Dynamic Asset Allocation with Time-Varying Investment Opportunities: How Costly are Deviations from the Optimal Investment Strategy?

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Abstract. The recent theoretical asset allocation literature has derived optimal dynamic investment strategies in a number of relatively advanced models of asset returns. But how important is it to get the intertemporal hedge absolutely correct or include the hedge term at all? Will unsophisticated investors do almost as well as sophisticated investors? More generally, how costly is it to deviate from the optimal investment strategy in some specific way? This paper provides some general results and a general framework for answering such questions and studies some specific model examples in detail. First, we discuss the importance of investing in long-term bonds that hedge interest rate risk. Second, we discuss the importance of investing in stock options that can hedge variations in stock price volatility. Third, we discuss the benefits from investing in growth stocks and value stocks in a framework with mean reversion in the returns of those stocks.

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1 Introduction

The seminal papers of Samuelson (1969) and Merton (1969) showed that the main conclusions of the static mean-variance theory of Markowitz (1952, 1959) carry over to a multi-period framework if investment opportunities (and relevant investor-specific variables) remain unchanged over time. On the other hand, in the presence of time-varying investment opportunities Merton (1971) demonstrated that long-term investors should generally deviate from the myopically optimal mean-variance strategy by a so-called intertemporal hedge term. Over the last decade numerous studies have derived the optimal intertemporal hedge demand and thus the optimal portfolio in closed form (or almost closed form) in various specific models of time-varying return dynamics incorporating, e.g., stock return predictability,[1] stochastic volatility,[2] and interest rate variations.[3] Only few studies, however, discuss how important it is to get the intertemporal hedge absolutely correct or include the hedge term at all. Will unsophisticated investors do almost as well as sophisticated investors? More generally, how costly is it to deviate from the optimal investment strategy in some specific way? How damaging is it to base your investment strategy on a given model of return dynamics when the actual return dynamics is different? How much is the utility of an investor affected by omitting a given asset from her portfolio? This paper provides some general results and a general framework for answering such questions and studies some specific model examples in detail.

Throughout the paper, we use continuous-time modeling and focus on investors who, equipped with some initial wealth, want to maximize expected power utility of terminal wealth. First, we consider a general model of return dynamics—equivalent to the setting of Liu (2007)—where both the short-term risk-free rate and the excess expected returns, volatilities, and covariances of the risky assets are governed by a general, possibly multi-dimensional, diffusion state variable. We show that any relevant investment strategy leads to an expected utility that can be characterized by the solution to a relatively simple partial differential equation (PDE). The utility loss associated with a sub-optimal investment strategy is defined as the percentage extra initial wealth required to bring the investor following the sub-optimal strategy to the same expected utility level that

[1] See, e.g., Kim and Omberg (1996), Barberis (2000), and Wachter (2002).

[2] See, e.g., Liu and Pan (2003), Chacko and Viceira (2005), and Liu (2007).

[3] See, e.g., Sørensen (1999), Brennan and Xia (2000, 2002), Campbell and Viceira (2001), Wachter (2003), Munk and Sørensen (2004), and Sangvinatsos and Wachter (2005).

can be obtained by following the optimal strategy. This utility loss can be computed from solving two PDEs, one for the given sub-optimal strategy and one for the optimal strategy. We specialize this result to three specific sub-optimal strategies: (i) the optimal strategy given that some assets are omitted from the portfolio, (ii) the myopic, "no hedge" strategy, and (iii) a "small" deviation from the optimal portfolio weights. We show that when the return dynamics have an "affine" or "quadratic" structure, the utility losses associated with these three sub-optimal strategies can be derived from solving appropriate ordinary differential equations.

In particular, the case (ii) allows us to address the importance of intertemporal hedging. Some authors report that, for the specific model of return dynamics they consider, the intertemporal hedging demand is quite small; see, e.g., Aït-Sahalia and Brandt (2001), Ang and Bekaert (2002), Brandt (1999), and Chacko and Viceira (2005). However, it is not clear that a small change in the long-term investment strategy cannot have a significant impact on the expected life-time utility. In fact in a model with a constant risk-free rate and a single stock index with constant expected return and time-varying volatility, Gomes (2007) reports small intertemporal hedging demands and significant—although not dramatically large—utility losses from ignoring the hedge term. The case (iii) allows us to gauge the robustness of the optimal investment strategy, e.g. deviations from the truly optimal strategy due to applying a slightly mis-specified model or slightly inaccurate parameter values. The size of the utility loss from small perturbations of the optimal strategy will also indicate how frequent the portfolio should be rebalanced in practical implementations.

We work through three examples in detail, two in affine models of return dynamics and one in a quadratic model. In the first example, the available assets are a stock (index), a bond, and cash (interpreted as deposits earning the short-term risk-free rate), bond prices and the short-term risk-free rate follows the Vasicek model, and the stock has a constant volatility and a constant Sharpe ratio. While the optimal investment strategy in this setting was derived by Sørensen (1999), the utility losses from sub-optimal investments seem unexplored. We compute and study utility losses for the three specific sub-optimal strategies described above using empirically reasonable parameter values. For example, we find that an investor with a relative risk aversion of 6 and a 20-year investment horizon suffers a wealth loss of about 29% if she completely ignores interest rate variations and, consequently, take the myopically optimal position in the stock and cash, but only a wealth loss of about 9% if she includes the bond into the speculative portfolio but does not hedge interest rate variations. Including bonds in long-term portfolios is thus primarily important due to the improved short-term risk-return trade-off, while the intertemporal hedging aspect is of secondary importance. Our results also show that around the optimal value the expected utility is much more sensitive to the portfolio weight of the stock than to the portfolio weight of the bond, i.e. it it much more to get the stock portfolio weight right.

Our second example revisits the asset allocation framework of Liu and Pan (2003) who consider a Heston-type model of stochastic stock price volatility. A single stock option will complete the market and we derive the optimal strategies with and without such an option. We measure the separate benefits of including the option as another speculative instrument to improve the risk-return tradeoff and of using the option as a hedge instrument against volatility risk.

Our third and final example addresses the importance of investing in value stocks and growth stocks in a setting allowing for mean reversion in the returns on those stocks. More precisely, we set up a model with three risky assets representing the market portfolio of stocks, an index of value stocks, and an index of growth stocks. We generalize the Kim and Omberg (1996) model of stock market mean reversion to this three-factor setting. We derive the optimal investment strategy and compute and discuss the utility loss incurred if value stocks and/or growth stocks are not included in the portfolio.

Utility costs calculations of sub-optimal behavior have been carried out in specific setting in various papers. Balduzzi and Lynch (1999) study consumption-investment problems of CRRA investors in a setting with transaction costs and/or predictability in stock returns. They use a discrete-time formulation and assume that the log-return on the stock market (represented by the value-weighted NYSE index) and its dividend (log-)yield follow a simple vector autoregressive model. They calibrate a discretized version of the model to the data and compute optimal strategies in the relevant utility maximization problems numerically using a grid-based recursive dynamic programming approach. Utility cost estimates are derived from the numerically computed approximations to the expected utilities from following various strategies. Brennan, Schwartz, and Lagnado (1997) present some related computations, but do not provide a measure of the utility loss of inappropriate intertemporal hedging.

The remainder of the paper is organized in the following way. Section 2 specifies the modeling framework and states a number of general results. Section 3 specializes the model and results to an affine framework and discusses two examples within that class, the Vasicek interest rate model and the Heston stochastic volatility model for stocks. Section 4 studies an example with a quadratic modeling structure, namely the model with time-varying market, growth, and value premia. Section 5 summarizes and concludes. The appendices contain proofs and various detailed computations.

2 General model and results

2.1 The general set-up and a characterization of the optimal investment strategy

Our general dynamic model of asset prices is equivalent to that of Liu (2007). The investors can invest in an instantaneously risk-free asset, interpreted as short-term cash deposits, which in the instant following time t yields a continuously compounded rate of return $r(x_t)$, where x is a state variable described below. The investor can also invest in d risky assets with time t prices gathered in the vector $P_t = (P_{1t}, \dots, P_{dt})^\top$ assumed to have dynamics

$$dP_t = \mathrm{diag}(P_t) \left[(r(x_t)\mathbf{1} + \sigma(x_t,t)\lambda(x_t))\; dt + \sigma(x_t,t)\, dz_t \right], \tag{1}$$

where $z = (z_1, \dots, z_d)^\top$ is a d-dimensional standard Brownian motion. The term $\mathrm{diag}(P_t)$ denotes the $(d \times d)$-matrix with the vector P_t along the main diagonal and zeros off the diagonal. The d-dimensional vector $\lambda(x_t)$ has the interpretation of a market price of risk (associated with the shock process z). The $(d \times d)$-matrix $\sigma(x_t,t)$ determines the variance-covariance matrix of the rates of return of the d risky assets over the next instant, $\sigma(x_t,t)\sigma(x_t,t)^\top$. For all (x,t), $\sigma(x,t)$ is assumed to be non-singular so that none of the assets are redundant.

The variable x_t is a state variable, say of dimension k, and is assumed to follow the diffusion process

$$dx_t = m(x_t)\, dt + v(x_t)\, dz_t + \hat{v}(x_t)\, d\hat{z}_t, \tag{2}$$

where m is a k-vector valued function, v is a $(k \times d)$-matrix valued function, $\hat{v}$ is a $(k \times k)$-matrix valued function, and $\hat{z}$ is a k-dimensional standard Brownian motion independent of z. The market will be incomplete whenever $\hat{v}(x_t)$ is not identically equal to zero. Let $\Sigma(x) = v(x)v(x)^\top + \hat{v}(x)\hat{v}(x)^\top$ denote the variance-covariance matrix of the state variable.

Note that we have assumed that the short-term interest rate and the market price of risk vector do not depend on calendar time directly. The fluctuations over time in these variables are presumably not due to the mere passage of time, but rather due to variations in some more fundamental economic variables. In contrast, the expected rates of returns and the price sensitivities of some assets will depend directly on time, e.g. the volatility and the expected rate of return on a bond will depend on the time-to-maturity of the bond and therefore on calendar time.

We represent an investment strategy by the d-dimensional process $\pi = (\pi_t)$, where $\pi_t = (\pi_{1t}, \dots, \pi_{dt})^\top$ is the vector of fractions of wealth ("portfolio weights")invested in the different risky assets at time t. The remaining fraction of wealth $1 - \pi_t^\top \mathbf{1}$ is invested in the instantaneously risk-free asset. For a given investment strategy π the value of the investment W_t^π will follow

$$dW_t^\pi = W_t^\pi \left[(r(x_t) + \pi_t^\top \sigma(x_t,t)\lambda(x_t))\; dt + \pi_t^\top \sigma(x_t,t)\, dz_t \right]. \tag{3}$$

We consider an investor with a power utility function of wealth at some future date T and ignore intermediate consumption and income other than financial returns. Any combination of an initial wealth W and an investment strategy π will give rise to a terminal wealth W_T^π (a partially controlled random variable) and the expected utility associated with that investment strategy is thus

$$J^\pi(W,x,t) = \mathrm{E}_t \left[\frac{1}{1-\gamma} \left(W_T^\pi\right)^{1-\gamma} \right], \tag{4}$$

where W is the initial (time t) wealth and $\gamma > 0$ is the constant relative risk aversion (CRRA) coefficient.[4] We assume $\gamma \geq 1$ to avoid problems with infinite expected utility that may arise for

[4]The case $\gamma = 1$ corresponds to logarithmic utility since $\frac{1}{1-\gamma} W^{1-\gamma}$ describes the same preferences as

$0 < \gamma < 1$, cf. Kim and Omberg (1996) and Korn and Kraft (2004). Clearly an optimal investment strategy π^* is one that maximizes expected utility,

$$J^*(W,x,t) \equiv J^{\pi^*}(W,x,t) = \sup_{\pi} J^{\pi}(W,x,t). \tag{5}$$

It is well-known that no matter what assumptions are made about the dynamics of investment opportunities, the optimal investment strategy for a CRRA investor will be independent of her wealth level. Hence we will focus on strategies of the form $\pi_t = \pi(x_t, t)$.

First, we characterize the optimal strategy and the associated expected utility via the solution to a partial differential equation. Liu (2007) has a similar result. For completeness, the proof is outlined in Appendix A.[5]

Theorem 1 *The expected utility generated by the optimal strategy is*

$$J^*(W,x,t) = \begin{cases} \frac{1}{1-\gamma}\left(We^{H^*(x,t)}\right)^{1-\gamma} & \text{for } \gamma > 1 \\ \ln W + H^*(x,t) & \text{for } \gamma = 1, \end{cases} \tag{6}$$

where $H^*(x,t)$ solves the PDE

$$\begin{aligned} &\frac{\partial H^*}{\partial t} + \left(m(x) + \frac{1-\gamma}{\gamma} v(x)\lambda(x)\right)^{\top} \frac{\partial H^*}{\partial x} + \frac{1}{2}\operatorname{tr}\left(\frac{\partial^2 H^*}{\partial x^2}\Sigma(x)\right) \\ &\quad + \frac{1-\gamma}{2}\left(\frac{\partial H^*}{\partial x}\right)^{\top}\left[\Sigma(x) - \left(1 - \frac{1}{\gamma}\right)v(x)v(x)^{\top}\right]\frac{\partial H^*}{\partial x} + r(x) + \frac{1}{2\gamma}\lambda(x)^{\top}\lambda(x) = 0 \end{aligned} \tag{7}$$

with the terminal condition $H^*(x,T) = 0$. The optimal strategy is

$$\pi^*(x,t) = \frac{1}{\gamma}\left(\sigma(x,t)^{\top}\right)^{-1}\lambda(x) + \frac{1-\gamma}{\gamma}\left(\sigma(x,t)^{\top}\right)^{-1} v(x)^{\top}\frac{\partial H^*}{\partial x}(x,t). \tag{8}$$

The first term in the expression (8) for the optimal portfolio is the speculative part corresponding to an investment in the standard tangency portfolio of risky assets. This term is present both for constant and time-varying investment opportunities. The second term represents the deviation from the speculative position due to time-varying investment opportunities and is therefore referred to as the intertemporal hedge term. Consistent with the well-known fund separation results, it can be decomposed into k terms, where each of the terms represents a position in a portfolio hedging variations in one of the components of the state variable vector x. The intertemporal hedge term disappears in three cases:

$\frac{1}{1-\gamma}(W^{1-\gamma} - 1)$ and from l'Hôspital's rule we have that

$$\lim_{\gamma\to 1}\frac{W^{1-\gamma}-1}{1-\gamma} = \lim_{\gamma\to 1}\frac{-W^{1-\gamma}\ln W}{-1} = \ln W.$$

[5] In a complete market not necessarily driven by a diffusion, Munk and Sørensen (2004) characterize the hedge term via the volatility of a particular stochastic process depending on the dynamics of the short rate and the market prices of risk (as well as the time horizon and risk aversion of the investor). Detemple, Garcia, and Rindisbacher (2003) represent the hedge term by integrals of the Malliavin derivatives of the short rate and the market prices of risk.

(i) $\gamma = 1$: a logarithmic investor does not hedge;

(ii) $v(x) = 0$: it is impossible to hedge the variations in investment opportunities;

(iii) if both r and $\lambda^\top \lambda$ are independent of x, the function $H^*(t) = (r + \frac{1}{2\gamma}\lambda^\top\lambda)(T-t)$ is a solution of (7) and independent of x so that the hedge term vanishes, confirming the conclusions of Nielsen and Vassalou (2006) and others that investors hedge only shifts in the instantaneous mean-variance efficient frontier (which has intercept r and slope $\sqrt{\lambda^\top\lambda}$).

2.2 The utility loss for sub-optimal investment strategies

The next theorem characterizes the expected utility generated by any given investment strategy $\pi(x,t)$. Appendix A gives the proof.

Theorem 2 *The expected utility generated by the investment strategy $\pi_t = \pi(x_t, t)$ is*

$$J^\pi(W,x,t) = \begin{cases} \frac{1}{1-\gamma}\left(We^{H^\pi(x,t)}\right)^{1-\gamma} & \text{for } \gamma > 1 \\ \ln W + H^\pi(x,t) & \text{for } \gamma = 1, \end{cases} \tag{9}$$

Here, for $\gamma \neq 1$, the function $H^\pi(x,t)$ is given by

$$H^\pi(x,t) = \frac{1}{1-\gamma}\ln\left\{\mathrm{E}_{x,t}^{\mathbb{Q}(\pi)}\left[e^{(1-\gamma)\int_t^T\left(r(x_s)+\pi(x_s,s)^\top\sigma(x_s,s)\left[\lambda(x_s)-\frac{\gamma}{2}\sigma(x_s,s)^\top\pi(x_s,s)\right]\right)ds}\right]\right\}, \tag{10}$$

where $\mathbb{Q}(\pi)$ is the equivalent probability measure under which the process $(z^\pi, \hat{z})$ with $dz_t^\pi = dz_t - (1-\gamma)\sigma(x_t,t)^\top\pi(x_t,t)\,dt$ is a standard Brownian motion. The function $H^\pi(x,t)$ satisfies the partial differential equation

$$\begin{aligned}\frac{\partial H^\pi}{\partial t} &+ (m(x) + (1-\gamma)v(x)\sigma(x,t)^\top\pi(x,t))^\top\frac{\partial H^\pi}{\partial x} + \frac{1}{2}\operatorname{tr}\left(\frac{\partial^2 H^\pi}{\partial x^2}\Sigma(x)\right) \\ &+ \frac{1-\gamma}{2}\left(\frac{\partial H^\pi}{\partial x}\right)^\top\Sigma(x)\frac{\partial H^\pi}{\partial x} + r(x) + \pi(x,t)^\top\sigma(x,t)\left[\lambda(x) - \frac{\gamma}{2}\sigma(x,t)^\top\pi(x,t)\right] = 0 \end{aligned} \tag{11}$$

with the terminal condition $H^\pi(x,T) = 0$.

By definition, with the same initial wealth, a suboptimal investment strategy will generate a lower level of expected utility than the optimal investment strategy. We measure the loss from following the suboptimal strategy as the percentage of extra initial wealth that is necessary to bring the investor to the utility level that can be obtained by following the optimal strategy with the original initial wealth. This measure of the utility loss is immune to increasing affine transformations of the utility function and is easy to relate to. In our setting, the loss L^π is determined from

$$J^\pi(W(1+L^\pi), x, t) = J^*(W,x,t), \tag{12}$$

which according to (6) and (9) for $\gamma \neq 1$ is equivalent to

$$\frac{1}{1-\gamma}\left(W(1+L^\pi)e^{H^\pi(x,t)}\right)^{1-\gamma} = \frac{1}{1-\gamma}\left(We^{H^*(x,t)}\right)^{1-\gamma} \tag{13}$$

and hence

$$L^\pi \equiv L^\pi(x,t) = e^{H^*(x,t)-H^\pi(x,t)} - 1 \approx H^*(x,t) - H^\pi(x,t). \tag{14}$$

We get exactly the same formula for $\gamma = 1$, i.e. (14) holds for all values of $\gamma > 0$. We refer to L^π as the *wealth loss* associated with the investment strategy π.

In affine or quadratic models of return dynamics, the function $H^*(x,t)$ associated with the optimal strategy is known in closed-form or easily computed by numerically solution of ordinary differential equations. To determine the wealth loss associated with a given investment strategy, we can compute the function H^π either by solving the PDE (11) or by Monte Carlo simulation based on (10). As we will see below, the PDE (11) associated with some relevant sub-optimal strategies can often be solved in closed form so that also the wealth loss is given by a closed-form expression.

The wealth loss just defined is not the only way to measure the utility loss.[6] An obvious alternative is to define the loss as the fraction of initial wealth that can be thrown away following the optimal strategy instead of a suboptimal strategy. This loss $l(x,t)$ is defined by

$$J^\pi(W,x,t) = J^*(W[1-l(x,t)],x,t) \quad \Rightarrow \quad l(x,t) = 1 - e^{H^\pi(x,t)-H^*(x,t)}$$

implying that $l(x,t) = L(x,t)/[1+L(x,t)]$ so that there is a one-to-one relation between the two measures and to a first-order approximation they are identical. Another alternative is to state the utility loss in terms of a return shortfall. We can interpret $\frac{1}{T-t}H^\pi(x,t)$ as the continuously compounded certainty equivalent return (used, e.g., in Liu and Pan (2003) and Haugh, Kogan, and Wang (2006)) associated with the investment strategy π and, consequently,

$$R^\pi(x,t) = \frac{1}{T-t}\ln(1+L^\pi(x,t)) = \frac{1}{T-t}(H^*(x,t) - H^\pi(x,t))$$

is the certainty equivalent return given up due to a sub-optimal investment strategy, clearly a utility loss measure equivalent to the measure L^π. While such a return shortfall measure has its merits, we will stick to the wealth loss in the remainder of the paper.

The loss defined above comes from following a given investment strategy instead of the optimal strategy. Similarly, one can define a wealth loss $L^{\pi,\tilde{\pi}}$ for following a given investment strategy π rather than another investment strategy $\tilde{\pi}$ via the equation

$$J^\pi(W(1+L^{\pi,\tilde{\pi}}),x,t) = J^{\tilde{\pi}}(W,x,t), \tag{15}$$

which again be expressed in return terms as $R^{\pi,\tilde{\pi}} = \frac{1}{T-t}\ln(1+L^{\pi,\tilde{\pi}})$. (If π is better than $\tilde{\pi}$ in utility terms, the loss $L^{\pi,\tilde{\pi}}$ will be negative.)

Does the loss associated with a sub-optimal investment in a given asset class depend on the investments in other asset classes? In general, the result is yes, but the next result shows that the

[6] Obviously, the wealth loss can be defined similarly in a setting with intermediate consumption. In such a setting, Gomes (2007) defines a seemingly different measure of the utility costs, but a few computations reveal that his measure is in fact equivalent to our measure.

answer is no if the asset classes are independent and the investments in the two asset classes are also independent. To be more precise, let us decompose the d risky assets into a class of (the first) d_1 assets and a class of $d_2 = d - d_1$ assets and decompose σ, λ, and v as

$$\sigma(x,t) = \begin{pmatrix} \sigma_{11}(x,t) & 0 \\ \sigma_{21}(x,t) & \sigma_{22}(x,t) \end{pmatrix}, \quad \lambda(x) = \begin{pmatrix} \lambda_1(x) \\ \lambda_2(x) \end{pmatrix}, \quad v(x) = \begin{pmatrix} v_1(x), \, v_2(x) \end{pmatrix}, \tag{16}$$

where σ_{ij} has dimension $d_i \times d_j$, λ_j has dimension d_j, and v_j has dimension $k \times d_j$. The two asset classes are said to be independent if (i) $\sigma_{21}(x,t) \equiv 0$ and (ii) the state variable x_t can be split in two, $x_t = (x_{1t}, x_{2t})$, so that x_1 and x_2 evolve independently, and the prices of the first d_1 assets only depend on x_1 and the prices of the other d_2 assets only depend on x_2. An investment strategy $\pi = (\pi_1, \pi_2)$, where π_i has dimension d_i $(i = 1, 2)$, is called *class-separated* if π_i at most depends on x_i and time.

Theorem 3 *In the above framework, suppose that the asset classes are independent and the short-rate dynamics either depend only on x_1 or on x_2. Then the wealth loss (and, consequently, the return loss) incurred due to following one class-separated investment strategy (π_1, π_2) instead of another class-separated investment strategy $(\pi_1, \tilde{\pi}_2)$, where only the investment in asset class 2 is different, is independent of π_1.*

This theorem implies that if you want to consider the costs of suboptimal investments in one asset class, you can only ignore other assets if their prices move independently and there is no dependence in portfolio weights across the asset classes. To give a concrete example, recall that recent literature has documented unspanned stochastic volatility (USV) in default-free interest rates (see e.g. Collin-Dufresne and Goldstein 2002), which implies that the appropriate inclusion of interest rate options into the long-term investment strategy will improve expected utility, but by how much? For a specific dynamic term structure model featuring USV, Trolle (2006) finds a substantial loss from investing in bonds only rather than bonds and an interest rate option. If, and only if, the term structure dynamics is independent of, say, stock price dynamics, this loss will carry over to the case where stock investments are also possible.

2.2.1 Omitted assets

What does it cost an investor not to have access to a given asset or a set of assets? Suppose that the investor can only invest in the first d_1 of the d assets and decompose σ, λ, and v as in (16). The expected utility generated by any strategy $\pi = (\pi_1, 0)$ follows from Theorem 2. With such a strategy, note that $\pi^\top \sigma = (\pi_1^\top \sigma_{11}, 0)$ in (10). The best the investor can do with the d_1 assets follows from Theorem 1. In particular, the optimal portfolio of the first d_1 assets when the other assets are omitted is given by

$$\pi_1(x,t) = \frac{1}{\gamma}\left(\sigma_1(x,t)^\top\right)^{-1} \lambda_1(x) + \frac{1-\gamma}{\gamma}\left(\sigma_1(x,t)^\top\right)^{-1} v_1(x)^\top \frac{\partial H_1}{\partial x}(x,t),$$

where $H_1(x,t)$ solves the PDE that comes from (7) when $v(x)$ is replaced by $v_1(x)$ and $\lambda(x)$ by $\lambda_1(x)$. The expected utility from that strategy is $J_1(W,x,t) = \frac{1}{1-\gamma}(We^{H_1(x,t)})^{1-\gamma}$ for $\gamma \neq 1$. The associated wealth loss is $L = \exp\{H^* - H_1\} - 1$.

2.2.2 No intertemporal hedging

What is the cost from ignoring stochasticity in the investment opportunities and hence following a purely speculative, "no hedge" strategy? Some investors may refrain from intertemporal hedging because they believe that investment opportunities are constant. In that case they may omit some assets from their portfolio considerations. For example, an investor trusting interest rates to be constant have no reason to invest in both short-term deposits and in bonds and may pursue a purely speculative strategy omitting bonds. As another example, an investor who firmly believes that the volatility of the stock market index is constant (or spanned by the index itself) sees no reason to invest in index options. The purely speculative strategy with potentially omitted assets is of the form

$$\pi_t = \begin{pmatrix} \frac{1}{\gamma} \left(\sigma_{11}(x_t,t)^\top\right)^{-1} \lambda_1(x_t) \\ 0 \end{pmatrix}. \tag{17}$$

Substituting this into the PDE (11) and simplifying, we obtain the following result.

Corollary 1 *The expected utility generated by the purely speculative strategy with potentially omitted assets is*

$$\bar{J}(W,x,t) = \frac{1}{1-\gamma}\left(We^{\bar{H}(x,t)}\right)^{1-\gamma}, \tag{18}$$

where the function $\bar{H}(x,t)$ is given by

$$\bar{H}(x,t) = \frac{1}{1-\gamma} \ln\left\{ \mathrm{E}^{\bar{\mathbb{Q}}}_{x,t}\left[e^{(1-\gamma)\int_t^T \left(r(x_s) + \frac{1}{2\gamma}\lambda_1(x_s,s)^\top \lambda_1(x_s,s)\right) ds}\right]\right\}, \tag{19}$$

where $\bar{\mathbb{Q}}$ is the equivalent probability measure under which the process $(\bar{z},\hat{z})$ with $d\bar{z}_t = dz_t - \frac{1-\gamma}{\gamma}\left(\lambda_1(x_t), 0\right)^\top dt$ is a standard Brownian motion. The function $\bar{H}(x,t)$ satisfies the partial differential equation

$$\frac{\partial \bar{H}}{\partial t} + \left(m(x) + \frac{1-\gamma}{\gamma} v_1(x)\lambda_1(x)\right)^\top \frac{\partial \bar{H}}{\partial x} + \frac{1}{2}\operatorname{tr}\left(\frac{\partial^2 \bar{H}}{\partial x^2}\Sigma(x)\right) + \frac{1-\gamma}{2}\left(\frac{\partial \bar{H}}{\partial x}\right)^\top \Sigma(x)\frac{\partial \bar{H}}{\partial x} + r(x) + \frac{1}{2\gamma}\lambda_1(x)^\top \lambda_1(x) = 0 \tag{20}$$

with the terminal condition $\bar{H}(x,T) = 0$.

The wealth loss associated with no hedging is

$$\bar{L}(x,t) = e^{H^*(x,t) - \bar{H}(x,t)} - 1.$$

For a moment assume that $d_1 = d$ so that no assets are omitted. Then, comparing (20) to (7), we see that the only difference between the relevant PDEs for the no-hedge strategy and the optimal strategy is the presence of $-\left(1-\frac{1}{\gamma}\right)v(x)v(x)^\top$ in the term involving both $\left(\frac{\partial H}{\partial x}\right)^\top$ and $\frac{\partial H}{\partial x}$ in the PDE associated with the optimal strategy. Again, if $\gamma = 1$ (log-utility) or $v = 0$ (hedging impossible), there is no difference and the wealth loss is zero.

2.2.3 Near-optimal investment strategies

Consider a trading strategy π^ε which deviates from the optimal strategy π^* in the sense that

$$\pi^\varepsilon(x_t, t) = \pi^*(x_t, t) + \left(\sigma(x_t, t)^\top\right)^{-1}\varepsilon(x_t, t) \tag{21}$$

for some $\varepsilon(x, t)$ that can be interpreted as the error made in the assessment of the optimal sensitivity of wealth with respect to the shocks to asset prices. Let $\Delta^\varepsilon(x, t) = H^*(x, t) - H^{\pi^\varepsilon}(x, t)$ so that the wealth loss is $L^{\pi^\varepsilon}(x, t) = \exp\{\Delta^\varepsilon(x, t)\} - 1 \approx \Delta^\varepsilon(x, t)$. Applying Theorem 2 with $\pi = \pi^\varepsilon$ gives the following result.

Corollary 2 *The loss associated with the strategy π^ε is characterized by the function $\Delta^\varepsilon(x, t) = H^*(x, t) - H^{\pi^\varepsilon}(x, t)$, which satisfies the PDE*

$$\frac{\partial \Delta^\varepsilon}{\partial t} + \left(m(x) + \frac{1-\gamma}{\gamma}v(x)\lambda(x) + (1-\gamma)v(x)\varepsilon(x,t) + \frac{1-\gamma}{\gamma}v(x)v(x)^\top\frac{\partial H^*}{\partial x} + (1-\gamma)\hat{v}(x)\hat{v}(x)^\top\frac{\partial H^*}{\partial x}\right)^\top\frac{\partial \Delta^\varepsilon}{\partial x}$$
$$+\frac{1}{2}\operatorname{tr}\left(\frac{\partial^2 \Delta^\varepsilon}{\partial x^2}\Sigma(x)\right) - \frac{1-\gamma}{2}\left(\frac{\partial \Delta^\varepsilon}{\partial x}\right)^\top\Sigma(x)\frac{\partial \Delta^\varepsilon}{\partial x} + \frac{\gamma}{2}\varepsilon(x,t)^\top\varepsilon(x,t) = 0 \tag{22}$$

with the terminal condition $\Delta^\varepsilon(x, T) = 0$.

We will use this result to evaluate the robustness of expected utility to small deviations from the optimal investment strategy.

In particular, note that if $\varepsilon(x, t)$ is independent of x, the solution $\Delta^\varepsilon(x, t) = \Delta^\varepsilon(t)$ to

$$\left(\Delta^\varepsilon\right)'(t) + \frac{\gamma}{2}\varepsilon(t)^\top\varepsilon(t) = 0, \quad \Delta^\varepsilon(T) = 0,$$

will also solve the full PDE (22). The solution is

$$\Delta^\varepsilon(t) = \frac{\gamma}{2}\int_t^T \varepsilon(s)^\top\varepsilon(s)\,ds. \tag{23}$$

Clearly the loss is increasing in the risk aversion, the time horizon, and the "squared error" $\varepsilon(s)^\top\varepsilon(s)$. The associated return loss is $\frac{\gamma}{2}\frac{1}{T-t}\int_t^T \varepsilon(s)^\top\varepsilon(s)\,ds$. Note that this does not depend on the specific return dynamics.

The costs of missing the optimal portfolio weight by a vector $\phi(x, t)$ can be found by letting $\varepsilon(x, t) = \sigma(x, t)^\top\phi(x, t)$. If both σ and ϕ are independent of x, the loss will be characterized by

$$\Delta(t) = \frac{\gamma}{2}\int_t^T \phi(s)^\top\sigma(s)\sigma(s)^\top\phi(s)\,ds, \tag{24}$$

which depends on the volatilities and covariances of the assets, but not the risk-free rate nor the market prices of risk.

3 Affine models

3.1 General analysis

Within the general framework of the previous section a model is said to be affine if $r(x)$, $m(x)$, $v(x)v(x)^\top$, $\hat{v}(x)\hat{v}(x)^\top$, $\lambda(x)^\top\lambda(x)$, and $v(x)\lambda(x)$ are all affine functions of x. In particular, the short rate is of the form

$$r(x) = R_0 + R^\top x, \tag{25}$$

for some scalar R_0 and k-vector R. The dynamics of the state variable x takes the form

$$dx_t = (M_0 + Mx_t)\, dt + D\sqrt{V(x_t)}\, dz_t + \hat{D}\sqrt{\hat{V}(x_t)}\, d\hat{z}_t, \tag{26}$$

where M_0 is a k-vector, M is a $k\times k$-matrix, D is a $k\times d$-matrix, $\hat{D}$ is a $k\times k$-matrix, and the $d\times d$-matrix $V(x)$ and the $k\times k$-matrix $\hat{V}(x)$ are diagonal matrices with elements

$$[V(x)]_{ii} = \nu_i + V_i^\top x, \qquad [\hat{V}(x)]_{ii} = \hat{\nu}_i + \hat{V}_i^\top x. \tag{27}$$

Furthermore, we must have

$$v(x)\lambda(x) = D\sqrt{V(x_t)}\lambda(x) = K_0 + Kx \tag{28}$$

for some k-vector K_0 and $(k\times k)$-matrix K, and

$$\lambda(x)^\top\lambda(x) = L_0 + L^\top x \tag{29}$$

for some scalar L_0 and k-vector L. Eqs. (28) and (29) are satisfied if $\lambda(x) = \sqrt{V(x)}\Lambda$ for some d-vector Λ but slightly more general specifications of $\lambda(x)$ are also possible.

If the state variable sensitivities $v(x)$ and $\hat{v}(x)$ are independent of x, x is a Gaussian process. In that case we can assume without loss of generality that the matrices $V(x)$ and $\hat{V}(x)$ are identity matrices (i.e. $V_i = \hat{V}_i = 0$ and $\nu_i = \hat{\nu}_i = 1$) so that $v(x) = D$ and $\hat{v}(x) = \hat{D}$. To ensure an affine set-up we need a constant $\lambda(x) = \lambda$ so that $K = L = 0$. As we shall see below, some results on utility losses are simpler for Gaussian models.

3.1.1 Optimal strategy

In an affine model, the PDE (7) corresponding to the optimal investment strategy has a solution of the form

$$H^*(x,t) = F^*(t) + G^*(t)^\top x \tag{30}$$

if the functions F^* and G^* satisfy $F^*(T) = 0$, $G^*(T) = 0$, and the system of ordinary differential equations

$$
\begin{aligned}
&(F^*)'(t) + \left(M_0 + \frac{1-\gamma}{\gamma} K_0\right)^\top G^*(t) + R_0 + \frac{L_0}{2\gamma} \\
&\quad + \frac{1-\gamma}{2}\left(\frac{1}{\gamma}\sum_{i=1}^{d}[D^\top G^*(t)]_i^2 \nu_i + \sum_{i=1}^{k}[\hat{D}^\top G^*(t)]_i^2 \hat{\nu}_i\right) = 0,
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
&(G^*)'(t) + \left(M + \frac{1-\gamma}{\gamma} K\right)^\top G^*(t) + R + \frac{L}{2\gamma} \\
&\quad + \frac{1-\gamma}{2}\left(\frac{1}{\gamma}\sum_{i=1}^{d}[D^\top G^*(t)]_i^2 V_i + \sum_{i=1}^{k}[\hat{D}^\top G^*(t)]_i^2 \hat{V}_i\right) = 0.
\end{aligned} \tag{32}
$$

Given G^*, F^* can be computed by integration,

$$
\begin{aligned}
F^*(t) = -\int_t^T (F^*)'(s)\,ds &= \left(M_0 + \frac{1-\gamma}{\gamma}K_0\right)^\top \int_t^T G^*(s)\,ds + \left(R_0 + \frac{L_0}{2\gamma}\right)(T-t) \\
&+ \frac{1-\gamma}{2}\left(\frac{1}{\gamma}\sum_{i=1}^{d}\nu_i \int_t^T [D^\top G^*(s)]_i^2\,ds + \sum_{i=1}^{k}\hat{\nu}_i \int_t^T [\hat{D}^\top G^*(s)]_i^2\,ds\right).
\end{aligned} \tag{33}
$$

From (8) and (30), the optimal portfolio in an affine setting is

$$
\pi^*(x,t) = \frac{1}{\gamma}\left(\sigma(x,t)^\top\right)^{-1}\lambda(x) + \frac{1-\gamma}{\gamma}\left(\sigma(x,t)^\top\right)^{-1}\sqrt{V(x)}D^\top G^*(t). \tag{34}
$$

In the Gaussian case, the ODE (32) simplifies to

$$
(G^*)'(t) + M^\top G^*(t) + R = 0, \tag{35}
$$

and the optimal strategy is

$$
\pi^*(x,t) = \frac{1}{\gamma}\left(\sigma(x,t)^\top\right)^{-1}\lambda) + \frac{1-\gamma}{\gamma}\left(\sigma(x,t)^\top\right)^{-1} D^\top G^*(t). \tag{36}
$$

3.1.2 Omitted assets

In order to cover the case with omitted assets, assume that

$$
v_1(x)\lambda_1(x) = K_{01} + K_1 x, \quad \lambda_1(x)^\top\lambda_1(x) = L_{01} + L_1^\top x, \tag{37}
$$

and decompose the matrix D as $D = (D_1, D_2)$ where D_i is $k \times d_i$. Then the function $H_1(x,t)$ that characterizes the maximal expected utility the investor can obtain by investing in only the first d_1 assets, is of the form

$$
H_1(x,t) = F_1(t) + G_1(t)^\top x,
$$

where

$$
\begin{aligned}
&F_1'(t) + \left(M_0 + \frac{1-\gamma}{\gamma}K_{01}\right)^\top G_1(t) + R_0 + \frac{L_{01}}{2\gamma} \\
&\quad + \frac{1-\gamma}{2}\left(\frac{1}{\gamma}\sum_{i=1}^{d_1}[D_1^\top G_1(t)]_i^2\nu_i + \sum_{i=d_1+1}^{d_2}[D_2^\top G_1(t)]_i^2\nu_i + \sum_{i=1}^{k}[\hat{D}^\top G_1(t)]_i^2\hat{\nu}_i\right) = 0,
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
&G_1'(t) + \left(M + \frac{1-\gamma}{\gamma}K_1\right)^\top G_1(t) + R + \frac{L_1}{2\gamma} \\
&\quad + \frac{1-\gamma}{2}\left(\frac{1}{\gamma}\sum_{i=1}^{d_1}[D_1^\top G_1(t)]_i^2 V_i + \sum_{i=d_1+1}^{d_2}[D_2^\top G_1(t)]_i^2 V_i + \sum_{i=1}^{k}[\hat{D}^\top G_1(t)]_i^2\hat{V}_i\right) = 0
\end{aligned} \tag{39}
$$

and $F_1(T) = G_1(T) = 0$. Again F_1 can be computed from G_1 by integration. The best investment strategy constrained to the first d_1 assets is

$$
\pi_1(x,t) = \frac{1}{\gamma}\left(\sigma_{11}(x,t)^\top\right)^{-1}\lambda_1(x) + \frac{1-\gamma}{\gamma}\left(\sigma_{11}(x,t)^\top\right)^{-1}\sqrt{V^{(1)}(x)}D_1^\top G_1(t), \tag{40}
$$

where $V^1(x)$ is the upper-left $d_1 \times d_1$ sub-matrix of $V(x)$.

3.1.3 No intertemporal hedging

The PDE (20) corresponding to the no-hedge strategy with potentially omitted assets has a solution of the form

$$
\bar{H}(x,t) = \bar{F}(t) + \bar{G}(t)^\top x \tag{41}
$$

if the functions $\bar{F}$ and $\bar{G}$ satisfy $\bar{F}(T) = 0$, $\bar{G}(T) = 0$, and the system of ODEs

$$
\bar{F}'(t) + \left(M_0 + \frac{1-\gamma}{\gamma}K_{01}\right)^\top \bar{G}(t) + R_0 + \frac{L_{01}}{2\gamma} + \frac{1-\gamma}{2}\left(\sum_{i=1}^{d}[D^\top \bar{G}(t)]_i^2\nu_i + \sum_{i=1}^{k}[\hat{D}^\top \bar{G}(t)]_i^2\hat{\nu}_i\right) = 0, \tag{42}
$$

$$
\bar{G}'(t) + \left(M + \frac{1-\gamma}{\gamma}K_1\right)^\top \bar{G}(t) + R + \frac{L_1}{2\gamma} + \frac{1-\gamma}{2}\left(\sum_{i=1}^{d}[D^\top \bar{G}(t)]_i^2 V_i + \sum_{i=1}^{k}[\hat{D}^\top \bar{G}(t)]_i^2\hat{V}_i\right) = 0. \tag{43}
$$

Given $\bar{G}$, $\bar{F}$ can be computed by integration,

$$
\begin{aligned}
\bar{F}(t) &= \left(M_0 + \frac{1-\gamma}{\gamma}K_{01}\right)^\top \int_t^T \bar{G}(s)\,ds + \left(R_0 + \frac{L_{01}}{2\gamma}\right)(T-t) \\
&\quad + \frac{1-\gamma}{2}\left(\sum_{i=1}^{d}\nu_i\int_t^T[D^\top \bar{G}(s)]_i^2\,ds + \sum_{i=1}^{k}\hat{\nu}_i\int_t^T[\hat{D}^\top \bar{G}(s)]_i^2\,ds\right).
\end{aligned} \tag{44}
$$

The utility loss due to no hedging is

$$
L(x,t) = e^{F^*(t)-\bar{F}(t)+\left(G^*(t)-\bar{G}(t)\right)^\top x} - 1. \tag{45}
$$

In the Gaussian case, we see that $\bar{G}(t) \equiv G^*(t)$ so the utility loss simplifies to

$$
L(t) = e^{F^*(t)-\bar{F}(t)} - 1, \tag{46}
$$

independent of the current state. Moreover, from (31) and (42) we conclude that

$$F^*(t) = \bar{F}(t) + \frac{(1-\gamma)^2}{2\gamma} \sum_{i=1}^{d} \int_t^T [D^\top G^*(s)]_i^2 \, ds + \frac{1-\gamma}{\gamma} \lambda_2^\top v_2^\top \int_t^T G^*(s) \, ds + \frac{\lambda_2^\top \lambda_2}{2\gamma}(T-t). \quad (47)$$

The term $\frac{\lambda_2^\top \lambda_2}{2\gamma}(T-t)$ is the loss due to the fact that the premia on the risks specific to the omitted assets are not picked up, while the term $\frac{1-\gamma}{\gamma}\lambda_2^\top v_2^\top \int_t^T G^*(s)\, ds$ reflects the reduction in diversification due to omitting the assets.

3.2 Example: one-factor Vasicek interest rates

Assume that investment opportunities vary with the short-term interest rate r_t which follows an Ornstein-Uhlenbeck process as in Vasicek (1977). In order to focus on interest rate uncertainty assume a single stock representing the stock market index with price dynamics

$$dS_t = S_t \left[(r_t + \Lambda_1 \sigma_S) \, dt + \sigma_S \, dz_{1t} \right],$$

where the volatility σ_S and the Sharpe ratio Λ_1 are constants. Write the dynamics of the short-term interest rate as

$$dr_t = \kappa[\bar{r} - r_t] \, dt - \rho \sigma_r \, dz_{1t} - \sqrt{1-\rho^2} \sigma_r \, dz_{2t}, \quad (48)$$

where κ, $\bar{r}$, and σ_r are positive constants and where $-\rho$ is the correlation between instantaneous changes in the short rate and the stock price. Let the constant Λ_2 be the market price of risk associated with the interest rate specific shock z_2. The dynamics of the price $B_t = B(r_t, t)$ of a bond (or any other asset only depending on interest rates) is then of the form

$$dB_t = B_t \left[(r_t + \Psi \sigma_{Bt}) \, dt + \rho \sigma_{Bt} \, dz_{1t} + \sqrt{1-\rho^2} \sigma_{Bt} \, dz_{2t} \right],$$

where $\Psi = \rho \Lambda_1 + \sqrt{1-\rho^2} \Lambda_2$ is the Sharpe ratio, $\sigma_{Bt} = -\sigma_r \frac{\partial B}{\partial r}(r_t, t)/B(r_t, t)$ is the volatility, and ρ is the correlation between the bond price and the stock price. In particular, for the zero-coupon bond maturing at time $\bar{T}$, the price is $B_t^{\bar{T}} = e^{-a(\bar{T}-t) - b(\bar{T}-t) r_t}$, where

$$b(\tau) = \frac{1}{\kappa} \left(1 - e^{-\kappa \tau} \right), \quad a(\tau) = y_\infty \left(\tau - b(\tau) \right) + \frac{\sigma_r^2}{4\kappa} b(\tau)^2,$$

and $y_\infty = \bar{r} + \frac{\Psi \sigma_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}$ is the asymptotic zero-coupon yield as time-to-maturity goes to infinity. The volatility is $\sigma_{Bt} = \sigma_r b(\bar{T} - t)$. It is well-known that any bond (or other fixed-income security) can be generated from an appropriate dynamic investment strategy in the bank account and in just one (arbitrary) bond (or other long-lived term structure derivative).

Our model fits into the affine set-up.[7] The ordinary differential equation (32) reduces to $(G^*)'(t) - \kappa G^*(t) + 1 = 0$, which with the condition $G^*(T) = 0$ has the unique solution

$$G^*(t) = \frac{1}{\kappa} \left(1 - e^{-\kappa (T-t)} \right) = b(T-t). \quad (49)$$

[7] Let $M_0 = \kappa \bar{r}$, $M = -\kappa$, $D = (-\rho \sigma_r, -\sqrt{1-\rho^2} \sigma_r)$, $\nu_1 = \nu_2 = 1$, $V_1 = V_2 = \hat{\nu} = \hat{V} = 0$, $\Lambda = (\Lambda_1, \Lambda_2)^\top$, $R_0 = 0$, and $R = 1$.

From (34) the optimal investment strategy is given by $\pi^* = (\pi_S^*, \pi_B^*)^\top$, where

$$\pi_S^* = \frac{\Lambda_1 - \frac{\rho}{\sqrt{1-\rho^2}}\Lambda_2}{\gamma\sigma_S}, \quad \pi_B^*(r,t) = \frac{\Lambda_2}{\gamma\sigma_B(r,t)\sqrt{1-\rho^2}} + \frac{\gamma-1}{\gamma}\frac{\sigma_r b(T-t)}{\sigma_B(r,t)}. \tag{50}$$

If the bond is the zero-coupon bond maturing at T, $\sigma_B(r,t) = \sigma_r b(T-t)$. The optimal portfolio choice in this setting was originally derived by Sørensen (1999).

Since the model is Gaussian it follows from (46) and (47) that the utility loss suffered in absence of intertemporal hedging is given by $L(t) = e^{f(t)} - 1$, where

$$\begin{aligned} f(t) &= \frac{(1-\gamma)^2}{2\gamma}\sigma_r^2 \int_t^T G^*(s)^2\,ds + \varepsilon\left(\frac{\Lambda_2^2}{2\gamma}(T-t) + \left(1-\frac{1}{\gamma}\right)\sqrt{1-\rho^2}\sigma_r\Lambda_2 \int_t^T G^*(s)\,ds\right) \\ &= \frac{(1-\gamma)^2\sigma_r^2}{2\gamma\kappa^2}\left[T-t-b(T-t)-\frac{\kappa}{2}b(T-t)^2\right] \\ &\quad + \varepsilon\left(\frac{\Lambda_2^2}{2\gamma}(T-t) + \left(1-\frac{1}{\gamma}\right)\sqrt{1-\rho^2}\sigma_r\Lambda_2\left[T-t-b(T-t)\right]\right). \end{aligned} \tag{51}$$

Here $\varepsilon = 0$ [$\varepsilon = 1$] if the bond is [is not] included in the speculative portfolio. The loss is determined by the risk aversion parameter γ, the investment horizon $T-t$, the volatility of the short rate σ_r, the mean reversion parameter κ, and—if bonds are completely omitted—the market price of bond-specific risk Λ_2.

The costs of missing the optimal portfolio weight by some percentage, (ϕ_S, ϕ_B) is according to (24) given by

$$\begin{aligned} \Delta(t) &= \frac{\gamma}{2}\int_t^T \left(\phi_S^2\,\sigma_S^2 + 2\,\rho\,\sigma_S\,\sigma_B(r,s)\,\phi_S\,\phi_B + \phi_B^2\,\sigma_B(r,s)^2\right) ds \\ &= \frac{\gamma}{2}\left(\phi_S^2\,\sigma_S^2(T-t) + 2\,\rho\,\phi_S\,\phi_B\,\frac{\sigma_S\,\sigma_r}{\kappa}\,(T-t-b(T-t))\right) \\ &\quad + \frac{\gamma\,\phi_B^2\,\sigma_r^2}{2}\left(\frac{1}{\kappa^2}\,(T-t-b(T-t)) - \frac{1}{2\kappa}\,b(T-t)^2\right), \end{aligned} \tag{52}$$

where we have assumed that the bond is a zero-coupon bond maturing at T. The loss is determined by the risk aversion parameter γ, the investment horizon $T-t$, the volatility of the stock σ_S, the volatility of the short rate σ_r, the mean reversion parameter κ, the correlation between the stock and bond market ρ, and of course by the percentage the optimal portfolio is missed (ϕ_S, ϕ_B).

Let us look at a numerical example. As benchmark estimates of the real interest rate we will use the estimates given in Brennan and Xia (2002), that is $\bar{r} = 0.01$, $\kappa = 0.63$, $\sigma_r = 0.03$, and $\Lambda_2 = 0.21$. Furthermore, we assume a market price of risk associated with the stock specific shock z_1 of $\Lambda_1 = 0.35$, a volatility of the stock price of $\sigma_S = 0.20$, and a correlation between the stock price and the bond price of $\rho = 0.20$. These estimates implies that the long yield becomes $y_\infty = 2.2\%$, the Sharpe ratio of the bond becomes $\Psi = 0.27$, and the volatility of a zero-coupon bond with 10 years to maturity becomes $\sigma_B(r,t) = \sigma_r b(10) = 4.8\%$

[Figure 1 about here.]

From Figure 1(a) we see that the loss from not hedging interest rate dynamics is convexly increasing in the remaining investment horizon $T-t$. For example an investor with a risk aversion of 6 and an investment horizon of 10 years will incur a loss of 3.7%, while an investor with the same risk aversion but an investment horizon of 20 year will incur a loss of almost 9% if he does not hedge the stochastic variations in the short-term interest rate. Furthermore we can see that for $\gamma > 1$ the loss increases with the investor's risk aversion. Remember that the loss from not hedging equals zero for a log-investor, $\gamma = 1$, so the loss is actual decreasing in the investor's risk aversion for a $\gamma < 1$.

In Figure 1(b) we have plotted the loss an investor incurs from completely omitting bonds in his portfolio. We can see that the loss is convexly increasing in the remaining investment horizon $T-t$. The loss from completely omitting the bonds are much larger than the loss from not hedging the interest rate risk. For example an investor with a risk aversion of 6 and an investment horizon of 20 years will incur a loss of almost 9% from not hedging interest rate dynamics, while he will incur a loss of almost 29% by completely omitting bonds in his portfolio. From the figure we can see that it is the investor with the highest value of γ who incurs the largest loss, while it is the investor with the lowest value of γ who incur the second largest lost. The investor with a small γ incur a big loss due to the missing risk premium when omitting bonds from the portfolio, while the investor with a high γ incur a big loss due to the missing hedge of the interest rate dynamics.

The losses are very sensitive to changes in the mean reversion parameter κ and the volatility parameter σ_r. Consider an investor with a risk aversion of 6 and an investment horizon of 20 years. By decreasing σ_r from 3% to 2% and keeping everything else constant, the loss from not hedging decreases from 8.68% to 3.77%, while the loss from completely omitting the bond in the portfolio decreases from 28.59% to 18.96%. On the other hand by decreasing κ from 0.63 to 0.5 increases the loss from not hedging the interest rate dynamics from 8.68% to 13.60%, while the loss from completely omitting the bond in the portfolio increases from 28.59% to 34.12%. For both parameters we get that the losses are much more sensitive to changes for longer investment horizons.

Figure 2 shows the optimal portfolio weight as a function of the investor'r risk aversion and investment horizon, respectively. Figure (a) shows the optimal portfolio weights as a function of the investor's risk aversion, assuming an investment horizon of $T - t = 20$. The figure shows that the weights on the stock and bond are decreasing in the risk aversion coefficient, while the weight on the cash is increasing. The weight on the hedge portfolio is increasing in γ, and for $\gamma < 1$ the investor takes a short position in the hedge portfolio. Surprisingly the investor takes a higher position in the bond than in stock, even if we subtract the hedge position from the total weight on the bond. Hence investors do not only use the bond as a hedge instrument, but also a speculative investment object. Figure (b) shows the optimal portfolio weight as a function of the investor's investment horizon, assuming a relative risk aversion of $\gamma = 6$. The stock weight is constant over time, the total weight on the bond decreases with the investment horizon, while the cash weight decreases. The weight on the hedge portfolio is constant and equals $(\gamma - 1)/\gamma = 5/6$. The reason we have a constant hedge term is that we have assumed that the available zero-coupon bond matures at the end of the investor's investment horizon, T. If we instead assume that the available bond is a 10-year to maturity bond, the weight on the bond as well as on the hedge portfolio will be increasing in the risk aversion coefficient, while the weight on the cash will be decreasing. As in Figure (a) we can see that the a relative big part of the investor's position in the bond is due to speculative reasons and not due to the investor's hedging demand.

[Figure 2 about here.]

Figure 3 shows the wealth loss an investor with a risk aversion of $\gamma = 6$ incurs from missing the optimal portfolio by some percentage (ϕ_S, ϕ_B). Figure (a) assumes an investment horizon of $T - t = 2$, while Figure (b) assumes an investment horizon of $T - t = 20$. From Figure 2 we have that the optimal portfolio strategy for an investor with a risk aversion of 6 and an investment horizon of 2 is to invest 25% in the stock, 188% in the bond, and take a short position of 113% in the bank account. An investor with an investment horizon of 20 should invest 25% of his wealth in the stock, 158% in the bond, and take a short position of 83% in the bank account. The position on the stock might seem a bit low compared to what is done in reality, while the position on the bond might seem a bit high. In Figure 3 we have plotted the wealth loss an investor with a risk aversion of $\gamma = 6$ incurs from missing the optimal portfolio by some percentage (ϕ_S, ϕ_B) for an investment horizon of 2 and 20 years, respectively. The loss is plotted for values of $\phi_S \in [-0.25; 0.5]$ and $\phi_B \in [-0.5; 0.25]$. The loss for an investment horizon of 20 years is approximately 10 times as big as the loss an investor with an investment horizon of 2 years incurs. For a positive ϕ_S the loss is convexly increasing in ϕ_S. While the loss is only slightly increasing in $\phi_B > 0$. For example an investor with an investment horizon of 20 years incurs a loss of only 0.13% if his portfolio weight in the bond deviate with 10% percentage points from the optimal weight, assuming he invests the optimal weight in the stock. On the other hand if he invests the optimal weight in the bond, but deviate with 10% from the optimal weight in the stock, he will incur a loss of 2.40%. Hence it is much more important for the investor to make sure that his weight on the stock is closer to the optimal portfolio weight than the weight on the bond.

It is interesting to note as long as the weight on the stock deviates with less than 12% percentage points and the weight on the bond deviates with less than 26% percentage points from the optimal portfolio the wealth loss is less than 5%, assuming an investment horizon of 20. One implication of this observation is that even though the optimal portfolio strategy requires investors to rebalance their portfolio continuously the loss investors incurs from rebalancing their portfolio in discrete time will be insignificant. Another implication is that the loss investors incur from not using the completely correct estimates for the parameters in the model will be insignificant as long as the deviations from the optimal portfolio weight do not get too big.

[Figure 3 about here.]

Due to the high values of the volatility of the short rate and the mean reversion parameter Brennan and Xia (2002) consider some alternative estimates. That is $\sigma_r = 0.013$ and $\kappa = 0.105$. For all of the above calculated losses these alternative estimates implies a significantly larger loss. For example with the alternative estimates an investor with a risk aversion of 6 and a investment horizon of 20 years will incur a loss of 29% from not hedging the interest rate risk compared to a loss of only 9% with the old estimates . He will incur a loss of 42.5% from completely omitting bonds in his portfolio compared to a loss of 29% with the old estimates.

3.3 Example: the Heston model of stochastic stock volatility

Following Heston (1993), we assume a constant short rate, r, and that the stock price follows a process with stochastic volatility. Indirectly, this means that we assume that the investment opportunities vary with the volatility of the stock price. Specifically, the dynamics of the stock price is given by

$$\begin{aligned} dS_t &= S_t \left[(r + \Lambda_1 v_t)\; dt + \sqrt{v_t}\, dz_{1t} \right], \\ dv_t &= \kappa \left(\bar{v} - v_t \right)\; dt + \rho \sigma_v \sqrt{v_t}\, dz_{1t} + \sqrt{1-\rho^2} \sigma_v \sqrt{v_t}\, dz_{2t}. \end{aligned} \tag{53}$$

The parameters κ, $\bar{v}$, and σ_v are positive constants and ρ is the correlation between instantaneous changes in the stock price and the volatility of the stock price. The market price of risk associated with z_1 is given by $\Lambda_1 \sqrt{v}$, while the market price of risk associated with z_2 is assumed to be given by $\Lambda_2 \sqrt{v}$.

The above market is incomplete, to complete the market we introduce a stock option. Hence, investors have access to trade in the stock, the risk-free asset and a stock option. Following the paper by Liu and Pan (2003) we let O_t denote the time-t price of the stock option. The price of the option depends on the underlying stock price S_t and the stock volatility v_t through $O_t = g(S, v, t)$ for some function g.[8] Itô's Lemma and the fundamental partial differential equation implies that the price dynamics of the stock option is given by

$$dO_t = rO_t\, dt + (g_s S_t + \sigma_v \rho g_v)\left(\Lambda_1 v_t\, dt + \sqrt{v_t}\, dz_{1t}\right) + \sigma_v \sqrt{1-\rho^2}\, g_v \left(\Lambda_2 v_t\, dt + \sqrt{v_t}\, dz_{2t}\right),$$

where g_s and g_v are the partial derivatives with respect to the stock price and volatility, respectively

$$g_s = \left.\frac{\partial g(s,v)}{\partial s}\right|_{(S_t, v_t)}, \qquad g_v = \left.\frac{\partial g(s,v)}{\partial v}\right|_{(S_t, v_t)}.$$

A derivative with $g_s \neq 0$ ensures exposure to the stock price shock z_1, while a derivative with $g_v \neq 0$ ensures exposure to the additional volatility shock z_2.

[8] For example, let K denote the exercise price and T the maturity date of the option, then $g(S_T, v_T, T) = (S_T - K)^+$ is a European call option, and $g(S_T, v_T, T) = (K - S_T)^+$ is a European put option.

The model fits into the affine set-up.[9] By solving the ordinary differential equation (32) we get that

$$G^*(t) = \frac{1}{\gamma} \frac{\left(\Lambda_1^2 + \Lambda_2^2\right)\left(e^{\sqrt{\theta}(T-t)} - 1\right)}{\left(\sqrt{\theta} + \eta\right)\left(e^{\sqrt{\theta}(T-t)} - 1\right) + 2\sqrt{\theta}}, \tag{54}$$

where we have introduced the parameters

$$\eta = \kappa - \frac{1-\gamma}{\gamma}\left(\rho\sigma_v\Lambda_1 + \sqrt{1-\rho^2}\sigma_v\Lambda_2\right), \quad \text{and} \quad \theta = \eta^2 + \frac{\sigma_v^2}{\gamma}\left(1 - \frac{1}{\gamma}\right)\left(\Lambda_1^2 + \Lambda_2^2\right).$$

To ensure that G^* is well defined we need to assume that $\theta > 0$, which is certainly true for $\gamma \geq 1$. From (34) it follows that the optimal investment strategy is given by $\pi^* = (\pi_S^*, \pi_O^*)^\top$, where

$$\pi_S^*(v,t) = \frac{1}{\gamma}\left(\Lambda_1 - \left(\frac{g_s S}{\sigma_v\sqrt{1-\rho^2}\,g_v} + \frac{\rho}{\sqrt{1-\rho^2}}\right)\Lambda_2\right) + \frac{\gamma-1}{\gamma}\frac{g_s S}{g_v}G^*(t) \tag{55}$$

$$\pi_O^*(v,t) = \frac{1}{\gamma}\frac{O}{\sigma_v\sqrt{1-\rho^2}\,g_v}\Lambda_2 + \frac{1-\gamma}{\gamma}\frac{O}{g_v}G^*(t). \tag{56}$$

Similar to Liu and Pan (2003) we find that the optimal position in the option is inversely proportional to g_v/O_t. g_v/O_t measures the volatility exposure for each dollar invested in the option, so a high value of g_v/O_t indicates that the option is effective as a vehicle to volatility risk. Hence a high value of g_v/O_t implies that the investor needs to invest a smaller portion of his wealth in the option, consistent with the above result.

Without loss of generality consider an option with $g_v > 0$, i.e. an option with positive exposure to volatility risk. A positive market price of volatility risk, $\Lambda_2 > 0$, makes the first term in (56) positive. This term reflects the investors speculative demand in the option. On the other hand a negative volatility risk premium induces the speculative investor to take a short position. If $\Lambda_2 = 0$ there is still a benefit from investing in options due to the second term in (56), which reflects the investors hedging demand. It is easy to verify that $G^*(t)$ is strictly positive for realistic parameter values. Hence an investor with $\gamma > 1$ will take a short position in the option to hedge the stochastic variations in the volatility, while an investor with $\gamma < 1$ will take a long position. These results are again similar to the results in Liu and Pan (2003).

The loss an investor incurs from following a no-hedge strategy with potentially omitted options is according to (45) given by

$$L(v,t) = e^{F^*(t) - \bar{F}(t) + \left(G^*(t) - \bar{G}(t)\right)^\top v} - 1, \tag{57}$$

where $G^*(t)$ is given by (54), and F^*, $\bar{F}$, and $\bar{G}$ can be determined from (33), (42) and (43),

[9] Let $M_0 = \kappa\bar{v}$, $M = -\kappa$, $D = \left(\rho\sigma_v, \sqrt{1-\rho^2}\,\sigma_v\right)$, $\nu_1 = \hat{\nu}_1 = \hat{V}_1 = 0$, $V_1 = V_2 = 1$, $\Lambda = (\Lambda_1, \Lambda_2)^\top$, $R_0 = r$, and $R = 0$.

respectively. In particular we get that

$$F^*(t) = r(T-t) + \frac{2\kappa\bar{v}\left(\Lambda_1^2+\Lambda_2^2\right)}{\gamma\left(\eta^2-\theta\right)}\left(\frac{\sqrt{\theta}+\eta}{2}(T-t)+\ln\left(\frac{2\sqrt{\theta}}{\left(\sqrt{\theta}+\eta\right)\left(e^{\sqrt{\theta}(T-t)}-1\right)+2\sqrt{\theta}}\right)\right),$$

$$\bar{F}(t) = r(T-t) + \frac{2\kappa\bar{v}\left(\Lambda_1^2+\varepsilon\Lambda_2^2\right)}{\gamma\left(\bar{\eta}^2-\bar{\theta}\right)}\left(\frac{\sqrt{\bar{\theta}}+\bar{\eta}}{2}(T-t)+\ln\left(\frac{2\sqrt{\bar{\theta}}}{\left(\sqrt{\bar{\theta}}+\bar{\eta}\right)\left(e^{\sqrt{\bar{\theta}}(T-t)}-1\right)+2\sqrt{\bar{\theta}}}\right)\right),$$

and

$$\bar{G}(t) = \frac{1}{\gamma}\frac{\left(\Lambda_1^2+\varepsilon\Lambda_2^2\right)\left(e^{\sqrt{\bar{\theta}}(T-t)}-1\right)}{\left(\sqrt{\bar{\theta}}+\bar{\eta}\right)\left(e^{\sqrt{\bar{\theta}}(T-t)}-1\right)+2\sqrt{\bar{\theta}}},$$

where we have introduced the parameters

$$\bar{\eta} = \kappa - \frac{1-\gamma}{\gamma}\left(\rho\sigma_v\Lambda_1+\varepsilon\sqrt{1-\rho^2}\sigma_v\Lambda_2\right), \qquad \bar{\theta} = \bar{\eta}^2+\sigma_v^2\left(1-\frac{1}{\gamma}\right)\left(\Lambda_1^2+\varepsilon\Lambda_2^2\right),$$

and $\varepsilon \in \{0,1\}$. Here $\varepsilon = 0$ [$\varepsilon = 1$] if the option is excluded [included] in the speculative portfolio. For $\bar{G}(t)$ to be well defined we need to assume that $\bar{\theta} > 0$, which is certainly satisfied whenever $\gamma > 1$. Note that the only difference between the formulas for $\bar{F}, \bar{G}$ relative to F^*, G^* is that θ is replaced by $\bar{\theta}$, and η is replaced by $\bar{\eta}$. It follows that the loss is determined by the risk aversion parameter γ, the investment horizon $T-t$, the volatility of the volatility σ_v, the mean reversion parameter κ, and the two market prices of risk Λ_1 and Λ_2.

Finally we can consider an investor who knows that the volatility of the stock is stochastic, but is restricted to trade in stocks only. Hence the investor will solve the optimal investment problem as if the market is incomplete.[10] From (34) it follows that the optimal investment in the stock is given by

$$\bar{\pi}_S^*(v,t) = \frac{\Lambda_1}{\gamma} + \frac{1-\gamma}{\gamma}\rho\sigma_v\bar{G}^*(t), \tag{58}$$

where

$$\bar{G}^*(t) = \frac{\Lambda_1^2}{\gamma}\frac{\left(e^{\sqrt{\tilde{\theta}}(T-t)}-1\right)}{\left(\sqrt{\tilde{\theta}}+\tilde{\eta}\right)\left(e^{\sqrt{\tilde{\theta}}(T-t)}-1\right)+2\sqrt{\tilde{\theta}}} \tag{59}$$

and

$$\tilde{\eta} = \kappa - \frac{1-\gamma}{\gamma}\rho\sigma_v\Lambda_1, \quad \text{and} \quad \tilde{\theta} = \tilde{\eta}^2+\sigma_v^2\Lambda_1^2\left(1-\frac{1}{\gamma}\right)\left(\frac{1-\gamma}{\gamma}\rho^2+1\right).$$

To ensure that $\bar{G}^*$ is well defined we assume that $\tilde{\theta} > 0$. Hence the investor will use the stock both as a hedge and speculative instrument. However he can not hedge the volatility risk perfectly due to the restriction so the investor will suffer a loss. Furthermore he will suffer a loss due to the

[10] The model will still be affine. Let $M_0 = \kappa\bar{v}$, $M = -\kappa$, $D = \rho\sigma_v$, $\hat{D} = \sqrt{1-\rho^2}\,\sigma_v$, $\nu_1 = \hat{\nu}_1 = 0$, $V_1 = \hat{V}_1 = 1$, $\Lambda = (\Lambda_1, \Lambda_2)^\top$, $R_0 = r$, and $R = 0$.

missing risk-and-return tradeoff from not investing in options. The entire loss he will suffers from the restriction is given by

$$L(v,t) = e^{F^*(t) - \bar{F}^*(t) + \left(G^*(t) - \bar{G}^*(t)\right)^\top v} - 1, \tag{60}$$

where

$$\bar{F}^*(t) = r(T-t) + \frac{2\kappa\bar{v}\Lambda_1^2}{\gamma\left(\tilde{\eta}^2 - \tilde{\theta}\right)}\left(\frac{\sqrt{\tilde{\theta}} + \tilde{\eta}}{2}(T-t) + \ln\left(\frac{2\sqrt{\tilde{\theta}}}{\left(\sqrt{\tilde{\theta}} + \tilde{\eta}\right)\left(e^{\sqrt{\tilde{\theta}}(T-t)} - 1\right) + 2\sqrt{\tilde{\theta}}}\right)\right).$$

Let us look at a numerical example. We use the same benchmark estimates as Liu and Pan (2003), that is $\bar{v} = 0.0169$, $\kappa = 5$, $\sigma_v = 0.25$ and $\rho = -0.40$. Liu and Pan (2003) let $\Lambda_1 = 4$, which yields an average equity risk premium of 6.76%, while they let the parameter Λ_2 vary. The latter is due to the missing consensus on reasonable values for the market price of volatility risk in the existing empirical literature. However, there is a strong support that volatility risk is priced, and several papers report that volatility risk is negatively priced.[11] Intuitively, a negative volatility risk premium can be supported by the fact that aggregate market volatility is typically high during recessions. For investors to take a short position in volatility therefore requires an additional risk premium as compensation for the loss of value during recessions.

[Figure 4 about here.]

Figure 4 shows the loss an investor incurs from not following the optimal investment strategy in the following three scenarios:

i) The investor does not believe it is important to hedge the stochastic variations in the volatility of the stock. Hence he only invest in the option for speculative reasons. The loss is determined from (57) with $\varepsilon = 1$.

ii) The investor believes that the volatility of the stock is constant and therefore only invest in the stock. The loss is determined from (57) with $\varepsilon = 0$.

iii) The investor knows the volatility of the stock is stochastic but is restricted to trade in stocks only. The loss is determined from (60).

Figure 4(a), (c) and (e) show the loss an investor incurs as a function of the investors investment horizon in Scenario i), ii) and iii) respectively. In all tree cases we can see that the loss is an increasing function in the investment horizon. However, the size of the loss in the three scenarios differs. The loss is not that significant in Scenario i). Even for an investment horizon of 20 years all the investors considered in the figure suffer a loss of less than 10%. The loss in Scenario ii) and iii) is significant. For example all the considered investors will incur a loss of more than 60% if they have a investment horizon of 10 years or more in both scenarios. It is hard to see any

[11] See among others Chernov and Ghysels (2000), Pan (2002), and Bakshi and Kapadia (2003).

difference between the losses in Figure 4(c) and Figure 4(e), i.e. in Scenario ii) and iii). However there is a small difference. The loss an investor in Scenario ii) incurs is insignificant larger than the corresponding loss an investor in Scenario iii) will incur. For example an investor with a risk aversion of 4 and an investment horizon of 5 years will suffer a loss of 73.71% in Scenario ii), while he will suffer a loss of 73.68% in Scenario iii). Hence, in accordance with Figure 4(a) it is not that important to hedge the stochastic variations in the volatility of the stock price as long as the investor realizes that the stock volatility is stochastic and he can gain from investing in options.

While the loss is monotonic decreasing in the investors risk aversion in Scenario ii) and iii), this is not the case in Scenario i). Figure 4(a) shows that it is the investor with the lowest risk aversion, $\gamma = 2$, who suffers the smallest loss, while it is the investor with the second lowest risk aversion, $\gamma = 4$, who suffers the largest loss. In particular it can be shown that the loss will decrease in the risk aversion for $\gamma < 1$, then increase up to some level of γ, after which the loss again decreases in the risk aversion. The explanation for this picture is as follows. An investor with a high risk aversion is more concerned about the stochastic variations in the investment opportunity set, hence an investor with a high risk aversion should lose more than an investor with a low risk aversion. On the other hand an investor with a high risk aversion takes a higher position in the bank account and hence a lower investment in the two risky assets - so even though an investor with a high gamma hedges more than an investor with a low gamma the actual loss he incurs will be lower due to the higher position in the bank account.

Figure 4(b), (d) and (f) show the loss an investor incurs as a function of the market price of risk factor Λ_2 in Scenario i), ii) and iii) respectively. We can see that the loss is very sensitive to the level of the market price of volatility risk, in particular when the volatility risk is negatively priced. For example an investor in Scenario i) with a risk aversion of 4 and an investment horizon of 10 years will incur a loss of 4.46% if the market price of risk is -6, while he will incur a loss of 38.25% if the market price of risk is -10. Empirically studies seems to agree on the fact that the volatility risk is priced and that the price is negative. However they disagree on the level of the price. According to Liu and Pan (2003) an estimate of $\Lambda_2 = -6$ can be seen as conservative estimate compared to other estimates reported in the literature.

Again it is hard to distinguish between Scenario ii) and iii), i.e. Figure 4(d) and (f). However, the loss in Scenario ii) is insignificantly larger than the corresponding loss in Scenario iii). For example for a zero volatility risk premium there is no myopic demand for derivatives, hence the investor in Scenario iii) does not suffer any loss, while the investor in Scenario ii) suffer a loss of 0.22% due to the missing hedge of the stochastic variations in the stock volatility.

4 Example with quadratic model: mean reversion in stock returns

Campbell and Shiller (1988), Fama and French (1989) and others have found evidence of predictability in asset returns. Poterba and Summers (1988) finds related evidence that stock returns exhibit mean reversion. Finally recently empirical findings by Campbell and Vuolteenaho (2004) indicates that it is important for a long-horizon investor to distinguish between value and growth stocks. They argue that value stocks are more risky than growth stocks from the perspective of a long-horizon risk averse investor. Jurek and Viceira (2006) verify this prediction by showing that a risk-averse investor, constrained to hold only value and growth stocks, decreases his allocation to value as his investment horizon increases.

We come up with a model which takes the above implications into account. Following Kim and Omberg (1996) we assume a constant short rate, r, but instead of allowing the investor to only trade in one stock index we allow the investor to trade in three different stock portfolios; the market portfolio, a growth stock portfolio and a value stock portfolio. Let $S = (S^M, S^G, S^V)^\top$ denote the price process of the three portfolios, where S^M is the price of the market portfolio, S^G is the price of the growth stock portfolio, and S^V is the price of the value stock portfolio. Assume that the price dynamics are given by

$$dS_t = \operatorname{diag}(S_t)\left[(r1 + \operatorname{diag}(\sigma_S)K\lambda_t)\,dt + \operatorname{diag}(\sigma_S)K\,dz_t\right], \tag{61}$$

where $\sigma_S = (\sigma_S^M, \sigma_S^G, \sigma_S^V)^\top$ is the vector of the volatilities corresponding to the three stock portfolios, respectively, and K is a (3×3)-matrix introducing correlations between the three portfolios

$$K = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{GM} & \sqrt{1-\rho_{GM}^2} & 0 \\ \rho_{VM} & \hat{\rho}_{GV} & \sqrt{1-\rho_{VM}^2-\hat{\rho}_{GV}^2} \end{pmatrix}.$$

Note ρ_{GM} and ρ_{VM} are the instantaneously correlations between the market portfolio and the growth portfolio, and the market portfolio and value portfolio, respectively. While $\hat{\rho}_{GV} = (\rho_{GV} - \rho_{GM}\rho_{VM})/\sqrt{1-\rho_{GM}^2}$, where ρ_{GV} is the instantaneously correlation between the value and growth portfolio. $\lambda_t = (\lambda_t^M, \lambda_t^G, \lambda_t^V)^\top$ is the time-varying market price of risk vector on the stocks. The market price of risk vector is described by an Ornstein-Uhlenbeck process

$$d\lambda_t = \operatorname{diag}(\kappa)\left(\bar{\lambda} - \lambda_t\right)\,dt + \operatorname{diag}(\sigma_\lambda)\left[L\,dz_t + \hat{L}d\hat{z}_t\right], \tag{62}$$

where $\sigma_\lambda = (\sigma_\lambda^M, \sigma_\lambda^G, \sigma_\lambda^V)^\top$ is the vector of volatilities corresponding to the three market prices of risk, $\kappa = (\kappa_M, \kappa_G, \kappa_V)^\top$, $\bar{\lambda} = (\bar{\lambda}_M, \bar{\lambda}_G, \bar{\lambda}_V)^\top$, L is (3×3)-matrix introducing correlations between the risk premiums and portfolios

$$L = \begin{pmatrix} \rho_{MM}^\lambda & 0 & 0 \\ \alpha_{GM}^\lambda & \alpha_{GG}^\lambda & 0 \\ \alpha_{VM}^\lambda & \alpha_{VG}^\lambda & \alpha_{VV}^\lambda \end{pmatrix},$$

and finally $\hat{L}$ is a (3×3) diagonal matrix with the vector

$$\left(\sqrt{1-(\rho_{MM}^\lambda)^2}, \sqrt{1-(\alpha_{MG}^\lambda)^2-(\alpha_{GG}^\lambda)^2}, \sqrt{1-(\alpha_{VM}^\lambda)^2-(\alpha_{VG}^\lambda)^2-(\alpha_{VV}^\lambda)^2}\right)^\top$$

down the diagonal. Note ρ^λ_{MM} in the L-matrix equals the correlation between the market portfolio and the associated market price of risk, while the α's only enters into the correlations between the risk premiums and portfolios. For example the correlation between the market portfolio and the risk premium on the growth portfolio is given by $\rho^\lambda_{GM} = \rho^\lambda_{MM}\alpha^\lambda_{GM}$.

Due to the stochastic market price of risk, this model is not affine. However it is still quite simple to determine the optimal portfolio strategy and the losses an investor suffers from following a sub-optimal investment strategy by the use of the general solutions. According to Theorem 1 we need to find the function $H^*(t)$ to determine the optimal investment strategy. $H^*(t)$ is the solution to the partial differential equation (7) with the boundary condition $H^*(T) = 0$. In the Appendix we have shown that

$$H^*(t,\lambda) = a^*(t) + b^*(t)^\top\lambda + \frac{1}{2}\lambda^\top c^*(t)\lambda, \tag{63}$$

where $a^*(t)$ is a deterministic function, $b^*(t)$ is a deterministic 3×1 vector-valued function, and $c^*(t)$ is a deterministic 3×3 matrix-valued function. The three deterministic functions solves a system of first order ordinary differential equations with boundary conditions $a^*(T) = b^*(T) = c^*(T) = 0$ which can be seen in the Appendix. The system of ordinary differential equations can easily be solved by using standard numerically techniques. It now follows from Theorem 1 that the optimal investment strategy is given by

$$\pi^*(\lambda,t) = \frac{1}{\gamma}\left(\sigma(\lambda,t)^\top\right)^{-1}\lambda + \frac{1-\gamma}{\gamma}\left(\sigma(\lambda,t)^\top\right)^{-1}v(\lambda)^\top\left(b^*(t) + c^*(t)\lambda\right), \tag{64}$$

where $v(\lambda) = \text{diag}(\sigma_\lambda)L$ and

$$\left(\sigma(\lambda,t)^\top\right)^{-1} = \begin{pmatrix} \frac{1}{\sigma^M_S} & -\frac{\rho_{GM}}{\sigma^M_S\sigma^G_S\sqrt{1-\rho^2_{GM}}} & \frac{\rho_{GM}\hat\rho_{GV}-\rho_{VM}\sqrt{1-\rho^2_{GM}}}{\sigma^M_S\sigma^G_S\sqrt{1-\rho^2_{GM}}\sqrt{1-\rho^2_{VM}-\hat\rho^2_{GV}}} \\ 0 & \frac{1}{\sigma^G_S\sqrt{1-\rho^2_{GM}}} & -\frac{\hat\rho_{GV}}{\sigma^G_S\sqrt{1-\rho^2_{GM}}\sqrt{1-\rho^2_{VM}-\hat\rho^2_{GV}}} \\ 0 & 0 & \frac{1}{\sigma^V_S\sqrt{1-\rho^2_{VM}-\hat\rho^2_{GV}}} \end{pmatrix}.$$

It is not easy to say anything general about the optimal investment strategy, but we will have a closer look at the strategy in the later numerical example.

To determine the loss an investor suffers from following a no-hedge strategy with potentially omitted assets we need to determine $\bar H(t)$ which is the solution to the partial differential equation (20) with boundary condition $\bar H(T) = 0$. In the Appendix we have shown that

$$\bar H(t,\lambda) = \bar a(t) + \bar b(t)^\top\lambda + \frac{1}{2}\lambda^\top\bar c(t)\lambda, \tag{65}$$

where $\bar a(t)$ is a deterministic function, $\bar b(t)$ is a deterministic 3×1 vector-valued function, and $\bar c(t)$ is a deterministic 3×3 matrix-valued function. The three deterministic functions solves a system of first order ordinary differential equations with boundary conditions $\bar a(T) = \bar b(T) = \bar c(T) = 0$ which can be seen in the Appendix. The system of ordinary differential equations can easily be solved by using standard numerically techniques. The loss can now be determined from (14), i.e.

the loss is given by

$$L(\lambda, t) = e^{H^*(\lambda,t) - \bar{H}(\lambda,t)} - 1. \tag{66}$$

Finally we want to determine the loss an investor suffers from following an sub-optimal investment strategy where the investor omits some of the available assets at the market. For example we want to determine the loss an investor suffers from only investing in the market portfolio, and the loss an investor suffers from omitting value and growth stocks, respectively. To determine the loss we need to determine $\bar{H}^*(t)$ which is the solution to the partial differential equation in Theorem 1 with boundary condition $\bar{H}^*(T) = 0$ and the further restriction that the investor only invest in the $d_1 \leq 3$ assets. The latter restriction implies that $\lambda = \left(\lambda^M, \lambda^G, \lambda^V\right)^\top$ need to be decomposed into two vectors and $v(\lambda) = \mathrm{diag}(\sigma_\lambda)L$ need to be decomposed into two matrixes. For a more detailed explanation see the Appendix. In the Appendix we have shown that

$$\bar{H}^*(t, \lambda) = \bar{a}^*(t) + \bar{b}^*(t)^\top \lambda + \frac{1}{2}\lambda^\top \bar{c}^*(t)\lambda, \tag{67}$$

where $\bar{a}^*(t)$ is a deterministic function, $\bar{b}^*(t)$ is a deterministic 3×1 vector-valued function, and $\bar{c}^*(t)$ is a deterministic 3×3 matrix-valued function. The three deterministic functions solves a system of first order ordinary differential equations with boundary conditions $\bar{a}^*(T) = \bar{b}^*(T) = \bar{c}^*(T) = 0$ which can be seen in the Appendix. The system of ordinary differential equations can easily be solved by using standard numerically techniques. The loss can now be determined from (14), i.e. the loss is given by

$$L(\lambda, t) = e^{H^*(\lambda,t) - \bar{H}^*(\lambda,t)} - 1. \tag{68}$$

NUMBERS AND ILLUSTRATIONS TO COME...

5 Conclusion

To come...

A Proofs of Theorems

Proof of Theorem 1.

The Hamilton-Jacobi-Bellman equation associated with the utility maximization problem (5) is

$$
\begin{aligned}
0 = \sup_{\pi\in\mathbb{R}^d} \Bigg\{ & \frac{\partial J}{\partial t} + W\frac{\partial J}{\partial W}\left[r(x) + \pi^\top \sigma(x,t)\lambda(x)\right] + \frac{1}{2}W^2\frac{\partial^2 J}{\partial W^2}\pi^\top\sigma(x,t)\sigma(x,t)^\top\pi + m(x)^\top\frac{\partial J}{\partial x} \\
& + \frac{1}{2}\operatorname{tr}\left(\frac{\partial^2 J}{\partial x^2}\Sigma(x)\right) + W\pi^\top\sigma(x,t)v(x)^\top\frac{\partial^2 J}{\partial W\partial x}\Bigg\}.
\end{aligned} \tag{69}
$$

Substituting in the qualified guess $J(W,x,t) = \frac{1}{1-\gamma}\left(We^{H^*(x,t)}\right)^{1-\gamma}$ and simplifying, we arrive at

$$
\begin{aligned}
0 = \sup_{\pi\in\mathbb{R}^d} \Bigg\{ & \frac{\partial H^*}{\partial t} + \left(m(x) + (1-\gamma)v(x)\sigma(x,t)^\top\pi(x,t)\right)^\top\frac{\partial H^*}{\partial x} + \frac{1-\gamma}{2}\left(\frac{\partial H^*}{\partial x}\right)^\top\Sigma(x)\frac{\partial H^*}{\partial x} \\
& + \frac{1}{2}\operatorname{tr}\left(\frac{\partial^2 H^*}{\partial x^2}\Sigma(x)\right) + r(x) + \pi(x,t)^\top\sigma(x,t)\left[\lambda(x) - \frac{\gamma}{2}\sigma(x,t)^\top\pi(x,t)\right]\Bigg\}.
\end{aligned} \tag{70}
$$

The first-order condition for π gives (8). Substituting that into (70) we obtain the PDE (7). □

Proof of Theorem 2.

The terminal wealth is

$$
\begin{aligned}
W_T^\pi = W_t^\pi \exp\Bigg\{ & \int_t^T \left(r(x_s) + \pi(x_s,s)^\top\sigma(x_s,s)\left[\lambda(x_s) - \frac{1}{2}\sigma(x_s,s)^\top\pi(x_s,s)\right]\right) ds \\
& + \int_t^T \pi(x_s,s)^\top\sigma(x_s,s)\,dz_s\Bigg\}
\end{aligned} \tag{71}
$$

so that

$$
J^\pi(W,x,t) = \frac{1}{1-\gamma}\,\mathrm{E}_{W,x,t}\left[(W_T^\pi)^{1-\gamma}\right] = \frac{1}{1-\gamma}W^{1-\gamma}e^{(1-\gamma)H^\pi(x,t)},
$$

where

$$
\begin{aligned}
e^{(1-\gamma)H^\pi(x,t)} = \mathrm{E}_{x,t}\Bigg[& e^{(1-\gamma)\int_t^T\left(r(x_s)+\pi(x_s,s)^\top\sigma(x_s,s)\left[\lambda(x_s)-\frac{1}{2}\sigma(x_s,s)^\top\pi(x_s,s)\right]\right)ds} \\
& \times e^{(1-\gamma)\int_t^T\pi(x_s,s)^\top\sigma(x_s,s)\,dz_s}\Bigg].
\end{aligned} \tag{72}
$$

We can rewrite this as

$$
\begin{aligned}
e^{(1-\gamma)H^\pi(x,t)} = \mathrm{E}_{x,t}\Bigg[& e^{\int_t^T(1-\gamma)\pi(x_s,s)^\top\sigma(x_s,s)\,dz_s - \frac{1}{2}\int_t^T(1-\gamma)^2\|\pi(x_s,s)^\top\sigma(x_s,s)\|^2\,ds} \\
& \times e^{(1-\gamma)\int_t^T\left(r(x_s)+\pi(x_s,s)^\top\sigma(x_s,s)\left[\lambda(x_s)-\frac{\gamma}{2}\sigma(x_s,s)^\top\pi(x_s,s)\right]\right)ds}\Bigg].
\end{aligned}
$$

By Girsanov's theorem this implies (10).

The dynamics of the state variable under the probability measure $\mathbb{Q}(\pi)$ is

$$
\begin{aligned}
dx_t &= m(x_t)\, dt + v(x_t)\, (dz_t^\pi + (1-\gamma)\sigma(x_t,t)^\top \pi(x_t,t)\, dt) + \hat{v}(x_t)\, d\hat{z}_t \\
&= (m(x_t) + (1-\gamma)v(x_t)\sigma(x_t,t)^\top \pi(x_t,t))\; dt + v(x_t)\, dz_t^\pi + \hat{v}(x_t)\, d\hat{z}_t.
\end{aligned}
$$

The PDE (11) now follows from an application of the Feynman-Kac theorem. □

Proof of Theorem 3. Substituting $\pi = (\pi_1, \pi_2)$ into (71), the terminal wealth can be written as

$$
\begin{aligned}
W_T^{(\pi_1,\pi_2)} = W_t \exp\left\{\int_t^T r_s\, ds\right\} &\exp\left\{\int_t^T \pi_{1s}^\top \sigma_{11s}(\lambda_{1s} - \frac{1}{2}\sigma_{11s}^\top \pi_{1s})\, ds + \int_t^T \pi_{1s}^\top \sigma_{11s}\, dz_{1s}\right\} \\
\times \exp&\left\{\int_t^T \pi_{2s}^\top \sigma_{22s}(\lambda_{2s} - \frac{1}{2}\sigma_{22s}^\top \pi_{2s})\, ds + \int_t^T \pi_{2s}^\top \sigma_{22s}\, dz_{2s}\right\}
\end{aligned}
\tag{73}
$$

assuming that $\sigma_{21s} \equiv 0$. Given the assumption on the interest rate dynamics, we can write the expected utility of any class-separated investment strategy on the form

$$
J^{(\pi_1,\pi_2)}(W,x,t) = \frac{1}{1-\gamma} W^{1-\gamma} V^{\pi_1}(x_1,t) V^{\pi_2}(x_2,t). \tag{74}
$$

Similarly, for the alternative strategy $(\pi_1, \tilde{\pi_2})$:

$$
J^{(\pi_1,\tilde{\pi}_2)}(W,x,t) = \frac{1}{1-\gamma} W^{1-\gamma} V^{\pi_1}(x_1,t) V^{\tilde{\pi}_2}(x_2,t). \tag{75}
$$

The wealth loss L from applying (π_1, π_2) instead of $(\pi_1, \tilde{\pi_2})$ is thus given by

$$
J^{(\pi_1,\pi_2)}(W[1+L],x,t) = J^{(\pi_1,\tilde{\pi}_2)}(W,x,t) \quad \Leftrightarrow \quad L = \left(\frac{V^{\tilde{\pi}_2}(x_2,t)}{V^{\pi_2}(x_2,t)}\right)^{1/(1-\gamma)} - 1, \tag{76}
$$

which is independent of π_1 as claimed. □

B Mean reversion in stock returns

Due to the stochastic market price of risk, the model is not affine, so we need to use the general solutions to find the optimal investment strategy. According to Theorem 1 we need to find the function $H^*(t)$ which is the solution to the partial differential equation (7) with the terminal condition $H^*(T) = 0$. A qualified guess of the form of the H^*-function is given by

$$
H^*(t,\lambda) = a^*(t) + b^*(t)^\top \lambda + \frac{1}{2}\lambda^\top c^*(t)\lambda, \tag{77}
$$

where $a^*(t)$ is a deterministic function, $b^*(t)$ is a deterministic 3×1 vector-valued function, and $c^*(t)$ is a deterministic 3×3 matrix-valued function. The terminal condition $H^*(T) = 0$ implies that $a^*(T) = b^*(T) = c^*(T) = 0$. Substituting the relevant derivatives of our guess into the PDE, simplifying, and finally matching the coefficient on $\lambda^\top[\cdot]\lambda$, $\lambda^\top$, and the constant terms lead to the following system of ordinary differential equations:

$$
\begin{aligned}
&(a^*)'(t) + \bar{\lambda}^\top \operatorname{diag}(\kappa) b^*(t) + \frac{1}{2}\operatorname{tr}\left(c^*(t) \operatorname{diag}(\sigma_\lambda)\left(LL^\top + \hat{L}\hat{L}^\top\right)\operatorname{diag}(\sigma_\lambda)\right) \\
&\qquad + \frac{1-\gamma}{2} b^*(t)^\top \operatorname{diag}(\sigma_\lambda)\left(\frac{1}{\gamma}LL^\top + \hat{L}\hat{L}^\top\right)\operatorname{diag}(\sigma_\lambda) b^*(t) + r = 0,
\end{aligned}
\tag{78}
$$

$$
\begin{aligned}
&(b^*)'(t) - \left(\operatorname{diag}(\kappa) - \frac{1-\gamma}{\gamma} L^\top \operatorname{diag}(\sigma_\lambda)\right) b^*(t) + \bar{\lambda}^\top \operatorname{diag}(\kappa) c^*(t) \\
&\qquad + (1-\gamma)\, c^*(t)^\top \operatorname{diag}(\sigma_\lambda)\left(\frac{1}{\gamma}LL^\top + \hat{L}\hat{L}^\top\right)\operatorname{diag}(\sigma_\lambda) b^*(t) = 0,
\end{aligned}
\tag{79}
$$

$$
\begin{aligned}
&(c^*)'(t) - 2\left(\operatorname{diag}(\kappa) - \frac{1-\gamma}{\gamma} L^\top \operatorname{diag}(\sigma_\lambda)\right) c^*(t) \\
&\qquad + (1-\gamma)\, c^*(t)^\top \operatorname{diag}(\sigma_\lambda)\left(\frac{1}{\gamma}LL^\top + \hat{L}\hat{L}^\top\right)\operatorname{diag}(\sigma_\lambda) c^*(t) + \frac{1}{\gamma} I = 0.
\end{aligned}
\tag{80}
$$

The system of first order differential equations with the boundary conditions $a^*(T) = b^*(T) = c^*(T) = 0$ can easily be solved by using standard numerically techniques.

To determine the loss an investor suffers from following a no-hedge strategy with potentially omitted assets we need to determine the function $\bar{H}(t)$ which is the solution to the partial differential equation (20) with the terminal condition $\bar{H}(T) = 0$. A qualified guess of the form of the $\bar{H}$-function is given by

$$
\bar{H}(t,\lambda) = \bar{a}(t) + \bar{b}(t)^\top \lambda + \frac{1}{2}\lambda^\top \bar{c}(t)\lambda, \tag{81}
$$

where $\bar{a}(t)$ is a deterministic function, $\bar{b}(t)$ is a deterministic 3×1 vector-valued function, and $\bar{c}(t)$ is a deterministic 3×3 matrix-valued function. The terminal condition $H^*(T) = 0$ implies that $\bar{a}(T) = \bar{b}(T) = \bar{c}(T) = 0$. Substituting the relevant derivatives of our guess into the PDE, simplifying, and finally matching the coefficient on $\lambda^\top[\cdot]\lambda$, $\lambda^\top$, and the constant terms lead to the following system of ordinary differential equations:

$$
\begin{aligned}
&(\bar{a})'(t) + \bar{\lambda}^\top \operatorname{diag}(\kappa)\bar{b}(t) + \frac{1}{2}\operatorname{tr}\left(\bar{c}(t)\Sigma(\lambda)\right) + \frac{1-\gamma}{2}\bar{b}(t)^\top \Sigma(\lambda)\bar{b}(t) + r = 0, \\
&(\bar{b})'(t) - \left(\operatorname{diag}(\kappa) - \frac{1-\gamma}{\gamma}\hat{v}_1(\lambda)^\top\right)\bar{b}(t) + \bar{\lambda}^\top \operatorname{diag}(\kappa)\bar{c}(t) + (1-\gamma)\,\bar{c}(t)^\top \Sigma(\lambda)\bar{b}(t) = 0, \\
&(\bar{c})'(t) - 2\left(\operatorname{diag}(\kappa) - \frac{1-\gamma}{\gamma}\hat{v}_1(\lambda)^\top\right)\bar{c}(t) + (1-\gamma)\,\bar{c}(t)^\top \Sigma(\lambda)\bar{c}(t) + \frac{1}{\gamma} I_0 = 0.
\end{aligned}
$$

Let $d_1 \leq 3$ be the number of assets the investor invest in, then I_0 is a 3×3 matrix where the first d_1 diagonal elements equal 1 and all other elements equal zero. Let $d_2 = d - d_1$ then $v(\lambda) = (v_1(\lambda), v_2(\lambda)) = \operatorname{diag}(\sigma_\lambda)L$ where v_i has dimension $k \times d_i$, and $\hat{v}_1(\lambda)$ equals $v_1(\lambda)$ but with $d - d_1$ extra columns of zeros. Finally

$$
\Sigma(\lambda) = \operatorname{diag}(\sigma_\lambda)\left(LL^\top + \hat{L}\hat{L}^\top\right)\operatorname{diag}(\sigma_\lambda).
$$

The system of first order differential equations with the boundary conditions $\bar{a}(T) = \bar{b}(T) = \bar{c}(T) = 0$ can easily be solved by using standard numerically techniques.

Finally we want to determine the loss an investor suffers from following an sub-optimal investment strategy where the investor omits some of the available assets at the market.To determine the loss we need to determine $\bar{H}^*(t)$ which is the solution to the partial differential equation in Theorem 1 with boundary condition $\bar{H}^*(T) = 0$ and the further restriction that the investor only invest in the $d_1 \leq 3$ assets. To capture the latter restriction we need to decompose λ and v as

$$\lambda = \begin{pmatrix} \lambda^M \\ \lambda^G \\ \lambda^V \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \qquad v(\lambda) = \operatorname{diag}(\sigma_\lambda) L = (v_1(\lambda),\, v_2(\lambda)).$$

Let $d_2 = 3 - d_1$ then λ_i has dimension d_i, and v_i has dimension $3 \times d_i$. It then follows from Theorem 1 that $\bar{H}^*(t)$ solves the PDE

$$\begin{aligned} &\frac{\partial \bar{H}^*}{\partial t} + \left(m(\lambda) + \frac{1-\gamma}{\gamma} v_1(\lambda)\lambda_1\right)^\top \frac{\partial \bar{H}^*}{\partial \lambda} + \frac{1}{2} \operatorname{tr}\left(\frac{\partial^2 H^*}{\partial \lambda^2} \Sigma(\lambda)\right) \\ &\quad + \frac{1-\gamma}{2} \left(\frac{\partial \bar{H}^*}{\partial \lambda}\right)^\top \left[\Sigma(\lambda) - \left(1 - \frac{1}{\gamma}\right) v_1(\lambda) v_1(\lambda)^\top\right] \frac{\partial \bar{H}^*}{\partial \lambda} + r + \frac{1}{2\gamma} \lambda_1^\top \lambda_1 = 0 \end{aligned}$$

with the terminal condition $\bar{H}^*(\lambda, T) = 0$. A qualified guess of the form of the $\bar{H}^*$-function is given by

$$H^*(t, \lambda) = \bar{a}^*(t) + \bar{b}^*(t)^\top \lambda + \frac{1}{2} \lambda^\top \bar{c}^*(t) \lambda, \tag{82}$$

where $\bar{a}^*(t)$ is a deterministic function, $\bar{b}^*(t)$ is a deterministic 3×1 vector-valued function, and $\bar{c}^*(t)$ is a deterministic 3×3 matrix-valued function. The terminal condition $\bar{H}^*(T) = 0$ implies that $\bar{a}^*(T) = \bar{b}^*(T) = \bar{c}^*(T) = 0$. Substituting the relevant derivatives of our guess into the PDE, simplifying, and finally matching the coefficient on $\lambda^\top [\cdot] \lambda$, $\lambda^\top$, and the constant terms lead to the following system of ordinary differential equations:

$$\begin{aligned} &(\bar{a}^*)'(t) + \bar{\lambda}^\top \operatorname{diag}(\kappa) \bar{b}^*(t) + \frac{1}{2} \operatorname{tr}\left(\bar{c}^*(t) \Sigma(\lambda)\right) \\ &\quad + \frac{1-\gamma}{2} \bar{b}^*(t)^\top \left(\Sigma(\lambda) - \left(1 - \frac{1}{\gamma}\right) v_1(\lambda) v_1(\lambda)^\top\right) \bar{b}^*(t) + r = 0, \\ &(\bar{b}^*)'(t) - \left(\operatorname{diag}(\kappa) - \frac{1-\gamma}{\gamma} \hat{v}_1(\lambda)^\top\right) \bar{b}^*(t) + \bar{c}^*(t) \operatorname{diag}(\kappa) \bar{\lambda} \\ &\quad + (1-\gamma)\, \bar{c}^*(t)^\top \left(\Sigma(\lambda) - \left(1 - \frac{1}{\gamma}\right) v_1(\lambda) v_1(\lambda)^\top\right) \bar{b}^*(t) = 0, \\ &(\bar{c}^*)'(t) - 2\left(\operatorname{diag}(\kappa) - \frac{1-\gamma}{\gamma} \hat{v}_1(\lambda)^\top\right) \bar{c}^*(t) \\ &\quad + (1-\gamma)\, \bar{c}^*(t)^\top \left(\Sigma(\lambda) - \left(1 - \frac{1}{\gamma}\right) v_1(\lambda) v_1(\lambda)^\top\right) \bar{c}^*(t) + \frac{1}{\gamma} I_0 = 0. \end{aligned}$$

I_0, $\hat{v}_1(\lambda)$, and $\Sigma(\lambda)$ have the same interpretations as above. The system of first order differential equations with the boundary conditions $a^*(T) = b^*(T) = c^*(T) = 0$ can easily be solved by using standard numerically techniques.

C Option Pricing

Following Heston (1993) the closed-form solution of a European call option on a non-dividend paying stock with maturity at time $\bar{T}$ and exercise price K is given by

$$C(S, v, t; K, \bar{T}) = SP_1 - Ke^{-r(\bar{T}-t)}P_2$$

where P_1 measures the probability of the call option expires in-the-money, while P_2 is the adjusted probability of the same event. In particular we have that

$$P_j(x, v, t) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\ln K} f_j(x, v, t; u)}{iu}\right] du$$

where $x = \ln S$, and

$$\begin{aligned}
f_j(x, v, t; u) &= \exp\left\{C(\bar{T} - t; u) + D(\bar{T} - t; u)v + iux\right\}, \\
C(\tau; u) &= rui\tau + \frac{\kappa\bar{v}}{\sigma_v^2}\left((b_j - \rho\sigma_v ui + d)\,\tau - 2\ln\left[\frac{1 - ge^{d\tau}}{1 - g}\right]\right), \\
D(\tau; u) &= \frac{b_j - \rho\sigma_v ui + d}{\sigma_v^2}\left(\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}\right), \\
g &= \frac{b_j - \rho\sigma_v ui + d}{b_j - \rho\sigma_v ui - d}, \\
d &= \sqrt{(\rho\sigma_v ui - b_j)^2 - \sigma_v^2\left(c_j ui - u^2\right)}
\end{aligned}$$

for $j = 1, 2$, and finally

$$c_1 = 1, \quad c_2 = -1, \quad b_1 = \kappa + \sigma_v\Lambda_2 - \rho\sigma_v, \quad b_2 = \kappa + \sigma_v\Lambda_2.$$

By the put-call parity the price of a European put option is given by

$$P(S, v, t; K, \bar{T}) = Ke^{-r(\bar{T}-t)}(1 - P_2) - S(1 - P_1).$$

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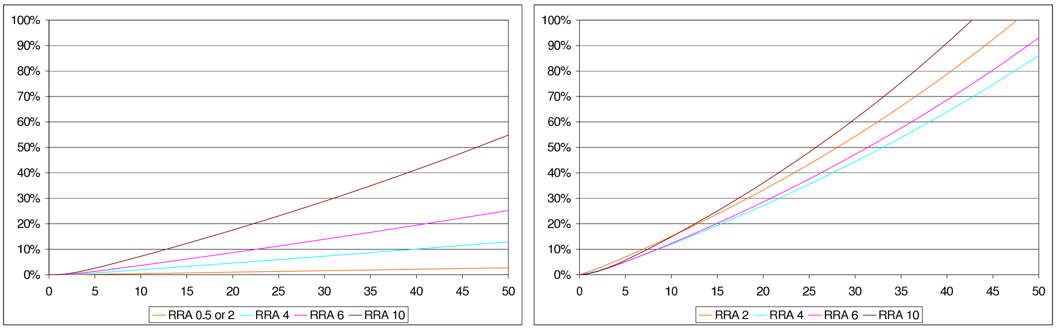
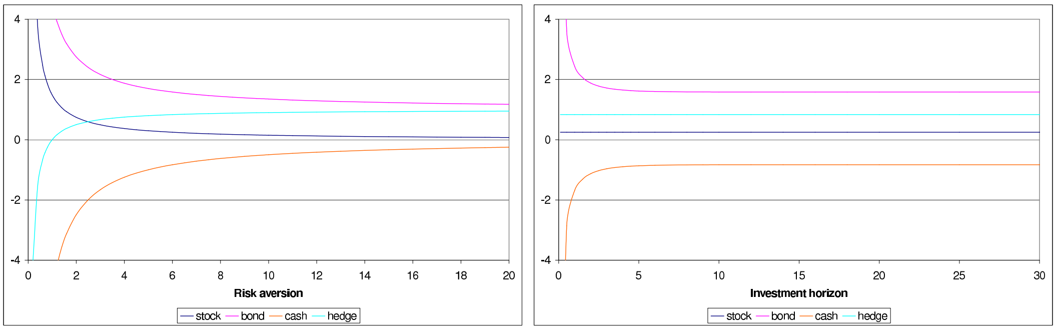


(a) The loss from not hedging interest rate dynamics

(b) The loss from not including bonds in the portfolio

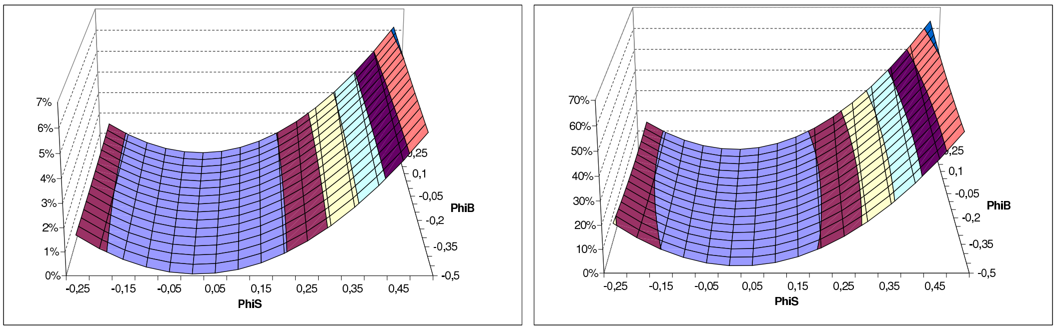
Figure 1: **The loss from not following the optimal strategy as a function of the investors investment horizon.** The short term interest rate is assumed to follow the model by Vasicek. The parameter values are as follows: $\bar{r} = 0.01$, $\kappa = 0.63$, $\sigma_r = 0.03$, $\Lambda_1 = 0.35$, $\Lambda_2 = 0.21$, and $\rho = 0.20$. In (a) $\varepsilon = 0$ and in (b) $\varepsilon = 1$.



(a) Optimal portfolio weights as a function of the investors risk aversion, $T - t = 20$.

(b) Optimal portfolio weights as a function of the investors investment horizon, $\gamma = 6$.

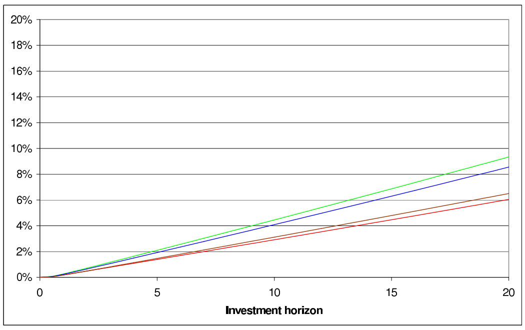
Figure 2: **Optimal portfolio weights.** The bond weight represents the total weight on a zero-coupon bond maturing at T, while the hedge weight represents the weight in the bond that is used to hedge the interest rate risk. The short term interest rate is assumed to follow the model by Vasicek. The parameter values are as follows: $\kappa = 0.63$, $\sigma_r = 0.03$, $\sigma_S = 0.20$, $\Lambda_1 = 0.35$, $\Lambda_2 = 0.21$, and $\rho = 0.20$.



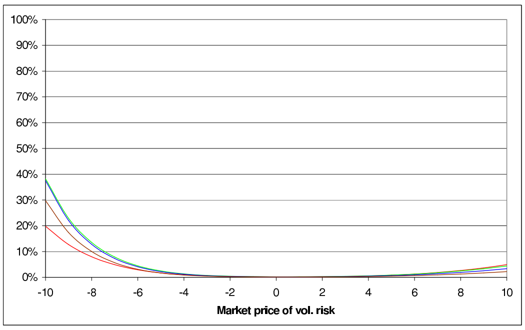
(a) The loss from missing the optimal portfolio by some percentage for an investor with $T = 2$.

(b) The loss form missing the optimal portfolio by some percentage for an investor with $T = 20$.

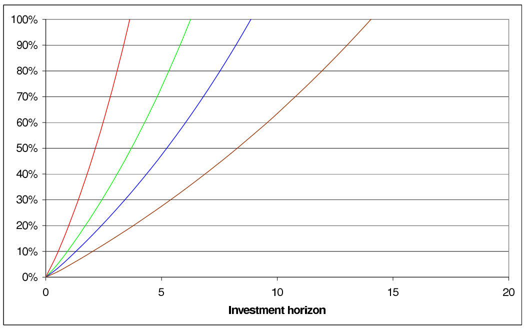
Figure 3: **The loss from missing the optimal portfolio by some percentage, (ϕ_S, ϕ_B).** The short term interest rate is assumed to follow the model by Vasicek. The parameter values are as follows: $\bar{r} = 0.01$, $\kappa = 0.63$, $\sigma_r = 0.03$, $\sigma_S = 0.20$, $\Lambda_1 = 0.35$, $\Lambda_2 = 0.21$, and $\rho = 0.20$. The investor is assumed to have a risk aversion of $\gamma = 6$.



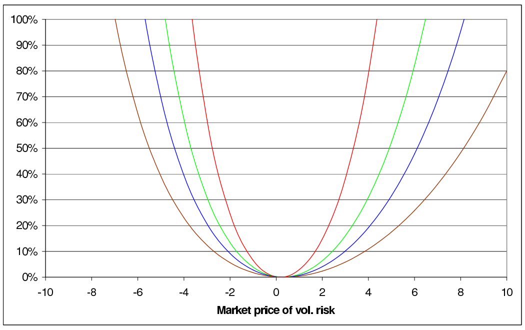
(a) The loss from not hedging the stochastic variations in the stock volatility, $\Lambda_2 = -6$.



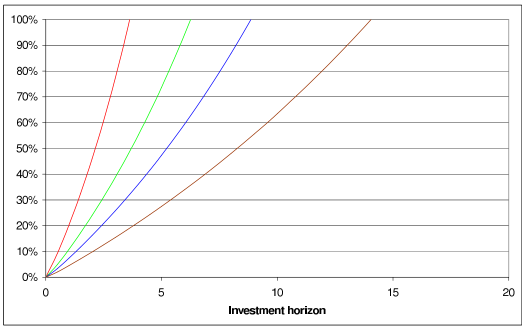
(b) The loss from not hedging the stochastic variations in the stock volatility, $T - t = 10$.



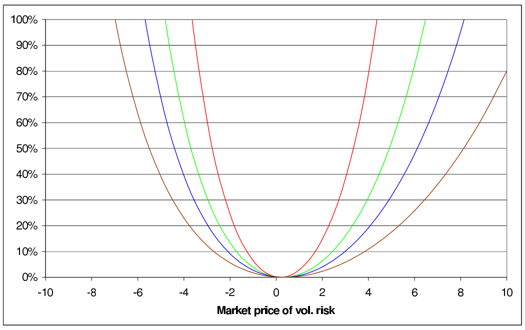
(c) The loss from investing as if the volatility of the stock is constant, $\Lambda_2 = -6$.



(d) The loss from investing as if the volatility of the stock is constant, , $T-t = 10$.



(e) The loss from being restricted to trade in stocks only, $\Lambda_2 = -6$.



(f) The loss from being restricted to trade in stocks only, $T - t = 10$.

Figure 4: **The loss from not following the optimal strategy as a function of the investors investment horizon and the market price volatility risk, respectively.** The stock price is assumed to follow the model by Heston. The parameter values are as follows: $\bar{v} = 0.0169$, $\kappa = 5$, $\sigma_v = 0.25$, $\Lambda_1 = 4$, and $\rho = -0.40$. The two figures in the top correspond to Scenario i), the two figures in the middle correspond to Scenario ii), while the two figures in the bottom correspond to Scenario iii). The red lines corresponds to an investor with $\gamma = 2$, the green lines correspond to an investor with $\gamma = 4$, the blue lines correspond to an investor with $\gamma = 6$, and the brown lines correspond to an investor with $\gamma = 10$.