Hedging and Pricing American Options with Static Hedging under Stochastic Volatility

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Abstract

Under stochastic volatility (SV), despite the abundant literature on American options pricing, there is little work on American options hedging. This paper develops a static hedging portfolio (SHP) method for hedging and pricing an American option under SV by constructing a portfolio of European vanilla options to match the payoff, delta, and vega of the target American option along its early exercise boundary. The novelty of the proposed SHP method is incorporating the expected variance conditional on the stock price into Chung and Shih's (2009) method and further improving their method by imposing the vega-matching condition. Our numerical analyses show the superiority of the proposed SHP method in effectively hedging and accurately pricing American options in the presence of SV; moreover, both the hedging and pricing performance are further increased after the introduction of the vega-matching condition.

Keywords: Static Hedging; American Options; Stochastic Volatility; Conditional Expected Variance; Vega Matching

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1. Introduction

The hedging and pricing of American options are classic and ongoing issues in the field of financial engineering; additional consideration of stochastic volatility (SV) makes the hedging and pricing of American options even more complex and difficult. Besides the classic multidimensional tree models and finite different methods (FDM), there are many other methods for the pricing of American options under the SV assumption. Most of these methods focus primarily on pricing issues, with little emphasis on hedging. Even when hedging is addressed, this is often through delta neutral dynamic hedging strategies. However, this paper focuses on using the static hedging portfolio (SHP) method to study the pricing and hedging issues of American options under SV.

The fundamental concept behind an SHP is to employ standard European options to create a portfolio whose value corresponds to the value of target options at some boundary conditions and at the expiration date, by determining the maturity date, strike price, and investment amount of each European option in the portfolio. *Static hedging* refers to a situation where, throughout the entire duration of the target option, regardless of stock price fluctuations, the issuer (hedger) does not need to make any adjustments to the SHP. Either at or before maturity, when it is time to pay the due compensation to the holder of the target option, the issuer (hedger) only needs to liquidate the SHP to generate the corresponding cash flow to pay the option holder. Although the main purpose of the SHP method is to achieve the hedging objective of saving on transaction costs, the SHP value can also closely approximate the value of the target option, as long as the SHP's payoffs at the boundary conditions are close to those of the target option. Theoretically speaking, since the target options and SHP share the same partial differential equation (PDE), if they have equal boundary-condition payoffs, their value today should be identical.

Static hedging was first introduced by Bowie and Carr (1994), Derman, Ergener and Kani (1995), and Carr, Ellis and Gupta (1998). It is used to hedge exotic options, such as European barrier options, under the assumption of the geometric Brownian motion (GBM) model. Contrary to the focus of most of the SHP literature, which predominantly examines barrier options or other exotic European options, Chung and

Shih (2009) are the first to apply the SHP method to the valuation of American options. Building upon the value-matching condition proposed by Derman, Ergener, and Kani (1995), they introduce an additional smooth-pasting condition to better align with the boundary conditions of American options. Using backward induction to solve for the investment proportions of European vanilla options in the SHP, they concurrently derive the critical exercise price for American options, which serves as an approximate estimate of the true early exercise boundary. Under both GBM and constant elasticity of variance (CEV) models, Chung and Shih (2009)'s SHP method accurately determines the price, delta, and gamma values of American options. Similarly, following this vein, Chung, Shih, and Tsai (2013a, 2013b) extend this methodology to hedge and evaluate American touch-in and touch-out options.

The literature on SHP methods under non-GBM models is sparse, but a summary of the key contributions is as follows. To our knowledge, Fink (2003) is the first to consider using an SHP to hedge and price European up-and-out call options under the Heston (1993) SV model.[1] To address the additional dimension of variance, Fink (2003) introduces one to four representative variances and includes plain call options with strike prices higher than the barrier into the SHP to ensure value consistency at the barrier boundaries under these representative variances. By introducing value matching at each time and variance node, theoretically, the hedging performance of the constructed SHP is better the finer the segmentation in time and variance dimensions. Nalholm and Poulsen (2006) extend the framework of Fink (2003) within a stochastic volatility with log-normal jumps. They focus on static hedging for European up-and-out barrier options, considering the inclusion of additional European vanilla options in the SHP, whose strike price is the most possible stock price level after the stock price penetrates the barrier from below due to a jump occurring. Takahashi and Yamazaki (2009) implement static hedging for European path-independent options in an SV model. They consider a pricing process with local volatility, identical to the stock price distribution in the SV model, based upon which an SHP, consisting of risk-free assets,

[1] To account for the SV model, Allen and Padovani (2002) also conduct optimized value matching at time and variance nodes along the barrier boundary. Their method, purely a hedging model, combines static and dynamic hedging to form a quasi-static hedge, aiming for optimized hedging of long-term or exotic European options. Since their method is not an exact SHP method, theoretically, it cannot be used to evaluate the target barrier option.

shorter-maturity forwards, European calls, and puts, can be determined. Tsai (2014) hedges and prices European barrier options under CEV models. Unlike Derman, Ergener, and Kani (1995) or earlier works, he considers both value-matching and theta-matching conditions, and introduces binary options into the SHP to significantly improve hedging effectiveness. Huh, Jeon, and Ma (2020) use an SHP to hedge and price European barrier options in a fast mean-reverting SV model. Leveraging the fast mean-reverting characteristic, they combine asymptotic expansion and perturbation theory, as suggested by Fouque, Papanicolaou, Sircar, and Sølna (2003), to transform the static hedging problem along the dimensions of both time and variance into two simpler static hedging problems along just the time dimension. Last, Guo and Chang (2020) evaluate European barrier options using an SHP in a generalized CEV model, where stock price volatility is an exponential function of the stock price, unrestricted in its exponent. Following Chung, Shih, and Tsai (2010, 2013a) and Tsai (2014), they not only consider the value-matching condition but also the theta-matching condition and incorporate binary options into the SHP. Moreover, they validate the use of repeated Richardson extrapolation to significantly enhance the accuracy of the SHP in evaluating European barrier options.

However, there are practical and extensional drawbacks to Fink's (2003) method. The first one is the choice of representative variances. In the time dimension, based on the option's time to maturity, such as half a year, appropriate and dispersed time points for value matching within this finite period, such as monthly or weekly intervals, can be sufficient. However, the appropriate upper and lower bounds of variance at each time point are unknown. Fink's (2003) method is limited to arbitrarily selecting a few representative variance values, which still remain constant throughout the duration of the target option,[2] casting doubt on whether this adequately represents the entire variance dimension. The second is the quantity of options in the SHP. When both variance and time dimensions are considered, the number of options included in the

[2] One potential method to take into account the probability distribution of different variances at each time point is to consider the 5th and 95th percentiles of the variance distribution at maturity as the upper and lower bounds at each checkpoint for matching testing. Within these bounds, an appropriate and diversified number of representative variances can be selected by which to construct the SHP. Theoretically, the more densely representative variances are chosen within these bounds, the greater the probability of accurately capturing the variations of the SHP in the variance dimension. A similar concept of considering the probability distribution can be found in Nalhom and Poulsen (2006).

SHP grows exponentially, which gradually erodes the benefit of lower hedging costs in the SHP method. Moreover, while theoretical research might assume the availability of vanilla European options with any expiration date and strike price, in practice, the possible expiration dates and (deeply out-of-the-money) strike prices for European options are limited. The third is the numerical issue when solving systems of equations. In Fink's method, out-of-the-money options with the same expiration date but different strike prices and variances may have very small and similarly scaled values, possibly leading to very positive or negative results when determining the investment amounts of plain vanilla European options. If large-scale trading for any single option is implemented, the SHP method's benefit of lower hedging costs could be further undermined, as discussed by Nalholm and Poulsen (2006), Huh, Jeon, and Ma (2020), and Fink (2003). Fourth, Fink's (2003) method is designed for European barrier options, utilizing the known (constant) barrier level to determine the strike prices of options in an SHP. It may not be feasible to combine Fink's method with the SHP methods developed by Chung and Shih (2009) or Chung, Shih, and Tsai (2013a, 2013b) for hedging and valuation of American options under SV, since their methods concurrently construct the SHP and determine the unknown early exercise boundary for the target American options.[3] Furthermore, under SV, the early exercise boundary at a given time point is a function of both stock price and variance, adding to the model complexity. To the best of our knowledge, there is currently no research that integrates or extends the methodologies of Fink (2003), Chung and Shih (2009), or Chung, Shih, and Tsai (2013a, 2013b) to hedge and price American options under SV.

Rather than arbitrarily considering a few fixed representative variances, this paper introduces the concept of expected variance conditional on the stock price into Chung and Shih's (2009) SHP method for hedging and pricing American options. Since it is impossible to increase the number of representative variances in an unlimited manner, we consider only the most probable occurring variance along the early exercise

[3] This characteristic allows Chung and Shih's (2009) and Chung, Shih, and Tsai's (2013a, 2013b) methods to not only hedge but also price American options. This is because the early exercise boundary of American options is a free boundary problem: solving for this boundary also yields the present value of the American option. In other words, if another American option valuation model is used first to obtain the early exercise boundary, the present value of the American option is also obtained. Therefore, subsequently using Fink's SHP method to fit the obtained boundary conditions of the American option and achieving an approximate valuation would be somewhat redundant.

boundary conditions, i.e., we consider the conditional expected variance given the stock price equal to the early exercise boundary. Note that with the use of conditional expected variance, the method proposed in this paper differs from the original concept of the SHP method, such as in Fink (2003), in accurately fitting boundary conditions for all considered time-variance nodes. We believe that if the proposed SHP method has a greater likelihood of fitting well when touching the early exercise boundary of the target American option, it probabilistically should produce more accurate valuation results for the target American option, and the average hedging performance for the target American option should be better. Furthermore, since the value and early exercise boundaries of American options are sensitive to changes in variance, and to mitigate the limitation of considering only one conditional expected variance at each examined time point, in addition to the value-matching condition and smooth-pasting condition proposed by Chung and Shih (2009), the SHP method proposed in this paper also considers the vega-matching condition.

We argue that there are some theoretical advantages of the proposed SHP method. First, Fink (2003) arbitrarily considers a few fixed representative volatilities; for example, he chooses the initial volatility level as one such representative volatility. However, it is critical to note that the probability of these representative volatilities occurring at the boundary conditions might be very low. For American puts, for example, the critical stock price at the early exercise boundary could be lower than the stock price today; moreover, due to the inverse relationship between the stock price and volatility processes, when the stock price moves toward the early exercise boundary, the accompanying volatility level could be substantially higher than its initial level. Even if Fink's (2003) method fits the boundary conditions under these low-probability representative volatilities, this might not necessarily enhance the hedging and valuation capabilities of the SHP method. Second, since this paper considers only a single variance value that is most likely to occur at the early exercise boundary, it is not difficult to incorporate our idea with Chung and Shih's (2009) method for hedging and pricing American options given SV. Third, under the SV model, the variance is stochastic as well as the stock price. A natural choice, therefore, is to consider the vega-matching condition at the early exercise surface in the time, stock price, and variance space. If the smooth-pasting condition ensures that the sensitivities of the target American option and the SHP to stock price changes at the early exercise boundary are consistent, then the vega-matching condition ensures that their sensitivities to variance changes at the early exercise boundary are consistent.

The remainder of the paper is organized as follows. Section 2 begins by reviewing the construction of an SHP for hedging American options in Chung and Shih (2009) and Heston's SV option pricing formula for European options. Section 3 details how to implement our conditional expectation of the variance under the SV model and integrate it into the SHP method of Chung and Shih (2009) for hedging and pricing American options. Section 4 analyzes the pricing, convergence, and hedging performance of the proposed SHP method. Section 5 concludes the paper.

2. Overview of Chung and Shih (2009) and Heston (1993)

2.1 Chung and Shih's (2009) SHP Method for Pricing American Options

Chung and Shih (2009) establish an SHP for American options under the assumption of the geometric Brownian for the underlying asset price, i.e.,

$$\frac{dS(t)}{S(t)} = (r-q)dt + \sigma dW(t), \quad (1)$$

where $S(t)$ is the underlying asset price at t, σ is a constant volatility, r is the risk-free rate, q is the dividend yield, and $W(t)$ denotes a standard Wiener process in the risk-neutral probability measure. Denote F as the value of any derivative asset on the underlying asset S. Then F satisfies the following PDE:

$$\frac{1}{2}\sigma^2 S^2 F_{SS} + (r-q)SF_S + F_t = rF, \quad (2)$$

where F_S and F_t denote the partial differentiation of F with respect to S and t, respectively, and F_{SS} denotes the second-order partial differentiation of F with respect to S. Based on Equation (2), for all derivative assets on S, their different values result from their different boundary conditions. This feature inspires the emergence of the SHP method: if one can construct a portfolio with more fundamental derivative assets on S (hedging position) by determining their strike prices and times to maturity to match the boundary conditions of a more complicated target derivative on S (hedged position), one can obtain the equality of values between the hedging and hedged positions not only at the boundary but also the theoretical values today. Therefore, the SHP method serves for both hedging and pricing purposes.

However, the early exercise boundary of American options is not known before conducting the SHP method; moreover, since it is a free boundary problem, the early exercise boundary of American options should be determined concurrently during the pricing process, i.e., during the process of constructing the SHP. To solve this problem, Chung and Shih (2009) impose value-matching as well as smooth-pasting conditions between the target American put and the SHP hedging positions at the early exercise boundary. During the construction of the SHP from the maturity backward toward today, the early exercise boundary of the target American put is also determined simultaneously and the information of the early exercise boundary at later time points affects the early exercise boundary at earlier time points.

Chung and Shih (2009) begin the construction of the SHP with one unit of the counterpart European vanilla put, with its strike price (X) and maturity (T) corresponding to the target American put. Then, n evenly-spaced time points before maturity are selected, i.e., t_0, $t_1 = t_0 + \delta t$, ..., $t_{n-1} = T - \delta t$, where $\delta t = \frac{T-t_0}{n}$, assuming that the SHP also matches the boundary condition of the American put at these time points. To determine the unknown boundary condition B_i at t_i, one must add to the SHP w_i units of standard European puts with maturity at t_{i+1} and a strike price of B_i. Taking the time point t_{n-1} for example, the value-matching and smooth-pasting conditions at the early exercise boundary are employed to solve the two unknowns, B_{n-1} and w_{n-1}, as

$$X - B_{n-1} = p(B_{n-1}, X, T - t_{n-1}, \sigma^2) + w_{n-1} p(B_{n-1}, B_{n-1}, T - t_{n-1}, \sigma^2), \quad (3)$$

$$-1 = p_S(B_{n-1}, X, T - t_{n-1}, \sigma^2) + w_{n-1} p_S(B_{n-1}, B_{n-1}, T - t_{n-1}, \sigma^2), (4)$$

where $p(S, X, TM, \sigma^2)$ and $p_S(S, X, TM, \sigma^2)$ are the Black–Scholes European put price and delta, respectively, with the inputs of the stock price (S), the strike price (X) and remaining time to maturity of the option (TM), and the stock return variance (σ^2). Since the risk-free interest rate (r) and dividend yield (q) are fixed as constants, we do not include them as the input parameters for simplicity. After acquiring the weight w_{n-1} and the early exercise boundary B_{n-1} at t_{n-1}, one should iteratively perform backward induction from t_{n-2} to t_0 to determine the weight and the boundary for the remaining time points. Finally, the SHP given n time points for the American put, $P_n^{SHP}(t_0)$, is formulated as

$$\begin{aligned} P_n^{SHP}(t_0) = p(S(t_0), X, T - t_0, \sigma^2) \\ +w_{n-1}p(S(t_0), B_{n-1}, T - t_0, \sigma^2) \\ +w_{n-2}p(S(t_0), B_{n-2}, t_{n-1} - t_0, \sigma^2) \\ +\cdots \\ +w_0 p(S(t_0), B_0, t_1 - t_0, \sigma^2). \end{aligned} \quad (5)$$

Chung and Shih (2009) show that when n increases, the SHP converges to the theoretical value of the examined American put.

2.2 Heston SV Model for Pricing European Options

After the 1987 crash, the assumption of constant volatility became unrealistic. Of the abundant SV literature, this paper focuses on the classical Heston (1993) model, which is an affine SV model and therefore prices European options efficiently with the analytic option formulas proposed by Heston (1993). Heston's assumptions of stochastic processes under the risk-neutral probability measure are

$$\frac{dS(t)}{S(t)} = (r - q)dt + \sqrt{v(t)}dW(t), \quad (6)$$

$$dv(t) = \kappa[\theta - v(t)]dt + \sigma_v\sqrt{v(t)}dZ(t), \quad (7)$$

$$dZ(t) = \rho dW(t) + \epsilon\sqrt{(1 - \rho^2)dt}, \quad (8)$$

where $v(t)$ is the variance process of stock return at time t, κ determines the reverting speed of the variance, θ is the long run mean of variance, σ_v is the volatility of the variance, $W(t)$ and $Z(t)$ are Wiener processes with correlation ρ, and ϵ is a standard normally distributed random variable which is independent of $dW(t)$. The value of any derivative asset F with S as the underlying asset, such as European and American options, satisfies the following PDE:

$$\frac{1}{2}vS^2F_{SS} + \rho\sigma_v vSF_{Sv} + \frac{1}{2}\sigma_v^2 vF_{vv} + (r-q)SF_S + \kappa(\theta - v)F_v + F_t = rF, \quad (9)$$

where F_s and F_{ss} (F_v and F_{vv}) represent the first- and second-order partial derivatives of value F with respect to stock price S (the variance of stock return v) and F_{sv} denotes the cross partial derivative with respect to S and v, whereas P_t signifies the partial derivative with respect to time t.

Based on the PDE in Equation (9), Heston (1993) derives the analytic-form pricing formula for the European call option as

$$H^c\big(S(t_0), X, T-t_0, v(t_0)\big) = \frac{1}{2}J(T-t_0; -i) - \frac{K}{2}J(T-t_0; 0)$$

$$-\frac{1}{\pi}\int_0^\infty \frac{Im[J(T-t_0; -i-u)\exp(iu \times \ln K)]}{u}\,du$$

$$+\frac{K}{\pi}\int_0^\infty \frac{Im[J(T-t_0; -u)\exp(iu \times \ln K)]}{u}\,du, \quad (10)$$

where

$$J(T-t_0; \phi) = S(t)^{i\phi}\exp[A(T-t_0; \phi) + B(T-t_0; \phi)v(t_0)],$$

$$A(T-t_0; \phi) = [\phi i(r-q) - r](T-t_0)$$

$$-\frac{\kappa\theta}{\sigma_v^2}\left[(\varepsilon - \rho\sigma_v\phi i + \kappa)(T-t_0) - 2\ln\left(1 - \frac{(\varepsilon + \rho\sigma_v\phi i - \kappa)(1-\exp(-\varepsilon(T-t_0)))}{2\varepsilon}\right)\right],$$

$$B(T-t_0; \phi) = \frac{i\phi(i\phi - 1)(1-\exp(-\varepsilon(T-t_0)))}{2\varepsilon - (\varepsilon + i\phi\sigma_v\rho - \kappa)(1-\exp(\varepsilon(T-t_0)))},$$

$$\varepsilon = \sqrt{(\rho\sigma_v\phi i - \kappa)^2 - \sigma_v^2\phi i(\phi i - 1)}.$$

The parameters in the function $H^c(S, X, TM, v)$ include the stock price, strike price and time to maturity of the examined options, and the initial variance level. We omit r, q, κ, and θ in the parameter list for simplicity. In addition, to obtain the value of European put options under the Heston model, we exploit the put-to-call parity equation, which is

$$H^p\big(S(t_0), X, T - t_0, v(t_0)\big) = H^c\big(S(t_0), X, T - t_0, v(t_0)\big) - Se^{-q(T-t_0)} + Xe^{-r(T-t_0)}. \quad (11)$$

3. Our Model

3.1 Stochastic Volatility Static Hedging Portfolio

When Heston's SV model is considered, it is not straightforward to hedge and price American options using the SHP method proposed in Chung and Shih (2009). The major problem is that the SHP method theoretically must solve the early exercise surface (rather than the early exercise boundary line) in the space of (S, v, t) and match conditions on it. However, it is nearly infeasible to extend Chung and Shih's (2009) method to achieve this. To address this problem, we examine matching conditions for only the expected variance conditional on the boundary level, the most probable variance level at the early exercise boundary, rather than for the constant representative variances discussed in the literature. As we argue in the introduction, if our SHP method has a greater likelihood of fitting well when touching the early exercise boundary of the target American option, it should on average produce more accurate valuation results for the target American option, and the average hedging performance for the target American option should be satisfactory. In addition to the value-matching and smooth-pasting conditions, we impose the vega-matching condition. According to Equation (9), since both F_S and F_v have roles to play, it is natural to consider the vega-matching condition (the sensitivity of the SHP with respect to v) in addition to the smooth-pasting condition (the sensitivity of the SHP with respect to S).

Following Chung and Shih (2009), we initiate the process with one unit of standard European put options, where all parameters correspond with the target American put option. Next, we evenly divide the remaining time before maturity into n time points, t_0, $t_1 = t_0 + \delta t$, ..., $t_{n-1} = T - \delta t$, where $\delta t = \frac{T-t_0}{n}$. Third, we add w_i units of standard European options with maturity at t_{i+1} and strike price at B_i into the SHP. However, different from Chung and Shih (2009), when solving the value-matching, smooth-pasting, and vega-matching conditions, we replace the constant variance σ^2 with the conditional expected variance, $E[v(t_i)|B_i]$, the details of which will be discussed in the following sections. To fulfill the vega-matching condition, we refer to Fink (2003) and add $\widehat{w}_i$ units of more deeply out-of-the-money European put options with maturity at t_{i+1} and strike price at $B_i - \gamma$, where $\gamma > 0$ is a given parameter. This modified SHP is still conducted using backward iteration, beginning at t_{n-1}:

$$\begin{aligned} X - B_{n-1} = {} & H^p(B_{n-1}, X, T - t_{n-1}, E[v(t_{n-1}|B_{n-1})]) \\ & + w_{n-1} H^p(B_{n-1}, B_{n-1}, T - t_{n-1}, E[v(t_{n-1}|B_{n-1})]) \\ & + \widehat{w}_{n-1} H^p(B_{n-1}, B_{n-1} - \gamma, T - t_{n-1}, E[v(t_{n-1}|B_{n-1})]), \end{aligned} \tag{12}$$

$$\begin{aligned} -1 = {} & \Delta^p(B_{n-1}, X, T - t_{n-1}, E[v(t_{n-1}|B_{n-1})]) \\ & + w_{n-1} \Delta^p(B_{n-1}, B_{n-1}, T - t_{n-1}, E[v(t_{n-1}|B_{n-1})]) \\ & + \widehat{w}_{n-1} \Delta^p(B_{n-1}, B_{n-1} - \gamma, T - t_{n-1}, E[v(t_{n-1}|B_{n-1})]), \end{aligned} \tag{13}$$

$$\begin{aligned} 0 = {} & \upsilon^p(B_{n-1}, X, T - t_{n-1}, E[v(t_{n-1}|B_{n-1})]) \\ & + w_{n-1} \upsilon^p(B_{n-1}, B_{n-1}, T - t_{n-1}, E[v(t_{n-1}|B_{n-1})]) \\ & + \widehat{w}_{n-1} \upsilon^p(B_{n-1}, B_{n-1} - \gamma, T - t_{n-1}, E[v(t_{n-1}|B_{n-1})]), \end{aligned} \tag{14}$$

where $H^p(S, X, TM, v)$, $\Delta^p(S, X, TM, v)$, and $\upsilon^p(S, X, TM, v)$ denote the price, delta, and vega (partial differentiation of $H^p(S, X, TM, v)$ with respect to the volatility, $\sqrt{v}$) of the European put under the Heston model; at the critical stock price B_{n-1} on the early exercise boundary, the payoff, delta, and vega of the target American put option are $X - B_{n-1}$, -1 , and 0, respectively. The vega is zero because if the American put is exercised, its payoff ($X - B_{n-1}$) is independent of the stock return variance. We solve the critical spot price B_{n-1} and the weights w_{n-1}, $\widehat{w}_{n-1}$ based on the above three equations. The first step involves expressing w_{n-1} and $\widehat{w}_{n-1}$ as functions of B_{n-1} based on Equations (13) and (14). The next step is to incorporate the obtained functions of w_{n-1} and $\widehat{w}_{n-1}$ into Equation (12). The final step entails determining the boundary root B_{n-1} through the application of the bisection method. In instances where the bisection method fails to solve a root, a brute force method is instead employed, which examines a sufficiently fine grid within the S-space to ascertain the solution.

For t_{n-2}, we add two more options that mature at t_{n-1} in the SHP for matching the three required conditions; the process is repeated backwards for t_{n-3}, t_{n-4}, ..., t_0. Since we always apply the SHP method to hedge and price an American option that is not yet early exercised at t_0 but can only be early exercised after t_0, it is a waste to solve the critical stock price B_0 at t_0, which is never touched when we conduct the hedging analysis. Therefore, we instead match the early exercise boundary at a time point slightly later than t_0, denoted as $\tilde{t}_0 = t_0 + 0.0001$. The system of equations examined at $\tilde{t}_0$ is

$$
\begin{aligned}
B_0 - X = {} & H^p(B_0, X, T - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]) \\
& + w_{n-1} H^p(B_0, B_{n-1}, T - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]) \\
& + \widehat{w}_{n-1} H^p(B_0, B_{n-1} - \gamma, T - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]) \\
& + w_{n-2} H^p(B_0, B_{n-2}, t_{n-1} - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]) \\
& + \widehat{w}_{n-2} H^p(B_0, B_{n-2} - \gamma, t_{n-1} - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]) \\
& + \cdots \\
& + w_1 H^p(B_0, B_1, t_2 - \widetilde{t_0}, E[v(\tilde{t}_0)|B_0]) \\
& + \widehat{w}_1 H^p(B_0, B_1 - \gamma, t_2 - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])
\end{aligned}
$$

$$+w_0 H^p(B_0, B_0, t_1 - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_0 H^p(B_0, B_0 - \gamma, t_1 - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]), \tag{15}$$

$$-1 = \Delta^p(B_0, X, T - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+w_{n-1}\Delta^p(B_0, B_{n-1}, T - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_{n-1}\Delta^p(B_0, B_{n-1} - \gamma, T - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+w_{n-2}\Delta^p(B_0, B_{n-2}, t_{n-1} - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_{n-2}\Delta^p(B_0, B_{n-2} - \gamma, t_{n-1} - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\cdots$$

$$+w_1\Delta^p(B_0, B_1, t_2 - \widetilde{t_0}, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_1\Delta^p(B_0, B_1 - \gamma, t_2 - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+w_0\Delta^p(B_0, B_0, t_1 - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_0\Delta^p(B_0, B_0 - \gamma, t_1 - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]), \tag{16}$$

$$0 = \upsilon^p(B_0, X, T - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+w_{n-1}\upsilon^p(B_0, B_{n-1}, T - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_{n-1}\upsilon^p(B_0, B_{n-1} - \gamma, T - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+w_{n-2}\upsilon^p(B_0, B_{n-2}, t_{n-1} - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_{n-2}\upsilon^p(B_0, B_{n-2} - \gamma, t_{n-1} - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\cdots$$

$$+w_1\upsilon^p(B_0, B_1, t_2 - \widetilde{t_0}, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_1\upsilon^p(B_0, B_1 - \gamma, t_2 - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+w_0 v^p(B_0, B_0, t_1 - \tilde{t}_0, E[v(\tilde{t}_0)|B_0])$$

$$+\widehat{w}_0 v^p(B_0, B_0 - \gamma, t_1 - \tilde{t}_0, E[v(\tilde{t}_0)|B_0]), \tag{17}$$

Here we continue to use the B_0 notation for convenience, but B_0 is actually the critical stock price on the early exercise boundary at $\tilde{t}_0$. The same logic should be applied to interpret w_0 and $\widehat{w}_0$. After solving all the unknowns B_i, w_i, $\widehat{w}_i$ at n different time points, the value of the SHP at t_0, $P_n^{SHP}(t_0)$ can be expressed as

$$\begin{aligned} P_n^{SHP}(t_0) = H^p(S(t_0), X, T - t_0, v(t_0)) \\ +w_{n-1} H^p(S(t_0), B_{n-1}, T - t_0, v(t_0)) \\ +\widehat{w}_{n-1} H^p(S(t_0), B_{n-1} - \gamma, T - t_0, v(t_0)) \\ +w_{n-2} H^p(S(t_0), B_{n-2}, t_{n-1} - t_0, v(t_0)) \\ +\widehat{w}_{n-2} H^p(S(t_0), B_{n-2} - \gamma, t_{n-1} - t_0, v(t_0)) \\ + \cdots \\ +w_1 H^p(S(t_0), B_1, t_2 - t_0, v(t_0)) \\ +\widehat{w}_1 H^p(S(t_0), B_1 - \gamma, t_0 - t_0, v(t_0)) \\ +w_0 H^p(S(t_0), B_0, t_1 - t_0, v(t_0)) \\ +\widehat{w}_0 H^p(S(t_0), B_0 - \gamma, t_1 - t_0, v(t_0)). \end{aligned} \tag{18}$$

3.2 Simulated Method for Conditional Expectation of Variance

Regarding the conditional expected variance, it is theoretically possible to determine the probability density function of the conditional variance if one knows the transition probability function $\phi(S(t), v(t)|S(t_0), v(t_0))$. This can be achieved through the following equation:

$$\phi(v(t)|S(t)) = \frac{\phi(S(t),v(t)|S(t_0),v(t_0))}{\phi(S(t)|S(t_0),v(t_0))} = \frac{\phi(S(t),v(t)|S(t_0),v(t_0))}{\int_0^\infty \phi(S(t),v(t)|S(t_0),v(t_0))dv(t)}. \quad (19)$$

Subsequently, the conditional expected variance can be derived as

$$E[v(t)|S(t)] = \int_0^\infty v(t)\phi(v(t)|S(t))dv(t). \quad (20)$$

Dragulescu and Yakovenko (2002) successfully combine the inverse Fourier and Laplace transformations to derive the transition probability function $\phi(S(t), v(t)|S(t_0), v(t_0))$ under the Heston (1993) model. Consequently, by numerically integrating the denominator of Equations (19) and (20), one can obtain $E[v(t)|S(t)]$. However, preliminary tests reveal that this method is not only excessively complex in a methodological sense, but also results in lengthy computation times for the backward induction described in the previous section. In an effort to enhance computational efficiency through numerical integration, a reduction in the number of segmentations can be considered. However, this inherently compromises precision.

One feasible way to estimate the most likely variance on the boundary at time t_i, $E[v(t_i)|B_i]$ is through simulation. We first simulate, for example, 15000 paths of stock price and variance with a time step of 5×10^{-5} based on the Heston model. Then we cluster the stock prices at time t_i according to a set of price intervals, where the stock prices are spaced, for example, by 5% of the initial stock price $S(t_0)$. After that, we calculate the average price and average corresponding variance in each cluster, i.e., $\bar{S}_j$ and $\bar{v}_j$ for $j = 1, \ldots, J$, which are then employed to estimate the conditional expected variance specifically at $\bar{S}_j$ as $E[v(t_i)|\bar{S}_j] = \bar{v}_j$ for $j = 1, \ldots, J$. Finally, we implement piece-wise linear interpolation across the J clusters (linear extrapolation based on the outermost two clusters) to obtain the most likely variance conditional on any value of B_i.

However, this workaround has a few disadvantages. First, it is time-consuming to simulate so many price and variance paths. Second, as the time draws nearer to t_0, the early exercise boundary of the American put (call) option falls (rises), but fewer price paths have the opportunity to cross the early exercise boundary. Therefore, the estimated conditional expected variance near the early exercise boundary close to t_0 is inaccurate. Consequently, the bias increases when building the SHP in a backward manner. In Panel A of Figure 1, since the range for the simulated stock price is wide enough at $t_{n-1} = 5/12$, we observe a substantial portion on which to apply piece-wise interpolation to estimate $E[v(t_{n-1})|S(t_{n-1})]$. In contrast, in Panel B of Figure 1, since the range for the simulated stock price is narrow at $t_1 = 1/12$, we must use linear extrapolation to generate less accurate estimations when the stock price is relatively low (the case for solving the early exercise boundary of American puts) and high (the corresponding case for calls).

[Figure 1 should be here]

3.3 Efficient Methods for Conditional Expectation of Variance

Due to the drawbacks of the above-mentioned simulation method, in this section we propose two methods by which to efficiently approximate the conditional expectation of variance that can be combined with the proposed SHP method under the SV model. First, we apply Ito's lemma to Equation (6) to obtain

$$d \ln S(t) = \left(r - q - \frac{v(t)}{2}\right) dt + \sqrt{v(t)} dW(t), \tag{21}$$

after which we consider the discrete-time counterparts of Equations (21), (7), and (8):

$$\Delta \ln S(t) = \left(r - q - \frac{v(t)}{2}\right) \Delta t + \sqrt{v(t)} \Delta W(t), \tag{22}$$

$$\Delta v(t) = \kappa[\theta - v(t)]\Delta t + \sigma_v \sqrt{v(t)} \Delta Z(t), \tag{23}$$

$$\Delta Z(t) = \rho \Delta W(t) + \epsilon \sqrt{(1 - \rho^2)\Delta t}. \tag{24}$$

In the first efficient method, we utilize Euler discretization to approximate the evolution of the stock price and variance processes, with the assumption that there is only one time step between t_0 and any time point t_i. Therefore, the conditional expectation of the variance at t_i can be expressed as

$$E[v(t_i)|S(t_i)] = E[v(t_0) + \Delta v(t_0)|S(t_i)] = v(t_0) + E[\Delta v(t_0)|S(t_i)]. \qquad (25)$$

According to Equation (23), we further approximate $E[\Delta v(t_0)|S(t_i)]$ as

$$E[\Delta v(t_0)|S(t_i)] = E\left[\kappa[\theta - v(t_0)](t_i - t_0) + \sigma_v\sqrt{v(t_0)}(Z(t_i) - Z(t_0))|S(t_i)\right]$$

$$= \kappa[\theta - v(t_0)](t_i - t_0) + \sigma_v\sqrt{v(t_0)}E[Z(t_i) - Z(t_0)|S(t_i)]. \qquad (26)$$

In addition, we rewrite Equation (24) as

$$Z(t_i) - Z(t_0) = \rho\big(W(t_i) - W(t_0)\big) + \epsilon\sqrt{(1-\rho^2)\Delta t};$$

given $S(t_i)$ is known, we approximate the spot innovation to be

$$W(t_i) - W(t_0) = \frac{\ln S(t_i) - \ln S(t_0) - \left(r - q - \frac{v(t_0)}{2}\right)(t_i - t_0)}{\sqrt{v(t_0)}} \qquad (27)$$

according to Equation (22). Finally, combining everything into Equation (26) yields

$$E[\Delta v(t_0)|S(t_i)] = \kappa[\theta - v(t_0)](t_i - t_0)$$

$$+\sigma_v\sqrt{v(t_0)}E\left[\rho\big(W(t_i) - W(t_0)\big) + \epsilon\sqrt{(1-\rho^2)\Delta t}|S(t_i)\right]$$

$$= \kappa[\theta - v(t_0)](t_i - t_0)$$

$$+\sigma_v\sqrt{v(t_0)}\rho\frac{\ln S(t_i) - \ln S(t_0) - \left(r - q - \frac{v(t_0)}{2}\right)(t_i - t_0)}{\sqrt{v(t_0)}} \qquad (28)$$

due to the independence of ϵ from $\Delta W(t)$; thus $E[\epsilon|S(t_i)] = 0$.

Hence we have the approximated conditional expectation of the variance at t_i:

$$E[v(t_i)|S(t_i)] = v(t_0) + \kappa[\theta - v(t_0)](t_i - t_0)$$

$$+\rho\sigma_v\left[\ln\frac{S(t_i)}{S(t_0)} - \left(r - q - \frac{v(t_0)}{2}\right)(t_i - t_0)\right]. \qquad (29)$$

The conditional expected variance estimation method outlined in Equation (29) is straightforward and convenient. In terms of computational efficiency for calculating

$E[v(t_i)|S(t_i)]$, the method described in equation (29) significantly outperforms Monte Carlo simulation combined with linear interpolation and extrapolation. However, it inherently carries a bias due to its neglect of the path-dependent nature of the SV model within the time interval between t_0 and any time point t_i. This error accumulates as t_i grows.

Although the accuracy of Euler discretization might raise concerns, in this paper we propose a second efficient method to enhance the approximation of the conditional expectation of variance by taking into account the time-dependent property of the variance to some extent in the stochastic variance model. To accomplish this, we employ the concept of drift interpolation proposed in van Haastrecht and Pelsser (2010) to approximate $\int_{t_0}^{t_i} v(\tau)d\tau$ as $\frac{v(t_i)+v(t_0)}{2}(t_i - t_0)$. In addition, we further fix $v(\tau)$ as $v(t_0)$ when evaluating $\int_{t_0}^{t_i}\sqrt{v(\tau)}dW(\tau)$ and $\int_{t_0}^{t_i}\sqrt{v(\tau)}dZ(\tau)$. Consequently, integrating Equation (21) over time, we have

$$\ln\frac{S(t_i)}{S(t_0)} = \int_{t_0}^{t_i}\left(r - q - \frac{v(\tau)}{2}\right)d\tau + \int_{t_0}^{t_i}\sqrt{v(\tau)}dW(\tau)$$

$$= (r-q)(t_i - t_0) - \frac{1}{2}\int_{t_0}^{t_i} v(\tau)d\tau + \int_{t_0}^{t_i}\sqrt{v(\tau)}dW(\tau)$$

$$= (r-q)(t_i - t_0) - \frac{1}{2}\left(\frac{v(t_i)+v(t_0)}{2}\right)(t_i - t_0) + \sqrt{v(t_0)}\int_{t_0}^{t_i}dW(\tau)$$

$$= (r-q)(t_i - t_0) - \frac{1}{2}\left(\frac{v(t_i)+v(t_0)}{2}\right)(t_i - t_0) + \sqrt{v(t_0)}\big(W(t_i) - W(t_0)\big). \quad (30)$$

Similarly, integrating Equation (7) over time yields

$$v(t_i) - v(t_0) = \int_{t_0}^{t_i}\kappa[\theta - v(\tau)]d\tau + \int_{t_0}^{t_i}\sigma_v\sqrt{v(\tau)}dZ(\tau)$$

$$= \kappa\theta(t_i - t_0) - \kappa\int_{t_0}^{t_i} v(\tau)d\tau + \sigma_v\int_{t_0}^{t_i}\sqrt{v(\tau)}dZ(\tau)$$

$$= \kappa\theta(t_i - t_0) - \kappa\left(\frac{v(t_i)+v(t_0)}{2}\right)(t_i - t_0) + \sigma_v\sqrt{v(t_0)}\int_{t_0}^{t_i} dZ(\tau)$$

$$= \kappa\theta(t_i - t_0) - \kappa\left(\frac{v(t_i)+v(t_0)}{2}\right)(t_i - t_0) + \sigma_v\sqrt{v(t_0)}\big(Z(t_i) - Z(t_0)\big). \quad (31)$$

Since Equation (30) implies

$$W(t_i) - W(t_0) = \frac{\ln\frac{S(t_i)}{S(t_0)} - \left[r - q - \frac{1}{2}\left(\frac{v(t_i)+v(t_0)}{2}\right)\right](t_i - t_0)}{\sqrt{v(t_0)}}, \quad (32)$$

we rewrite Equation (31) as

$$v(t_i) - v(t_0) = \kappa\theta(t_i - t_0) - \kappa\left(\frac{v(t_i)+v(t_0)}{2}\right)(t_i - t_0)$$

$$+\sigma_v\sqrt{v(t_0)}\left(\rho\big(W(t_i) - W(t_0)\big) + \epsilon\sqrt{(1-\rho^2)(t_i - t_0)}\right)$$

$$= \kappa\theta(t_i - t_0) - \kappa\left(\frac{v(t_i)+v(t_0)}{2}\right)(t_i - t_0)$$

$$+\sigma_v\sqrt{v(t_0)}\left\{\rho\frac{\ln\frac{S(t_i)}{S(t_0)} - \left[r - q - \frac{1}{2}\left(\frac{v(t_i)+v(t_0)}{2}\right)\right](t_i - t_0)}{\sqrt{v(t_0)}} + \epsilon\sqrt{(1-\rho^2)(t_i - t_0)}\right\}. \quad (33)$$

Therefore,

$$v(t_i)\left[1 + \left(\frac{\kappa}{2} - \frac{\rho\sigma_v}{4}\right)(t_i - t_0)\right] = v(0) + \kappa\theta(t_i - t_0) - \kappa\frac{v(t_0)}{2}(t_i - t_0)$$

$$+\rho\sigma_v\left[\ln\frac{S(t_i)}{S(t_0)} - \left(r - q - \frac{v(t_0)}{4}\right)(t_i - t_0)\right]$$

$$+\epsilon\sigma_v\sqrt{v(t_0)}\sqrt{(1-\rho^2)(t_i - t_0)}. \quad (34)$$

Finally, by taking the expectation conditional on $S(t_i)$ on both sides of the above equation, we obtain the second efficient approximations for the conditional expectation of the variance:

$$E[v(t_i)|S(t_i)] = \frac{v(0)+\left(\kappa\theta-\kappa\frac{v(t_0)}{2}\right)(t_i-t_0)+\rho\sigma_v\left[\ln\frac{S(t_i)}{S(t_0)}-\left(r-q-\frac{v(t_0)}{4}\right)(t_i-t_0)\right]}{1+(\frac{\kappa}{2}-\frac{\rho\sigma_v}{4})(t_i-t_0)}. \tag{35}$$

4. Numerical Results

This section analyzes the pricing and hedging performance of the three methods for calculating the conditional expected variance. Moreover, we focus on the degree of performance improvement caused by introducing the vega-matching condition. As a robustness test, we also investigate the impact of constructing the SHP portfolio using only European puts with standard strike prices that are really traded in the market.

4.1 Pricing Performance

We first compare the pricing performance of the SHP method with the Euler-method conditional expectation variance (Method 1^*), drift-interpolated conditional expected variance (Method 2^*), and simulated conditional expected variance (Method 3^*). Additionally, the corresponding no-vega-matching models—Method 1, Method 2 and Method 3—are also included for comparison to determine whether the vega-match condition is worth taking into account. When there is no vega-matching condition, it is not necessary to add $\widehat{w}_i$ units of European puts with the strike price to $B_i - \gamma$ into the SHP; thus our SHP method degenerates to Chung and Shih's (2009) SHP method except that the constant variance used in Chung and Shih's (2009) SHP method is replaced with the conditional expected variance.

Two sets of American puts are examined, where the size of the first set is small, appropriate for detailed analyses, and the other set of American put contracts is large and randomly generated, used to measure the average real-world performance of our SHP method. Finally, the FDM is employed to calculate the benchmark, where the minimal step sizes in time and variance are 5×10^{-5} and 0.005, respectively.

The parameter values of American puts in Set 1 are $S_0 = 100$, $r = 0.05$, $T = 0.5$, $\sigma_v = 0.3$, $\kappa = 1$, $\theta = 0.09$, $\rho = -0.7$, $K = \{90,100,110\}$, $q \in \{0.02,0.05,0.08\}$, and $v_0 \in \{0.04,0.09,0.16\}$. The detailed pricing results of the methods are presented in Table 1. When implementing the proposed SHP methods, $\gamma =$

2.5 ($2.5\% \times S_0$) and $n = 6$. Table 2 shows the root mean square percentage error (RMSPE) and root mean square error (RMSE) for the pricing results in Table 1. Panel A of Table 2 shows generally that simulation (Methods 3^* and 3) performs the best, followed by drift interpolation (Methods 2^* and 2), followed by the Euler method (Methods 1^* and 1). In addition, the implementation of the vega-matching condition yields a substantial reduction in the RMSE. The reductions are quantified by 66.8%, 77.4%, and 73.3% respectively, for Methods 1^*, 2^*, and 3^*, highlighting the importance of considering the vega-matching condition. Although Methods 1^* and 2^*, which estimate the conditional expected variance efficiently, do not perform as well as Method 3^*, the pricing performance of Methods 1^* and 2^* is still impressive. The RMSPE and RMSE for Method 1^* (Method 2^*) are 0.1283% and 0.0115 (0.0595% and 0.0061), both of which are extremely small. In addition, Panel B of Table 2 presents the RMSE under $r \geq q$ and $r < q$. Note that in instances where $r \geq q$, we still observe a substantial reduction in RMSE for Methods 1^*, 2^*, and 3^*, quantified by 66.7%, 77.6%, and 74.0% respectively; however, when $r < q$, there is no substantial difference in RMSE regardless of whether the vega-matching condition is used or not. It is well-known that the early exercise boundary for $r \geq q$ is higher than that for $r < q$, while all else being held the same. The results in Panel B of Table 2 suggest that the vega-matching condition is more effective when $r \geq q$, where the current stock price is closer to the early exercise boundary and thus it is likely to early exercise an American option. In fact, when the current stock price is further from the early exercise boundary, the probability of early exercise is smaller and the American put is more likely to resemble its European counterpart. As a result, it is of little use to consider the vega-matching condition at the early exercise boundary. In this case it is not as necessary to construct a complicated SHP for hedging: indeed, the issuer (hedger) can achieve satisfactory and cheaper hedging by considering only the counterpart European put in the SHP.

[Table 1 should be here]

[Table 2 should be here]

To clearly illustrate that including the vega-matching condition enhances the adherence between the proposed SHP and the target American puts in terms of both value and vega, we select one of the simulated stock prices from the contracts with $S_0 = 100,\ X = 110,\ r = 0.05,\ q = 0.05,\ T = 0.5,\ v_0 = 0.16,\ \sigma_v = 0.3,\ \kappa = 1,$ $\theta = 0.09$, and $\rho = -0.7$ to observe the error series of value and vega for each method. In Panel A of Figure 2, the vega errors between the SHPs and the FDM benchmark gradually increase with time for this option contract, especially for the SHPs without vega-matching. At the same time, in Panel B of Figure 2, although the value differences between the SHPs (particularly those with vega-matching) and the FDM benchmark are small, they also increase with time for this option contract. Finally, we present the average computational time required to implement the different SHP methods for calculating the conditional expected variance in Table 3, where simulation (Methods 3^* and 3) is more time consuming, and for the SHPs without imposing vega matching, the bisection method for solving the critical stock price often fails to obtain reliable results, necessitating instead the use of the brute force method, resulting in longer computation times.

[Figure 2 should be here]

[Table 3 should be here]

The basic idea of the SHP method implies that when the hedging time points n increases, the portfolio value should converge to the theoretical value of the target option. However, we modify the SHP by introducing the conditional expected variance under Heston's SV model. Therefore, we are interested in whether convergence occurs when n increases. Here we analyze the RMSPE among 27 contracts in Set 1 given different values of n to examine the convergence property of Methods $1^*,\ 1,\ 2^*$ and 2. For each panel in Figure 3, given a different n, in addition to reporting the figure of the RMSPE of Set 1 and marking it with a black dot, the top of the vertical line, the top of the vertical bar, the bottom of the vertical bar, and the bottom of the vertical line represent the maximum, 75% quantile, 25% quantile, and minimum absolute percentage error. Regardless of whether the matter vega-matching condition is considered or not, the proposed SHP methods converge quickly when $n = 4$. However, non-vega-matching SHPs are less reliable as the SHP values begin to diverge from the FDM benchmark after $n = 4$. In addition, comparing the vertical line and bar between Methods 1^* and 1 (Methods 2^* and 2), we see that the variances of the 27-contract pricing results in Set 1 based on the SHPs without vega matching (Panels B and D) are significantly higher and tend to increase with n. The results in Figure 3 demonstrate that as long as the vega-matching condition is imposed, even though the proposed SHP methods only examine one variance value—the conditional expected variance—the convergence pattern with respect to n is largely unchanged.

[Figure 3 should be here]

Note that in practice, European strike prices are not continuous. Therefore, European options with strike prices B_i in the SHP may not exist in the market because B_i are not standard strike prices. We seek to test the proposed model by considering the realistic constraint of trading only European options with standard strike prices. Therefore, in contrast to the above analyses, we assume that the existing strike prices are multiples of, say, 5% of the initial stock price and then examine the pricing and hedging performance of the demonstrated option contracts in Set 1. These experiments show how well the SHP replicates real-world American puts.

To determine a standard-strike critical boundary ($\ddot{B}_i$), we consider only strike prices that are divisible by 5 (=5%× S_0) and the set of $\Theta = [40,45,\ldots,95,100]$ to represent standard strike prices. We iteratively search from high to low for the appropriate critical boundary at time t_i. During this optimization process, we further impose two additional constraints. At time t_{n-1}, the optimization problem is formulated as

$$\min_{\ddot{B}_{n-1}, w_{n-1}, \widehat{w}_{n-1}} (\text{error of value} - \text{matching})^2$$

$$+(\text{error of smooth} - \text{pasting})^2$$

$$+(\text{error of vega} - \text{matching })^2$$

$$\text{s.t. } \ddot{B}_{n-1} \in \Theta, \ddot{B}_{n-1} \le \min(X, \frac{r}{q}X), \tag{36}$$

where $\min(X, \frac{r}{q}X)$ is the theoretical early exercise boundary of an American put when the time is the maturity date T, and the errors of value-matching, smooth-pasting, and vega-matching correspond to the mismatches between the left-hand- and right-hand-side of Equations (12), (13), and (14), respectively. For values of time t_i other than t_{n-1}, the optimization problem is formulated as

$$\min_{\ddot{B}_i, w_i, \widehat{w}_i} (\text{error of value} - \text{matching})^2$$

$$+(\text{error of smooth} - \text{pasting})^2$$

$$+(\text{error of vega} - \text{matching})^2$$

$$\text{s.t. } \ddot{B}_i \in \Theta,\ \ddot{B}_i \le \ddot{B}_{i+1}. \tag{37}$$

Moreover, when analyzing the constraints of using only standard strike prices, the parameter γ is set to the minimal interval between standard strike prices, i.e., $\gamma = 5$ in our experiments. The pricing performance results are presented in Tables 4 and 5: the RMSPEs and RMSEs both increase since the standard strike prices for the European options in the SHP are not the exact solution when solving the value-matching, smooth-pasting, or vega-matching condition. Additionally, when using only standard strike prices, the methods with the vega-matching condition remain superior: the vega-matching condition reduces RMSEs by 38.4%, 77.1%, and 21.5%, respectively for Methods 1^*, 2^*, and 3^*. In contrast, the performance of the simulated method to approximate the conditional expected variance seems to degrade. Finally, similar to Panel B of Table 2, the vega-matching condition is significantly more effective when $r \ge q$.

[Table 4 should be here]

[Table 5 should be here]

To test the robustness of the proposed SHP methods, we randomly generate 600 parameter combinations to test their pricing and hedging performance. The possible values for each parameter are summarized in Table 6. For simplicity and also for the purposes of further comparison, we fix $S_0 = 100$, $r = 0.05$, and $\theta = 0.13$ and examine 30 combinations of (X, T), where $X \in \{90,95,100,105,110\}$ and $T \in \left\{\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}\right\}$. We randomly draw 20 sets of $(q, \kappa, v_0, \sigma_v, \rho)$ from each parameter's individual uniform distribution within a reasonable range. Each combination of (X, T) is combined with the simulated 20 sets of $(q, \kappa, v_0, \sigma_v, \rho)$ to form 20 examined contracts. To differentiate the 27 option contracts examined in Table 1, these 600 option contracts are referred to as Set 2.

[Table 6 should be here]

In Panel A of Table 7, Method 2^* is the best-performing method. Second, the SHP methods with the vega-matching condition continue to be more accurate for pricing American put options, except Method 3^*. Although Method 3^* has the lowest RMSE, it exhibits the highest RMSPE, indicating significant errors in some cheaper option contracts. To determine the underlying reason, we conduct further analysis. We group and analyze different maturities to show that Method 3^* exhibits poor pricing capability for the shortest maturity; i.e., when maturity = $\frac{1}{12}$ in Set 2, it yields an RMSPE of 4.60% and an RMSE of 0.0143, compared to an RMSPE of 0.3708% (0.3621%) and an RMSE of 0.0069 (0.0055) based on Method 1^* (Method 2^*), the results of which are consistent with the theoretical drawbacks of simulation to estimate the conditional expected variance mentioned in Figure 1. Similarly, in Panel B of Table 7, when $r \geq q$, the pricing errors of the SHP methods with the vega-matching condition (Methods 1^*, 2^*, and 3^*) are substantially lower than the SHP methods without the vega-matching conditions (Methods 1, 2, and 3), but the effect from the vega-matching condition is minor when $r < q$.

[Table 7 should be here]

Finally, we draw on the method of Chung, Hung, and Wang (2010) to present an accuracy–calculation time plot for different pricing methods. The trade-off between computation time and pricing accuracy can be found in Figure 4, where accuracy is measured as $RMSE \times 100$. In this figure, in addition to FDM 1 (with a time step of 5×10^{-5} and a tick size in variance of 0.005) serving as the benchmark of the pricing result, we also compare the results of FDM 2 (with a time step of 10×10^{-5}) and FDM 3 (with a time step of 20×10^{-5}). Given the same method of estimating conditional expected variance, SHPs with vega matching always outperform their counterparts without vega matching in terms not only of smaller errors but also of less computation time. In addition, FDM 2 requires a longer computation time than Methods 1^* and 2^*, whereas its improvement in pricing accuracy is not pronounced. In contrast, FDM 3, although faster in computation, incurs significantly larger errors then the other methods. Given Figure 4, for the sake of balancing the accuracy and calculation time, we recommend the use of Method 2^* to construct the SHP under Heston's SV model.

[Figure 4 should be here]

4.2 Hedging Risk Analysis

In this section, we discuss the core value of this paper, which is the hedging performance of the SHP methods from the viewpoint of American put issuers. For each contract, we simulate 1000 stock price-variance paths based on Equations (22) and (23) with $\Delta t = 5 \times 10^{-5}$ and then calculate a cumulative hedging error of each path either when the American put is early exercised or matured at T. Suppose that as long as a stock price-variance path hits the early exercise surface generated from the benchmark FDM, the American put is exercised by its holder.

Following Chung, Huang, Shih, and Wang (2019), when selling an American put option at $P(t_0)$, we simultaneously use the sales proceeds to construct our static hedging portfolio $P_n^{SHP}(t_0)$, after which we deposit (borrow) the margin (lost) into (from) the bank account to earn (pay) the interest rate r, i.e.,

$$\eta_0 = P(t_0) - P_n^{SHP}(t_0), \tag{38}$$

where η_0 denotes the initial hedging error. If the American option does not exercise before t_1, the balance of the bank account grows at the risk-free interest rate:

$$\eta_1^- = \eta_0 e^{r(t_1 - t_0)}. \tag{39}$$

We then add the payoffs from the European options expired at t_1 in the SHP into the bank account, i.e.,

$$\eta_1 = \eta_1^- + w_0(B_0 - S_1)^+ + \widehat{w}_0(B_0 - \gamma - S_1)^+. \tag{40}$$

The bank account is continually updated in this manner until the American put option is exercised. At this time, we liquidate the SHP and define the hedging error as

$$HE_\tau = e^{-r\tau}[P_n^{SHP}(\tau) + \eta_\tau - (X - S_\tau)^+], \tag{41}$$

where τ is the American put option exercise time, $P_n^{SHP}(\tau)$ is the value of the static hedging portfolio at τ, and $(X - S_\tau)^+$ is the value paid out by the issuer. If $P_n^{SHP}(\tau) + \eta_\tau$ is larger than $(X - S_\tau)^+$, the issuer has positive hedging error (HE_τ); otherwise, the issuer has negative HE_τ. To evaluate the extreme loss of the hedging risk, we adopt four measures suggested by Siven and Poulsen (2009). The first risk measure is the 5% value at risk, defined as $VaR_{0.05} = -inf\{z \in R; Pr(HE_\tau \leq z) \leq 0.05\}$. The second risk is the expected shortfall, defined as $ES_{0.05} = -E[HE_\tau | HE_\tau \leq VaR_{0.05}]$. The third risk measure is the expected squared hedging error, defined as $ESHE_\tau = E[HE_\tau^2]$. The last risk measure is the expected loss, defined as $EL_\tau = -E[HE_\tau | HE_\tau \leq 0]$. For all four risk measures, smaller results indicate better hedging performance.

We analyze the hedging performance of the 27 data of Set 1, using both non-standard and standard European options and reported in Panels A and B, respectively, in Table 8. Panel A in Table 8 shows that the hedging ability of the proposed SHP methods is in general excellent, particularly for Methods 2^* and 3^*. Compared to the initial stock price $S_0 = 100$, the $VaR_{0.05}$ of Methods 1^*, 2^*, and 3^* (Methods 1, 2, and 3) is only 0.0834, 0.0752, and 0.0731 (0.1090, 0.1002, and 0.0934), respectively. Moreover, the vega-matching SHPs further reduce the hedging risk, compared to the corresponding non-vega-matching SHPs. As expected, due to the non-optimality of considering only standard strike prices as potential critical stock prices that solve the value-matching, smooth-pasting, or vega-matching condition, the hedging performance is slightly weakened in Panel B in Table 8. Furthermore, the decline of the performance of Method 3^* is significantly greater than those of Methods 1^* and 2^*.

[Table 8 should be here]

Last, we examine the hedging performance of the proposed SHP methods for Set 2. Among the 600 option contracts in Set 2, the performance of Methods 2^* and 3^* is almost equal for the four risk measurements. In addition, the SHPs with vega matching dominate the SHPs without vega matching in terms of the examined four risk measurements. Compared with Panel of Table 8, the hedging performance for Set 1, the superiority of the introduced vega-matching condition weakens in Table 9. Perhaps this is due to the short maturities in Set 2, ranging from $\frac{1}{12}$ to $\frac{6}{12}$; in Set 1 the time to maturity is fixed at 0.5. Recall that in Figure 2, the mismatching problem with respect to vega is less severe when the passage of time is short, which implies that the marginal benefit of introducing the vega-matching condition in the SHP method to hedge American puts with short maturities is relatively small. Table 10 reports the hedging measurement for early exercised paths in Set 2. We conduct this analysis because the issuers or hedgers who employ our SHP method are most concerned about hedging risk when target American puts are early exercised. For early exercise paths, the performance of Methods 2^* and 3^* is still equal. The four risk measurements of Methods 1^*, 2^*, and 3^* are better than those of Methods 1, 2, and 3. For example, the $VaR_{0.05}$ decreases by 28.2%, 32.8%, and 31.6% and $ES_{0.05}$ decreases by 12.2%, 17.7%, and 15.1% due to the vega-matching condition. The results in Table 10 demonstrate the superiority of using Methods 1^*, 2^*, and 3^* to hedge American puts that are finally exercised.

[Table 9 should be here]

[Table 10 should be here]

5. Conclusion

In this paper, we propose a novel method to construct an SHP under Heston's SV model for American puts. First, to extend Chung and Shih's (2009) method under SV, we replace the constant variance with the conditional expected variance in their method. Furthermore, we examine three methods to estimate the conditional expected variance: the Euler method, drift interpolation, and simulation. Moreover, since the purpose of the smooth-pasting condition is to align the sensitivity with respect to the stock price change of the SHP and the target option at the early exercise boundary, this paper incorporates the vega-matching condition in the SHP method, which minimizes the sensitivity mismatch with respect to the variance change between the SHP and the target option at the early exercise boundary. To implement the vega-matching condition, we adopt Fink's (2003) method by involving a further out-of-the-money European option at each examined time point.

Our numerical results indicate that the proposed SHP methods show excellent pricing accuracy, particularly those with the vega-matching condition. The vega-matching condition reduces the pricing error by 66.8%, 77.4%, and 73.3% (38.4%, 77.1%, and 21.5%) for the Euler method, drift interpolation, and simulation to calculate the conditional expected variance for the illustrative 27 option contracts (when only standard strike prices are allowed to be traded). Furthermore, the inclusion of the vega-matching condition in the SHP methods not only enhances their stability but also ensures convergence toward the theoretical values of the target American puts when the number of examined time points increases.

The hedging performance of the proposed SHP methods is also impressive. The advantage of introducing vega-matching is pronounced in hedging risks. For the illustrative 27 option contracts, we show that the hedging risk is significantly smaller in the vega-matching SHPs, resulting in the reduction of $VaR_{0.05}$ ($ES_{0.05}$) by 23.5%, 25.0%, and 21.7% (18.0%, 17.7%, and 14.3%) for the Euler method, drift interpolation, and simulation to calculate the conditional expected variance, respectively.

Last, even when examining randomly generated option contracts, the pricing and hedging performance of the proposed SHP methods is still satisfactory, particularly when the vega-matching condition is imposed. As a compromise between accuracy and computation time, we suggest using the proposed SHP with drift interpolation to calculate the conditional expected variance under Heston's SV model.

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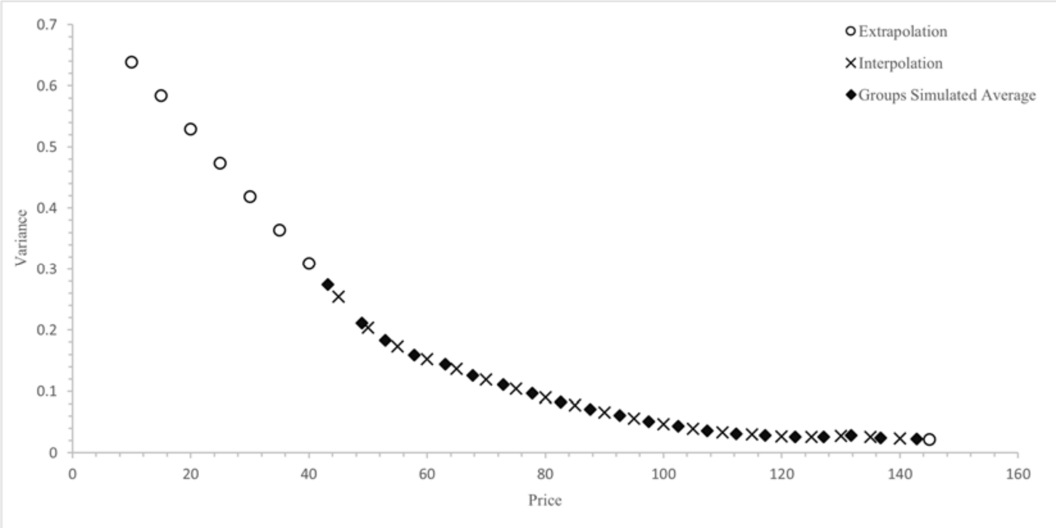
Panel A: Simulated variance conditional on stock price at $t = \frac{5}{12}$



Panel B: Simulated variance conditional on stock price at $t = \frac{1}{12}$

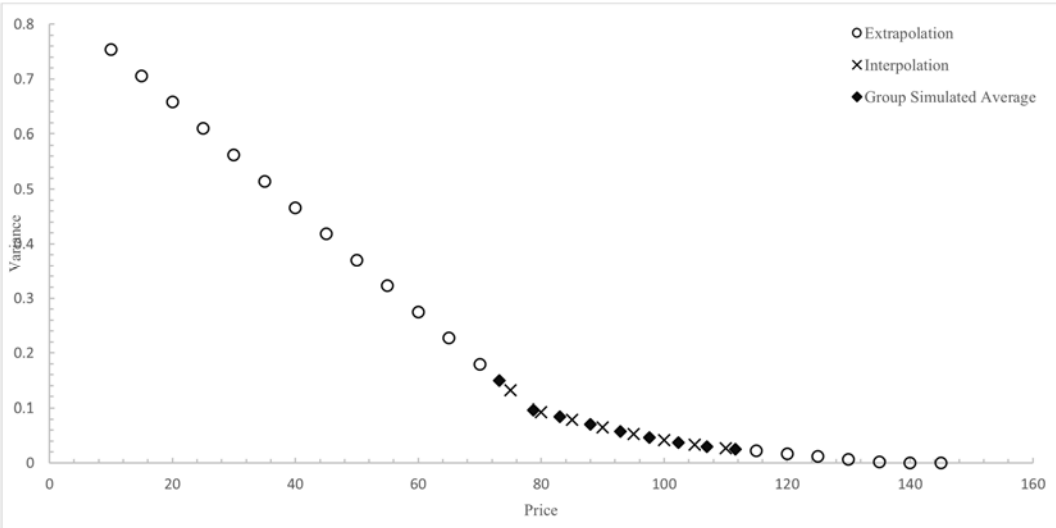
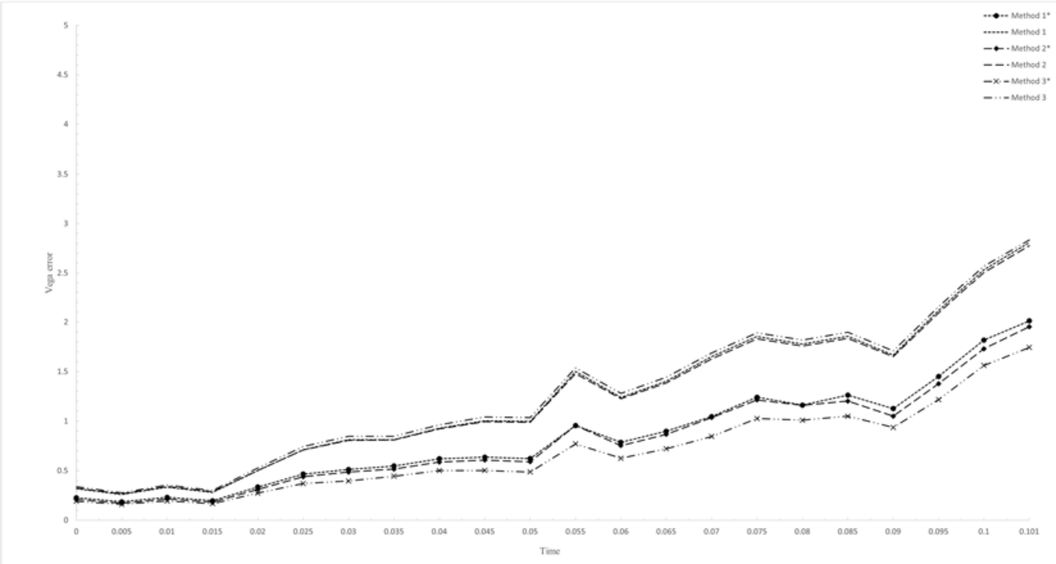


Figure 1. Simulated conditional expected variance. We conduct 15,000 simulated paths with a time step of 5×10^{-5} based on Equations (22) and (23) and then group the price paths into fixed price intervals at each examined time t_i. The parameters are $S_0 = 100$, $X = 90$, $r = 0.05$, $q = 0.02$, $T = 0.5$, $v_0 = 0.04$, $\sigma_v = 0.3$, $\kappa = 1$, $\theta = 0.09$, $\rho = -0.7$, and $n = 6$. In the figure, the diamonds represent the average variance at t_i across all corresponding variance paths within each price interval, with the interval's average stock price serving as the representative stock value. The crosses indicate the variance values piece-wise interpolated from the simulated average variance and representative stock value. The circles represent the stock prices outside the bounds of the stock price paths, necessitating the estimation of the conditional expected variance with linear extrapolation based on the outermost two simulated points.

Panel A: Vega Error Trends



Panel B: Value Error Trends

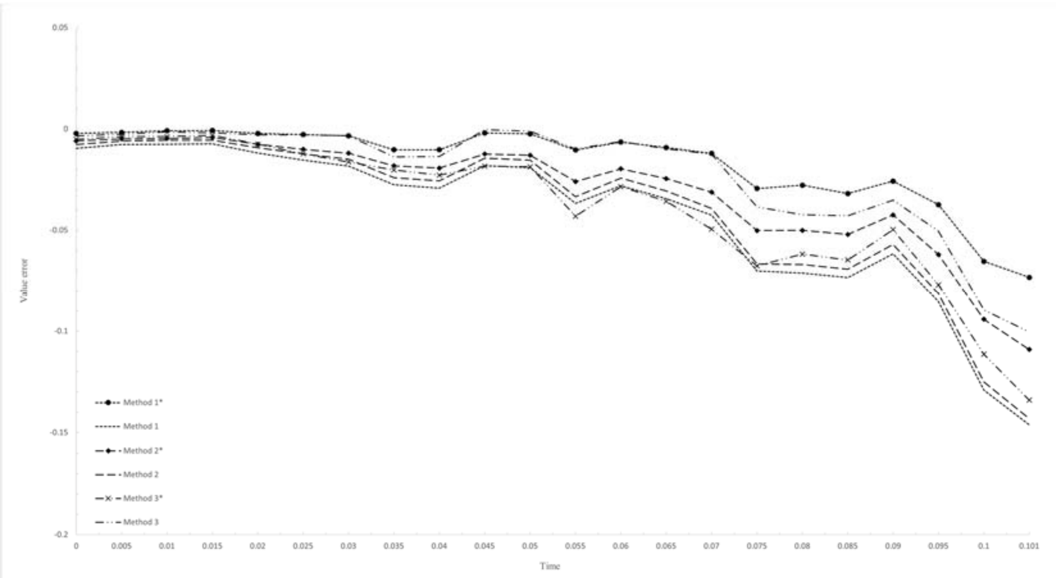
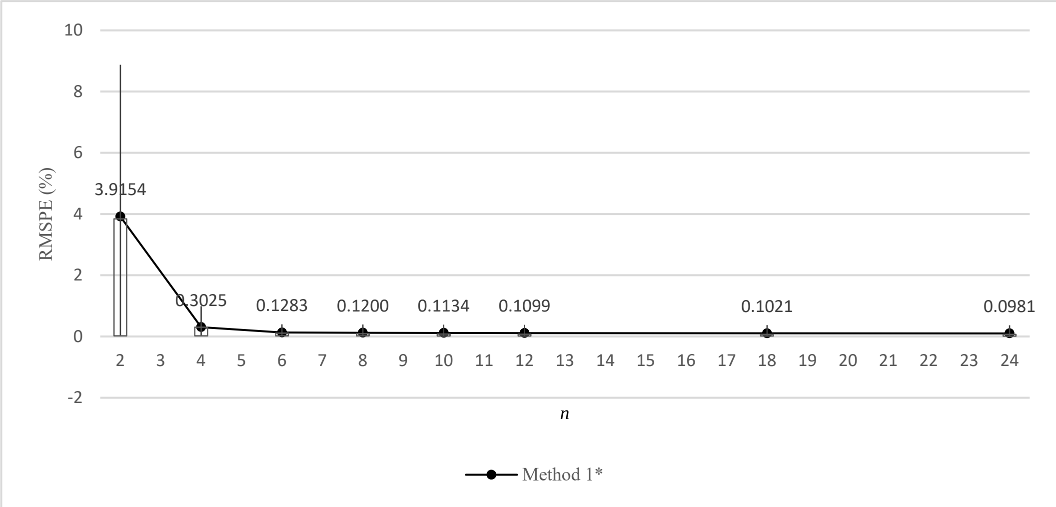
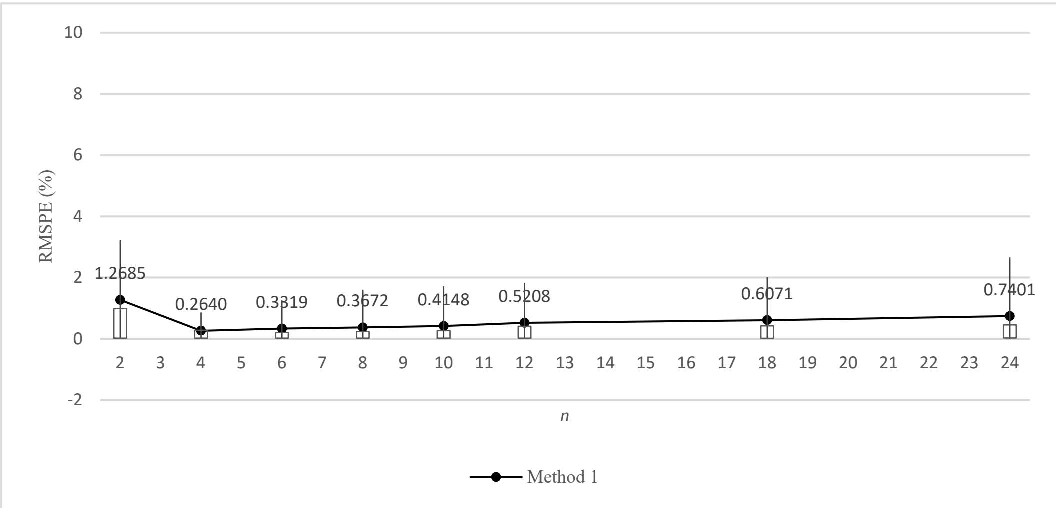


Figure 2. Vega and value error trends. This figure depicts the time-series disparities in the vega and value of the different methods for a simulated price path that is early exercised at a time to be 0.101. The benchmarks are derived based on the FDM. Since differences of the vega and value results of different methods are minor, particularly when the passage of time is short, to enhance the detectability, we subtract the benchmark option vegas (values) from the SHP (vegas) values for each method at different time points. The parameters are $S_0 = 100$, $X = 110$, $r = 0.05$, $q = 0.05$, $T = 0.5$, $v_0 = 0.16$, $\sigma_v = 0.3$, $\kappa = 1$, $\theta = 0.09$, $\rho = -0.7$, $\gamma = 2.5$, and $n = 6$.

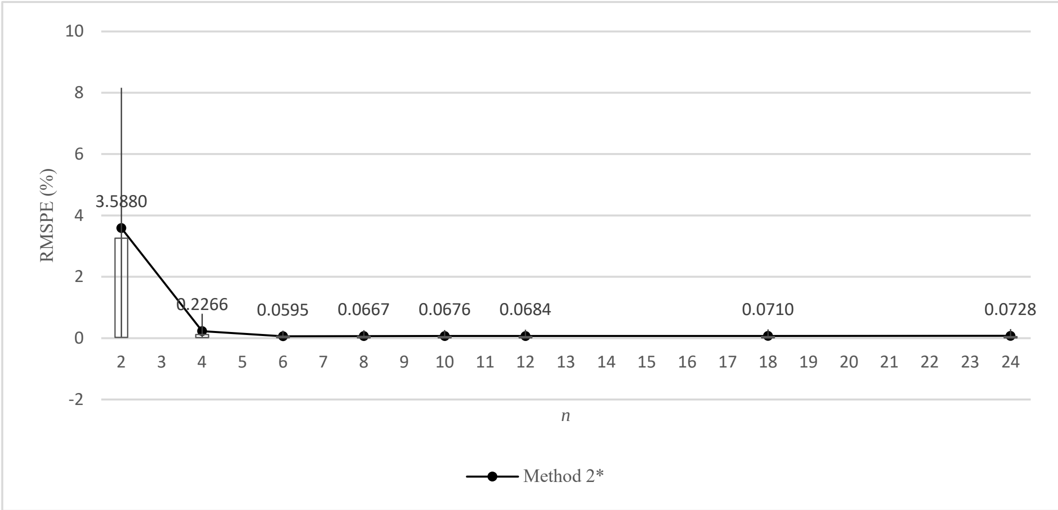
Panel A: Method 1^* convergence analysis of Set 1



Panel B: Method 1 convergence analysis of Set 1



Panel C: Method 2^* convergence analysis of Set 1



Panel D: Method 2 convergence analysis of Set 1

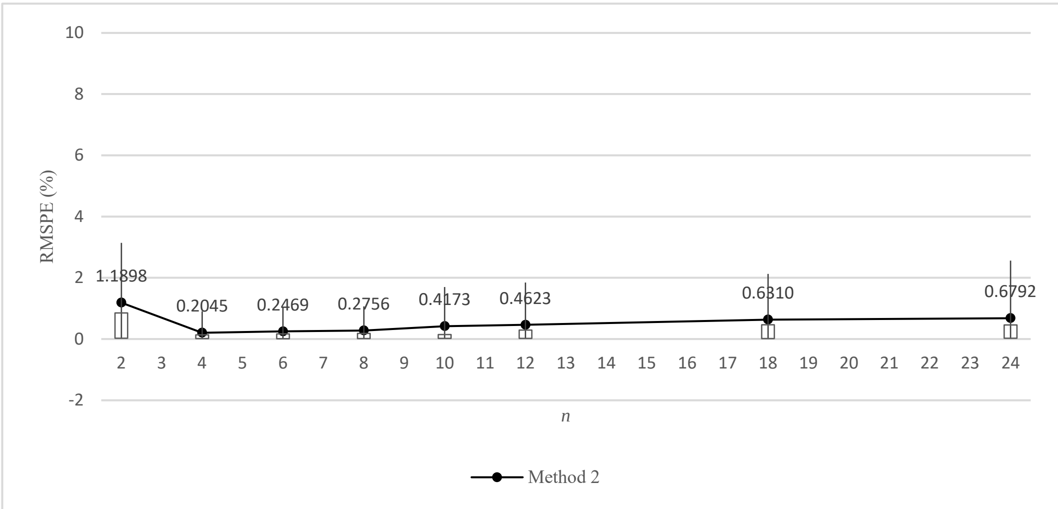


Figure 3. Convergence analyses with respect to n. Based on the option contracts in Table 1, this figure illustrates the RMSPE of the SHP Method 1^*, 1, 2^*, and 2 given various values for n. When implementing these SHP methods, γ is fixed at 2.5. Taking $n = 6$ as an example, the black dots represent the RMSPE among the option contracts in Set 1 given $n = 6$. For a given n, the top of the vertical line, the top of the vertical bar, the bottom of the vertical bar, and the bottom of the vertical line represent the maximum, 75th percentile, 25th percentile, and minimum absolute percentage error among the 27 contracts in Set 1, respectively. If there is no vega-matching condition, the pricing results apparently do not improve with n: not only do the RMSPEs of the SHP methods clearly increase with n, but the variances of the errors among Set 1 also become larger with n.

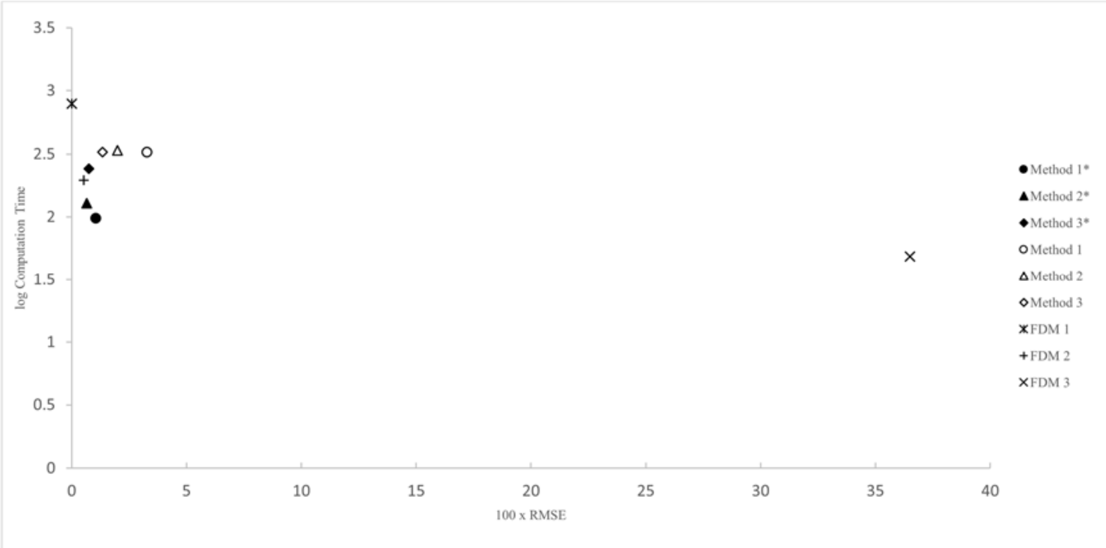


Figure 4. Accuracy–computation time comparison based on Set 2. In this figure, in addition to FDM 1 (with a time step of 5×10^{-5} and a tick size in variance of 0.005) serving as the benchmark of the pricing result, we also compare the results of FDM 2 (with a time step of 10×10^{-5}) and FDM 3 (with a time step of 20×10^{-5}). When implementing the SHP methods, $\gamma = 2.5$ and $n = 6$. A leftward trend indicates a decrease in error, while a downward trend signifies shorter computation time. Two key observations can be drawn from the graph: First, FDM 2 requires a longer computation time than Methods 1* and 2*, whereas the improvement in pricing accuracy is not substantial. In contrast, FDM 3, although faster in computation, incurs significantly larger errors then the other methods. Secondly, within the same method of estimating conditional expected variance, SHPs with vega matching outperform those without vega matching in terms of not only smaller errors but also shorter computation times.

X	q	v_0	FDM	Method 1*	Method 2*	Method 3*	Method 1	Method 2	Method 3
90	0.02	0.04	2.2485	2.2430	2.2472	2.2465	2.2396	2.2446	2.2470
90	0.02	0.09	3.7897	3.7835	3.7876	3.7868	3.7787	3.7833	3.7872
90	0.02	0.16	5.5979	5.5920	5.5953	5.5947	5.5875	5.5900	5.5945
100	0.02	0.04	5.5089	5.4931	5.5042	5.5060	5.4742	5.4876	5.5085
100	0.02	0.09	7.5786	7.5666	7.5742	7.5748	7.5510	7.5581	7.5764
100	0.02	0.16	9.7996	9.7914	9.7959	9.7958	9.7826	9.7938	9.7935
110	0.02	0.04	11.3155	11.2690	11.2893	11.3006	11.1735	11.1982	11.2259
110	0.02	0.09	13.2721	13.2495	13.2597	13.2585	13.1797	13.2140	13.2591
110	0.02	0.16	15.4979	15.4862	15.4914	15.4894	15.4808	15.4657	15.4942
90	0.05	0.04	2.5017	2.5005	2.5016	2.5012	2.5003	2.5010	2.5013
90	0.05	0.09	4.0961	4.0945	4.0958	4.0954	4.0942	4.0958	4.0957
90	0.05	0.16	5.9396	5.9458	5.9438	5.9428	5.9375	5.9390	5.9390
100	0.05	0.04	6.0045	6.0026	6.0054	6.0057	5.9990	6.0020	6.0066
100	0.05	0.09	8.0734	8.0712	8.0736	8.0735	8.0705	8.0736	8.0742
100	0.05	0.16	10.2929	10.2973	10.2968	10.2935	10.2902	10.2925	10.2926
110	0.05	0.04	11.9836	11.9752	11.9809	11.9818	11.9582	11.9643	11.9864
110	0.05	0.09	13.9124	13.9068	13.9106	13.9109	13.9055	13.9102	13.9068
110	0.05	0.16	16.1134	16.1118	16.1136	16.1125	16.1090	16.1118	16.1140
90	0.08	0.04	2.7997	2.8000	2.8000	2.7999	2.8000	2.8000	2.8000
90	0.08	0.09	4.4480	4.4484	4.4485	4.4484	4.4484	4.4484	4.4484
90	0.08	0.16	6.3234	6.3239	6.3240	6.3239	6.3239	6.3240	6.3239
100	0.08	0.04	6.6083	6.6107	6.6108	6.6107	6.6107	6.6107	6.6107
100	0.08	0.09	8.6580	8.6598	8.6599	8.6599	8.6598	8.6599	8.6599
100	0.08	0.16	10.8579	10.8595	10.8596	10.8596	10.8593	10.8595	10.8595
110	0.08	0.04	12.9184	12.9210	12.9211	12.9210	12.9209	12.9211	12.9210
110	0.08	0.09	14.7239	14.7259	14.7261	14.7260	14.7258	14.7260	14.7259
110	0.08	0.16	16.8457	16.8475	16.8478	16.8476	16.8472	16.8476	16.8474

Table 1. Pricing results for Set 1. The parameters are $S_0 = 100$, $r = 0.05$, $T = 0.5$, $\sigma_v = 0.3$, $\kappa = 1$, $\theta = 0.09$, $\rho = -0.7$, $X \in \{90,100,110\}$, $q \in \{0.02,0.05,0.08\}$, $v_0 \in \{0.04,0.09,0.16\}$, $\gamma = 2.5$, and $n = 6$.

Panel A: Pricing errors for Set 1

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
RMSPE	0.1283%	0.0595%	0.0477%	0.3319%	0.2469%	0.1565%
RMSE	0.0115	0.0061	0.0047	0.0346	0.0270	0.0176

Panel B: RMSE given $r \geq q$ or $r < q$

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$r \geq q$	0.0141	0.0074	0.0056	0.0423	0.0330	0.0215
$r < q$	0.0017	0.0018	0.0018	0.0016	0.0018	0.0017

Table 2. Pricing errors under different methods of estimating conditional expected variance for Set 1. The definitions of RMSPE and RMSE are as follows: $RMSPE = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(\frac{P_i - P_i^b}{P_i^b} \times 100\right)^2}$, $RMSE = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(P_i - P_i^b)^2}$, where P_i^b is the benchmark pricing result and $m = 27$ for Set 1. When implementing the SHP methods, $\gamma = 2.5$ and $n = 6$. Panel A shows that under the same method for estimating conditional expected variance, vega-matching SHPs exhibit smaller errors compared to non-vega-matching SHPs. In addition, simulation (Methods 3* and 3) performs the best, followed by drift interpolation (Methods 2* and 2), followed by the Euler method (Method 1* and 1). In Panel B, under $r \geq q$, corresponding to a higher early exercise boundary and consequently greater probability of early exercise, our results show a significant reduction in errors for SHPs with vega matching in this scenario. For reference, within Set 1, focusing solely on the difference between the target American puts and the counterpart European puts with the same strike price and time to maturity, their RMSEs are 0.1156 and 0.0029 when $r \geq q$ and $r < q$, respectively.

Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
85.52	85.03	326.81	207.15	216.56	375.81

Table 3. Average computation time (in seconds) for Set 1. The data presented in the table reveal that the time taken for simulation is longer than that required for computing the conditional expected variance based on the Euler and drift interpolation methods. Additionally, we find that non-vega-matching portfolios are more likely to resort to the brute force method when solving the value-match and smooth-pasting conditions during backward induction, resulting in longer computation times

X	q	v_0	FDM	Method 1*	Method 2*	Method 3*	Method 1	Method 2	Method 3
90	0.02	0.04	2.2485	2.2182	2.2187	2.2574	2.2180	2.2180	2.2180
90	0.02	0.09	3.7897	3.7868	3.7777	3.7781	3.7392	3.7396	3.7396
90	0.02	0.16	5.5979	5.5785	5.5289	5.5296	5.5286	5.5286	5.5286
100	0.02	0.04	5.5089	5.3955	5.3981	5.4937	5.3946	5.3946	5.3946
100	0.02	0.09	7.5786	7.5846	7.5562	7.5527	7.4412	7.4412	7.4412
100	0.02	0.16	9.7996	9.7915	9.7795	9.6483	9.6457	9.6457	9.6457
110	0.02	0.04	11.3155	10.9250	11.2869	10.9327	10.9214	10.9214	10.9214
110	0.02	0.09	13.2721	13.2774	13.2371	12.9454	12.9386	12.9385	12.9385
110	0.02	0.16	15.4979	15.5602	15.5376	15.5206	15.1916	15.1915	15.1915
90	0.05	0.04	2.5017	2.4973	2.5015	2.5021	2.4946	2.4952	2.4952
90	0.05	0.09	4.0961	4.0897	4.0905	4.0905	4.0825	4.0825	4.0825
90	0.05	0.16	5.9396	5.9270	5.9347	5.9274	5.9168	5.9167	5.9167
100	0.05	0.04	6.0045	5.9945	6.0055	6.0042	5.9804	5.9826	5.9826
100	0.05	0.09	8.0734	8.0767	8.0680	8.0676	8.0356	8.0383	8.0382
100	0.05	0.16	10.2929	10.3105	10.3092	10.3067	10.2442	10.2441	10.2441
110	0.05	0.04	11.9836	11.9906	11.9685	11.9573	11.9030	11.9093	11.9090
110	0.05	0.09	13.9124	13.9308	13.9135	13.9081	13.8210	13.8279	13.8276
110	0.05	0.16	16.1134	16.0851	16.1338	16.1004	16.0106	16.0180	16.0178
90	0.08	0.04	2.7997	2.8000	2.8000	2.8000	2.7999	2.7999	2.7999
90	0.08	0.09	4.4480	4.4486	4.4485	4.4486	4.4482	4.4482	4.4482
90	0.08	0.16	6.3234	6.3238	6.3247	6.3233	6.3232	6.3232	6.3232
100	0.08	0.04	6.6083	6.6109	6.6108	6.6108	6.6105	6.6105	6.6105
100	0.08	0.09	8.6580	8.6599	8.6598	8.6599	8.6589	8.6589	8.6589
100	0.08	0.16	10.8579	10.8607	10.8603	10.8595	10.8569	10.8569	10.8569
110	0.08	0.04	12.9184	12.9213	12.9211	12.9210	12.9201	12.9210	12.9201
110	0.08	0.09	14.7239	14.7268	14.7265	14.7265	14.7229	14.7228	14.7228
110	0.08	0.16	16.8457	16.8482	16.8476	16.8475	16.8405	16.8405	16.8405

Table 4. Pricing results for Set 1 using standard strike prices. The parameters are $S_0 = 100$, $r = 0.05$, $T = 0.5$, $\sigma_v = 0.3$, $\kappa = 1$, $\theta = 0.09$, $\rho = -0.7$, $X \in \{90,100,110\}$, $q \in \{0.02,0.05,0.08\}$, $v_0 \in \{0.04,0.09,0.16\}$, $\gamma = 5$, and $n = 6$. In the experiment, the critical early exercise boundary ($\ddot{B}_i$), serving as the strike price of the European option in the SHP, is restricted to be in the set of $\Theta = [40,45,\dots,95,100]$.

Panel A: Pricing errors for Set 1 using standard strike prices

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
RMSPE	0.8274%	0.5388%	0.9024%	1.2139%	1.2098%	1.2099%
RMSE	0.0799	0.0296	0.1015	0.1298	0.1292	0.1293

Panel B: RMSE using standard strike prices given $r \geq q$ or $r < q$

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$r \geq q$	0.0979	0.0362	0.1256	0.1590	0.1583	0.1583
$r < q$	0.0022	0.0020	0.0018	0.0020	0.0022	0.0021

Table 5. Pricing errors under different methods using standard strike prices for Set 1. When implementing the SHP methods, $\gamma = 5$ and $n = 6$. In addition, the critical early exercise boundary ($\ddot{B}_i$), serving as the strike price of the European option in the SHP, is restricted to be in the set of $\Theta = [40,45,\dots,95,100]$. Upon comparison with the data presented in Table 2, it is evident that the use of standard strike prices, which are not exact for solving value-matching, smooth-pasting, or vega-matching conditions, results in larger, but still acceptable pricing errors. However, the superiority of methods with the vega-matching condition still holds. Moreover, simulation seems to lose its advantage when approximating the conditional expected variance.

S	X	r	q	κ
100	{90, 95, 100, 105, 110}	0.05	unif(0, 0.08)	unif(0.1, 5)
v_0	θ	σ_v	ρ	T
unif(0.01, 0.25)	0.13	unif(0.1, 0.5)	unif(-0.9, -0.5)	$\{\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}\}$

Table 6. Parameter value pools for Set 2. To facilitate further comparison, we randomly drew 20 sets of data for q, κ, v_0, σ_v, and ρ, where unif(a,b) is defined as the uniform distribution between a and b. Then each combination of X and T, for example, $(X, T) = (90,3/12)$, was combined with the simulated 20 sets of q, κ, v_0, σ_v, and ρ to form 20 examined contracts. We generated a total of 600 option contracts in Set 2.

Panel A: Pricing errors for Set 2

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
RMSPE	0.1944%	0.1617%	1.8775%	0.3090%	0.1930%	0.1306%
RMSE	0.0104	0.0066	0.0075	0.0326	0.0199	0.0134

Panel B: RMSE given $r \geq q$ or $r < q$

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$r \geq q$	0.0110	0.0069	0.0079	0.0344	0.0395	0.0141
$r < q$	0.0022	0.0019	0.0020	0.0029	0.0035	0.0019

Table 7. Pricing errors under different methods for 600 randomly generated option contracts in Set 2. This table displays the pricing errors compared to a benchmark for the proposed SHPs formed with and without the vega-matching condition, under various methods of estimating conditional expected variance. When implementing the SHP methods, $\gamma = 2.5$ and $n = 6$. Overall speaking, Method 2^* demonstrates the best pricing performance. In Panel A, it is still best to include the vega-matching condition. Although Method 3^* performs well in terms of RMSE, it exhibits the largest RMSPE, indicating significant errors for some cheaper option contracts. After removing the cheaper option contracts with the shortest maturity of $T = \frac{1}{12}$, the resulting RMSPE and RMSE of Method 3^* becomes 0.6342% and 0.0070, respectively. This observation underscores the inherent instability associated with simulation to estimate the conditional expected variance, particularly when the time to maturity is relatively short.

Panel A: Hedging risk measurements for Set 1

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$VaR_{0.05}$	0.0834	0.0752	0.0731	0.1090	0.1002	0.0934
$ES_{0.05}$	0.1158	0.1085	0.1069	0.1413	0.1319	0.1248
$ESHE$	0.0432	0.0430	0.0426	0.0457	0.0449	0.0449
EL	0.0411	0.0376	0.0369	0.0551	0.0505	0.0465

Panel B: Hedging risk measurements for Set 1 using standard strike prices

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$VaR_{0.05}$	0.1252	0.1268	0.1511	0.2856	0.2809	0.2818
$ES_{0.05}$	0.2013	0.2057	0.2245	0.3703	0.3654	0.3660
$ESHE$	0.0569	0.0540	0.0623	0.0776	0.0771	0.0772
EL	0.0672	0.0672	0.0839	0.1581	0.1558	0.1559

Table 8. Hedging risk measurements for Set 1. When implementing the SHP methods, $n = 6$ and $\gamma = 2.5$ (5) for Panel A (B). First, Methods 2^* and 3^* perform the best in Panel A, but Method 3^* performs poorly when using only standard strike prices. Second, both panels consistently demonstrate that the proposed SHP methods with vega matching exhibit smaller (better) hedging risk compared to those without vega matching. Additionally, due to the constraint of using only standard strike prices, Panel B exhibits increased hedging risk.

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$VaR_{0.05}$	0.0677	0.0607	0.0601	0.0865	0.0812	0.0789
$ES_{0.05}$	0.1412	0.1324	0.1329	0.1508	0.1462	0.1443
$ESHE$	0.0071	0.0066	0.0065	0.0083	0.0073	0.0072
EL	0.0365	0.0314	0.0319	0.0460	0.0428	0.0405

Table 9. Hedging risk measurements for Set 2. When implementing the SHP methods, $\gamma = 2.5$ and $n = 6$. The best performing methods are Methods 2^* and 3^*: their performance on the examined four risk measurements is almost equal. This table still demonstrates that the SHPs with vega matching exhibit smaller (better) hedging risk compared to those without vega matching. In contrast to Panel A of Table 8, since the average time to maturity in Set 2 is shorter and thus the mismatching problem with respect to vega is less severe (see Figure2 for illustration), the advantage of introducing the vega-matching condition is less evident in this table.

	Method 1^*	Method 2^*	Method 3^*	Method 1	Method 2	Method 3
$VaR_{0.05}$	0.0490	0.0419	0.0412	0.0682	0.0624	0.0602
$ES_{0.05}$	0.0877	0.0776	0.0782	0.0999	0.0943	0.0921
$ESHE$	0.0051	0.0043	0.0041	0.0063	0.0050	0.0050
EL	0.0320	0.0290	0.0290	0.0400	0.0364	0.0352

Table 10. Hedging risk measurements for early exercised paths of Set 2. When implementing the SHP methods, $\gamma = 2.5$ and $n = 6$, the performance of Methods 2^* and 3^* is still equal in terms of the examined four risk measurements. Within these paths, for Method 1^*, 2^*, and 3^*, their $VaR_{0.05}$ decreases by 28.2%, 32.8%, and 31.6% and $ES_{0.05}$ decreases by 12.2%, 17.7%, and 15.1% because of the vega-matching condition, which demonstrates their superior hedging ability for American puts that are finally exercised.