific volatility.

I next calculate the mean and volatility of stock returns in this augmented stochastic volatility model which confirms the intuition of the main model in the main body of the text: productivity elasticity can still be decomposed into an operating leverage factor and a second factor attributable to growth options. While the formula for the return variance includes more terms, only the responsiveness to total productivity matters quantitatively.

PROPOSITION 6. *The conditional expectation of firms' excess return per unit of time is*

$$\frac{\mathbb{E}_t[\mathrm{d}R_{i,t} - r\mathrm{d}t]}{\mathrm{d}t} = \Omega_{i,t}^{(\theta)}(\mu - r) + \Omega_{i,t}^{(v^X)}\lambda, \tag{B4}$$

and the conditional variance of firms' excess return per unit of time is

$$\begin{aligned}\frac{\mathbb{V}\mathrm{ar}_t[\mathrm{d}R_{i,t} - r\mathrm{d}t]}{\mathrm{d}t} &= \left(\Omega_{i,t}^{(\theta)}\right)^2 (v_t^X + v_{i,t}^Z) + \left(\Omega_{i,t}^{(v^X)}\right)^2 \frac{(\xi^X)^2}{v_t^X} + \left(\Omega_{i,t}^{(v^Z)}\right)^2 \frac{(\xi^Z)^2}{v_{i,t}^Z} \\ &+ 2\Omega_{i,t}^{(\theta)}\Omega_{i,t}^{(v^X)}\rho^X\xi^X + 2\Omega_{i,t}^{(\theta)}\Omega_{i,t}^{(v^Z)}\rho^Z\xi^Z + 2\Omega_{i,t}^{(v^X)}\Omega_{i,t}^{(v^Z)}\frac{\rho^v\xi^X\xi^Z}{\sqrt{v_t^X v_{i,t}^Z}},\end{aligned} \tag{B5}$$

where $\Omega_{i,t}^{(\theta)} = \frac{\partial W_{i,t}/W_{i,t}}{\partial\theta_{i,t}/\theta_{i,t}}$, $\Omega_{i,t}^{(v^X)} = \frac{\partial W_{i,t}/W_{i,t}}{\partial v_t^X/v_t^X}$, *and* $\Omega_{i,t}^{(v^Z)} = \frac{\partial W_{i,t}/W_{i,t}}{\partial v_{i,t}^Z/v_{i,t}^Z}$ *denote firm-level elasticities with respect to productivity, aggregate variance, and idiosyncratic variance, respectively.*

Proof. Similar to Appendix A.4. □

Because idiosyncratic productivity variance bears no risk premium, the expected return has the same two-factor structure as in Proposition 2. However, the return variance resembles the variance of a three-asset portfolio containing of a "productivity component," an "aggregate productivity variance component," and an "idiosyncratic productivity variance component" and is now impacted by all three non-zero correlations. Operating leverage created by fixed maintenance costs and procyclical growth option continue to create a U-shaped productivity elasticity. The variance elasticity $\Omega^{(v^A)}$ continues to decrease as a function of book-to-market ratio, and thereby lowers the expected return of growth stocks. Because variance elasticities are of small magnitude, the return volatility remains dominated by the U-shaped sensitivity to the business cycle.