Estimating Asset Correlations From Stock Prices or Default Rates – Which Method is Superior?

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Abstract

This paper sets out to help explain a result from previous empirical research, finding that estimates of asset correlations based on equity prices tend to be considerably higher than correlation estimates from default rates. Both data sources are used in the industry and the correlation estimates are a key factor driving the risk measures of credit risk in a loan portfolio. Therefore, sorting out this empirical puzzle is highly important for credit risk modelling.

Keywords: Asset correlation, single risk factor model, small sample properties, structural model, Basel II

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1. Introduction

This paper sets out to help explain the empirical puzzle, why estimates of asset correlations based on equity prices tend to be considerably higher than correlation estimates from default rates. Since correlation estimates arguably constitute the most important factor driving the credit risk of a loan portfolio, sorting out the empirical puzzle is highly important for credit risk modelling.

We explore the hypothesis that the empirical puzzle is created by the use of different data sets which require also different estimation methodologies. For this purpose we compare the accuracy of estimating asset correlations from default rates and from stock prices in a clinical simulation study. This setup avoids typical limitations of comparing asset correlation estimates from historical default rates and equity returns. More precisely, the sample of firms for which stock prices are available is typically too small for a meaningful comparison with the corresponding default rates because default events are rare. Even if default rates are available for a sufficiently large sample of firms, this sample typically includes non-listed firms which precludes estimating their asset correlation from equity returns. In a simulation study both equity prices and default rates are instead generated consistently, i. e. by the same data generating process (*DGP*).

Our results may also give guidance in which circumstances equity returns and in which default rates are more appropriate for the estimation of asset correlations. Finding that differences in estimation methodologies cannot explain the variation in the correlation estimates would suggest that the previously observed differences are instead caused by other reasons, for example, by sample mismatches.

It is not obvious from the outset if the estimation from stock prices or from default rates is more efficient. On the one hand defaults are rare events which require a longer time interval between two observation dates compared to equity data which are available on a daily basis. As a consequence, time series of default rates are based on yearly counted defaults and often contain not more than ten to twenty observations. As they cover a relatively long time period, they are vulnerable to regime shifts, for example structural breaks in the evolution of the economy or the introduction of a new bankruptcy code. On the other hand, trading activities can generate additional noise in the stock prices. In this case, correlation estimates based on market prices may be perturbed by factors unrelated to credit risk, for example a sudden drop in the demand or the supply of assets.

The key advantage of a simulation study is the possibility to estimate correlations from stock prices and default rates *consistently* since both time series are generated by the same DGP, based on the same model. In order to avoid an inadvertently preferential treatment of stock returns, perturbations of stock prices from factors unrelated to credit risk also need to be accounted for. More precisely, we introduce stochastic, mean-reverting correlations as model error in the stock returns while still employing an estimation methodology based on a Merton-type model that assumes constant correlations. Although this model error affects also the estimation from default rates, we expect that its effect is smaller than for equity prices. Therefore, this procedure should provide a more realistic comparison between the correlation estimation from default rates and from stock prices.

For the estimation of asset correlations from stock prices we employ the usual estimation methodologies from the literature. In the case of correlation estimation from default rates, various methodologies have been applied in the literature. The most general approach would be to use model-free estimation techniques, employed first by Lucas (1995) and later refined by De Servigny and Renault (2002). However, it has been noted, for example by Gordy and Heitfield (2000), that a model-based estimation may provide more efficient estimates, given the model describes the true DGP well enough. Therefore, we employ the asymptotic single risk factor (ASRF) model[4] which is consistent with a Merton-type model and which has gained great popularity as theoretic foundation of the internal ratings based approach in the Basel II framework.[5]

Grundke (2007) includes a comprehensive overview of the numerous empirical work on the estimation of asset correlations, which has so far produced quite diverse results. De Servigny and Renault (2002) compare sample default correlations and default correlations inferred from a factor model, which uses empirical equity correlations as proxies for the asset correlation. They find that the link between both default correlation estimates, although positive as expected, is rather weak, which adds to the puzzle that we intend to explain. The result is robust against replacing the Gaussian copula by a t–copula. Furthermore, the volatility of the riskless interest rates appears not to have a direct impact on asset correlations. The authors neither give an explanation for the weak link between default-rate based and stock price-based correlation estimates nor do they test the assumption that equity correlations are good proxies for asset correlations. Roesch (2003) reports to our knowledge the lowest estimates of asset correlations which are estimated from default frequencies of German corporates in a factor model and vary between 0.5% and 3.5%, dependent on the business sector.[6] Duellmann and Scheule (2003) employ a similar sample of German firms but differentiated between buckets characterized by size and default probability. The correlation estimates for the buckets considerably vary between one and fourteen percent. Dietsch and Petey (2004) use samples of

[4]See Gordy (2001).

[5]See Basel Committee on Banking Supervision (2005). The ASRF model provides under certain assumptions an asymptotic justification that capital charges for single exposures are portfolio invariant as they add up to the capital charge of the total portfolio. For the purpose of estimating asset correlations we do not need this asymptotic argument but we still refer to the model as the ASRF model because it has become widely known under this name.

[6]Modelling the (unconditional) default probabilities as time-varying and driven by macroeconomic factors may have contributed to the relatively low correlation estimates.

French and German firms and obtain results which are between those of Roesch and those of Duellmann and Scheule.

Lopez (2002) instead estimates asset correlations from stock prices. For this purpose he employs a version of the ASRF model which is calibrated to the multi-factor model of the KMV Portfolio Manager software. His correlation estimates for a sample of medium and large US corporates lie between 10 and 26 percent.[7] In the Basel II framework the asset correlation parameter depends on the probability of default (PD) and for medium-size corporate borrowers also on firm size. Depending on the PD and neglecting the regulatory size adjustment, the asset correlation varies between twelve and twenty four percent. Although the asset correlation is a supervisory set parameter, it was originally calibrated to the economic capital of credit portfolios of large international banks which motivates their inclusion in the list of previous empirical results. In summary the above mentioned studies show that correlation estimates vary considerably. They are usually higher if they are based on stock prices than if they are based on default-rates. In this paper we will explore possible explanations for this result.

The paper is organized as follows. In Section 2 we present the Merton-type credit risk model together with the DGP both for the stock prices as well as for the time series of default rates. The model and the DGP form the basis of our simulation study. Section 3 comprises the estimation from stock prices and Section 4 from default rates. The simulation results including a comparison of the estimation methodologies and an analysis of a model error in the form of stochastic asset correlations are presented in Section 5. Section 6 summarizes and concludes.

2. The Simulation Model

Selecting the Merton Model as the underlying model of the DGP is motivated by the fact that it provides a structural link between default events and stock price returns. More specificly we use a multi-firm extension of the Merton (1974) model with N identical firms. Furthermore, this extension is consistent with the ASRF model.[8] Later, the model is perturbed by a model error due to stochastic correlations in the DGP.

[7]This asset correlation levels are confirmed by Zeng and Zhang (2001), also estimating from KMV data.

[8]See Pitts (2004).

2.1. Asset Value Model and Data Generating Process

In the following we consider a portfolio of N homogenous firms. In the Merton (1974) model the asset value of every firm i is assumed to follow under the physical measure $\mathbb{P}$ a geometric Brownian motion of the form

$$dV_{i,t} = \mu V_{i,t}\, dt + \sigma V_{i,t}\, dW_{i,t}, \tag{1}$$

where $V_{i,t}$ denotes the asset value at time t of firm i, μ the drift and σ the volatility parameter of the stochastic process. Since we assume all firms to be homogenous, μ, σ and the initial firm value $V_{i,0} = V_0$ are identical for every firm. In order to capture dependencies between firms, the Brownian motion $W_{i,t}$ is decomposed into two independent Brownian motions, X_t and $B_{i,t}$:

$$dW_{i,t} = \sqrt{\rho}\, dX_t + \sqrt{1-\rho}\, dB_{i,t}. \tag{2}$$

Equation (2) gives the standard representation of a single factor model and explains why the parameter ρ is commonly referred to as the asset correlation. Note that the model assumes that the asset correlation ρ is the same for all pairs of firms. This assumption is typical for empirical studies as it allows this parameter to be estimated from a cross section of firms.

Simulating the time-continuous asset value process requires an appropriate discretisation. The common Euler scheme only gives good numerical results if the drift of the diffusion coefficients is constant. Since we allow later on for stochastic correlations, the use of a higher order scheme is recommended.[9] Starting with constant correlation, the asset value of firm i at time $t + \Delta t$ is defined as follows:

$$V_{i,t+\Delta t} = V_{i,t} + V_{i,t}(r + \sigma\lambda)\Delta t + V_{i,t}\, \sigma \sqrt{\Delta t}\, W_{i,t} + V_{i,t} \frac{1}{2}\sigma^2 \Delta t (W_{i,t}^2 - 1), \tag{3}$$
$$\text{with } W_{i,t} = \sqrt{\rho} X_t + \sqrt{1-\rho} B_{i,t}.$$

In Merton-type models, the equity value $E_i(V_{i,t}, h)$ of firm i at time t represents a call option on the firm's asset value $V_{i,t}$ with the time to maturity h. It is given by the well-known Black and Scholes (1973) formula:

$$E_i(V_{i,t}, h) \;=\; V_{i,t}\, \Phi(d_1) - e^{-r\,h} D\, \Phi(d_2) \tag{4}$$

$$\text{with } d_1 = \frac{\log\left(\frac{V_{i,t}}{D}\right) + (r + \frac{1}{2}\sigma^2)\, h}{\sigma\sqrt{h}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{h}.$$

[9] In Kloeden and Platen (1992), chapter 10, the Milshtein scheme is recommended for short time intervals as it increases the order of strong convergence from 0.5 in the Euler scheme to 1. The increase in convergence is loosely spoken caused by accounting for the second order term of the Itô-Taylor Expansion.

While the asset values are simulated from the DGP given by equation (3), the corresponding equity values, which are the basis of the correlation estimation, are inferred from equation (4).

Equation (4) requires that the riskless short-term interest rate r is deterministic. This implicit assumption can be motivated *inter alia* by the findings of De Servigny and Renault (2002) that interest rate volatility does not significantly affect asset correlations, the estimation of which is the purpose of our study. The assumption of a constant risk horizon h and a constant debt value D is justified as the estimation of asset correlations from stock prices does not depend on h and D, given that debt is deterministic.

2.2. Simulation of Default Events

The default rates need to be simulated in line with the DGP from the previous subsection. For this purpose, they are based on a portfolio of N borrowers whose asset values follow the DGP given by equation (3). Borrower-dependent PDs can negatively affect correlation estimates as standard models assume some kind of PD homogeneity across borrowers. In order to study the best case in terms of estimation accuracy, all firms in the portfolio are assigned the same PD.

Since borrowers can only either default or survive, holding the set of borrowers fixed introduces a survivorship bias. More precisely, the credit quality of the borrowers in the sample slowly increases over time as the borrowers with a negative evolution of their asset values drop out which also causes the sample size to decline.[10] In order to control for these effects, we do not use a fixed set of borrowers but assume a homogeneous portfolio in which borrowers are reassigned every year to rating classes defined by fixed PDs. This procedure is implemented in the simulations by setting asset values and portfolio size back to the original values V_0 and N from the beginning of the first year.

The number of defaulted firms is given in every year by the number of firms of which the asset values falls below the outstanding debt value D at the end of the year. The default probability PD at time t of a default at time $t+h$ given the information about the firm value for each firm in the homogeneous portfolio is consistent with the Merton (1974) model and given by

$$PD = \mathbb{P}(V_{i,t+h} < D) = \mathbb{P}\left(V_{i,t}\ \exp\left[(\mu - \frac{\sigma^2}{2}) \cdot h + \sigma(W_{i,t+h} - W_{i,t})\right] < D\right),$$

[10]See Duan et al. (2003).

which leads to

$$PD = \mathbb{P}\left(\frac{W_{i,t+h} - W_{i,t}}{\sqrt{h}} < \frac{\log\left(\frac{D}{V_{i,t}}\right) - (\mu - \frac{\sigma^2}{2})h}{\sigma\sqrt{h}}\right) = \Phi\left(\frac{c}{\sqrt{h}}\right) \tag{5}$$

with

$$c = \frac{\log\left(\frac{D}{V_{i,0}}\right) - (\mu - \frac{\sigma^2}{2})h}{\sigma}. \tag{6}$$

Following standard procedure, the risk horizon h is in our simulation constant and set to one year. As the one–year default probability of each firm equals the probability that the asset return $W_{i,t+h} - W_{i,t}$ of the period with length h falls below the distance-to-default c, counting these events in every simulation run provides the numerator of the default rate of the portfolio. Note that with the static risk horizon h, cases in which the asset value falls below the default threshold before the end of the period but the borrower is "cured" before the year end, are not counted as defaults. This is a restriction of the standard implementation of the Merton model which we copy in order to be consistent with empirical work on correlation estimation.

2.3. Stochastic Asset Correlation

Comparing estimates of asset correlations from equity prices and default rates may be considered as giving an undue preference to the first method. The reason is it uses much more observations since stock prices are available with a higher frequency than default rates. In order to balance this effect we introduce a misspecification that is likely to appear also in real equity prices but which affects the estimation from default rates less.

For this purpose we allow for stochastic asset correlations. The results can also contribute to answer the question how fluctuations in asset correlations over time affect the estimation accuracy given that their stochastic nature is not accounted for in the estimation method.

Stochastic asset correlations are not the only way to produce estimation noise and – if systematic – to influences default correlations. Stochastic volatility of asset values offers an alternative approach which can also be motivated by empirical findings in stock markets.[11] In this case, equation (4) which relates asset values to equity values no longer holds which implies a considerable technical burden for the simulation analysis to generate equity values. Therefore, we prefer stochastic asset correlations which do not affect (4).

[11] See Bakshi et al. (1997).

The case of stochastic asset correlations is implemented as follows. The deterministic asset correlation ρ in equation (2) is replaced by a stochastic variable ρ_t such that the stochastic innovations $dZ_{i,t}$ are given by

$$dW_{i,t} = \sqrt{\rho_t}\, dX_t + \sqrt{1-\rho_t}\, dB_{i,t}. \tag{7}$$

Consistency with the case of a deterministic correlation ρ suggests that the long-run mean of the stochastic correlation equals ρ. For this purpose, we impose an Ornstein-Uhlenbeck process for the stochastic asset correlation with long term mean $\theta = \rho$. Given that asset correlations – contrary to interest rates – are not necessarily positive, we assume the following Vasicek process under the physical measure $\mathbb{P}$:

$$d\rho_t = \kappa(\rho - \rho_t)dt + \sigma_\rho dZ_t. \tag{8}$$

The parameter κ denotes the mean reversion parameter and σ_ρ the volatility of the mean reversion process. The stochastic innovations dX_t, $dB_{i,t}$ and dZ_t are at any time t pairwise independent increments of Brownian motions. Following the Milshtein scheme given in equation (3), the stochastic process of the asset correlation in discrete form is given by

$$\rho_{t+\Delta t} = \rho_t + \kappa(\rho - \rho_t)\Delta t + \sigma_\rho \sqrt{\Delta t}\, Z_t. \tag{9}$$

3. Correlation Estimation from Stock Prices

Asset correlations can be estimated from stock price returns or from default rates. The first approach is presented in the following section, the second in Section 4.

The estimation from stock prices is based on the structural model described in Section 2. We differentiate between a *direct* estimation method which estimates asset correlations directly from equity returns and an *indirect* and conceptually better founded method, which requires in the first step to estimate the asset returns from which in the second step asset correlations are estimated.

Following Duan et al. (2003), asset correlations are approximated in the first, direct method by pairwise equity correlations, which are estimated from stock returns. Estimating asset correlations directly from equity prices is quite common in empirical studies and can be motivated by the equivalence of using equity and asset values in the limiting case when the length of the time horizon approaches zero. Using equity returns to estimate asset correlations has nevertheless been often criticized because it ignores the leverage in the capital structure. This is, however, considered a lesser concern for high-grade borrowers.[12] Since correlation estimation from market prices

[12] See Mashal et al. (2003).

competes with an estimation from default rates particularly for less risky borrowers for which default events are even more scarce, we focus exactly on this segment for which the leverage argument should be a lesser concern.

The second, indirect estimation method consists of two steps. In the first step, the asset values are estimated from stock prices and liabilities and transformed into log-returns. In the second step, the asset correlations are estimated from the asset returns of the first step. This procedure was also employed by Lopez (2002), Pitts (2004) and Duellmann et al. (2007) who obtained the asset values of non–financial companies from the MKMV model. We employ instead the classic Merton (1974) model but follow the MKMV method of estimating asset values from equity prices and balance sheet information. As given in Bohn and Crosbie (2003), MKMV uses a two step algorithm with m iteration steps to estimate the asset value and its standard deviation σ from a time series of equity values.

(1.) Set $m = 0$ and use $\hat{\sigma}^{(0)} = 0.3$ as a starting value, with r, T, D given exogenously.

(2.) Compute $\hat{V}_{i,t}^{(m+1)} = BS^{-1}(E_{i,t}, \hat{\sigma}^{(m)})$ from the Black/Scholes formula BS(.), given by equation (4), for all t.

(3.) Compute the standard deviation $\hat{\sigma}^{(m+1)}$ of the logarithmic asset value changes.

(4.) Stop if $|\hat{\sigma}^{(m+1)} - \hat{\sigma}^{(m)}| \leq \epsilon$, else increase m by one and return to (2.).

The algorithm produces Maximum Likelihood (ML) estimates of the volatility and the asset value[13] and is superior to the method employed before by Jones et al. (1984).[14]

After having estimated the asset values in the first step and following Pitts (2004), a random effects model[15] is employed in the second step to estimate the asset correlations. Besides the time-dependent random effect $\tilde{X}_t$ and the idiosyncratic disturbance term $\tilde{B}_{i,t}$, the only explanatory variable in the parsimonious model is a firm-dependent intercept a_i for every firm:

$$\Delta \log[V_{i,t}] = a_i + \sigma_X \tilde{X}_t + \sigma_B \tilde{B}_{i,t}, \text{ where } \tilde{X}_t, \tilde{B}_{i,t} \sim \text{iid. } \mathcal{N}(0,1). \tag{10}$$

Since we assume a homogenous portfolio, a_i is constant across firms. Pitts (2004) uses the ML estimates from the random effects model to estimate the asset correlation by taking into account that $\hat{\sigma}_X^2$ corresponds to $\sigma^2 \Delta t \rho$ and $\hat{\sigma}_B^2$ to $\sigma^2 \Delta t(1-\rho)$ in equation (3). The asset correlation estimator $\hat{\rho}$ is then given by

$$\hat{\rho} = \frac{\hat{\sigma}_X^2}{\hat{\sigma}_X^2 + \hat{\sigma}_B^2}. \tag{11}$$

[13]See Duan et al. (2004).

[14]See Duan et al. (2003) for a discussion of the ML estimation in this context.

[15]See Hsiao (2003) for further details on random effects models.

Following the common procedure that asset correlations are directly estimated from equity returns, we apply the random effects estimator as well directly to stock price returns.

As an alternative to the random effects estimator, asset correlations are additionally estimated by the mean of the pair-wise correlations of all firms:

$$\hat{\rho}_{PW} = \frac{2}{N(N-1)} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} corr[\log[\vec{E}_i], \log[\vec{E}_j]], \tag{12}$$

where $\vec{E}_i$ denotes the vector that collects the equity returns of firm i over time.

Summarizing, we employ two approaches for the estimation of asset correlations, both for those estimated from equity returns and for those estimated from asset returns (the latter being inferred before from equity returns). The first approach is based on the sample correlation of a time series of equity or asset returns and the second approach is based on the random effects model.

Finally, we replace the constant asset correlation by a stochastic variable which follows an Ornstein-Uhlenbeck process. Since we investigate this case as an example of model risk and a robustness check, we do not account for this feature in the parameter estimation.

4. Asset Correlation Estimation from Default Rates

4.1. Estimation of Asset Correlation Within the ASRF Framework

In Section 2.2 we have outlined how default events in the Merton Model can be simulated to be consistent with the ASRF model. This section describes the ML and the Method-of-Moments (MM) methodologies which we employ to estimate asset correlations from sampled default rates.

In principle, default correlations can be estimated directly from observed default events as in Lucas (1995) and then reverted back into asset correlations. However, to improve estimation efficiency we make use of the model structure in the estimation procedure. From equation (3) follows for the probability of default:

$$\begin{aligned} PD &= \mathbb{P}\left(\frac{W_{i,t+h} - W_{i,t}}{\sqrt{h}} < \frac{c}{\sqrt{h}}\right) \\ &= \mathbb{P}\left(\sqrt{\rho}\,\frac{\overbrace{(X_{t+h} - X_t)}^{=\Delta X_t}}{\sqrt{h}} + \sqrt{1-\rho}\,\frac{\overbrace{(B_{i,t+h} - B_{i,t})}^{=\Delta B_{i,t}}}{\sqrt{h}} < \frac{c}{\sqrt{h}}\right). \end{aligned} \tag{13}$$

The law of large numbers implies that the default rate of the time period from t to $t+h$ converges for large portfolios and long time series to the conditional default probability

$$\mathbb{P}\left(\frac{W_{i,t+h}-W_{i,t}}{\sqrt{h}} < \frac{c}{\sqrt{h}}\middle| \Delta X_t = x\right) = \mathbb{P}\left(\frac{\Delta B_{i,t}}{\sqrt{h}} < \frac{c-\sqrt{\rho}x}{\sqrt{h}\sqrt{1-\rho}}\right) = g(x;\rho,c), \quad (14)$$

which is obviously a function depending on the realization x given ρ and c.

Since $\Delta B_{i,t}$ is an increment of a Brownian motion, it follows that $\frac{\Delta B_{i,t}}{\sqrt{h}}$ is standard normal distributed. Therefore, we get for the conditional default probability

$$g(x;\rho,c) = \Phi\left(\frac{c-\sqrt{\rho}x}{\sqrt{h}\sqrt{1-\rho}}\right).$$

The corresponding density of the default frequency DF_t is given in the limit by

$$f(DF_t;\rho,PD) = \frac{\sqrt{1-\rho}}{\rho}\exp\left(-\frac{(1-2\rho)\delta_t^2 - 2\sqrt{1-\rho}\delta_t\gamma + \gamma^2}{2\rho}\right), \quad (15)$$

$$\text{where } \delta_t = \Phi^{-1}(DF_t) \text{ and } \gamma = \Phi^{-1}(PD). \quad (16)$$

Note that the time intervals for which the series of default frequencies $(DF_t)_{t=1,\ldots,T}$ is computed do not overlap.

Maximizing the log-likelihood function

$$LL(PD,\rho,DF_1,\ldots,DF_T) = \sum_{t=1}^{T} log\left[f(DF_t;\rho,PD)\right] \quad (17)$$

leads to the following ML estimator:[16]

$$\hat{\rho} = \frac{\frac{m_2}{T} - \frac{m_1^2}{T^2}}{1 + \frac{m_2}{T} - \frac{m_1^2}{T^2}}$$

$$\widehat{PD} = \Phi(T^{-1}\sqrt{1-\hat{\rho}}\, m_1)$$

$$m_1 = \sum_{t=1}^{T}\delta_t, \qquad m_2 = \sum_{t=1}^{T}(\delta_t)^2. \quad (18)$$

The estimator is called the Asymptotic Maximum Likelihood (AML) estimator as it requires a large bucket of firms and a long time series.

Besides the ML methodology, Gordy (2000) also employs an MM estimator for the expected default rate $\bar{p}$ and the asset correlation ρ. It is based on matching the

[16]The standard errors of the estimator are given in the Appendix of Duellmann and Trapp (2004).

first and second moment of $g(X)$ with the empirical first and second moment of the default rates:[17]

$$\mathbb{E}[g(X)] = \bar{p} \tag{19}$$

$$\mathbb{E}[g(X)^2] = \Phi_2(\Phi^{-1}(\bar{p}), \Phi^{-1}(\bar{p}), \rho), \tag{20}$$

where $\Phi_2(.)$ denotes the cumulative bivariate Gaussian distribution function. The left hand side of (20) is computed by the sample variance of the default frequencies and ρ is backed out numerically. We refer to this estimation as the Asymptotic Method-of-Moments (AMM) estimator.[18]

5. Comparative Static Analysis

5.1. Simulation Setup and Performance Measures

The different methodologies to estimate asset correlations are analyzed in a comparative static analysis. Its setup is designed to be fair to both estimation methodologies while minimizing the computational workload to the extent possible. The asset correlation estimators are applied to the same homogeneous portfolio of firms in each simulation run. Stock prices and default rates are generated consistently. In order to achieve generality of the results, the model parameters of the simulated portfolio, namely the asset correlation ρ, the probability of default PD, the number of borrowers or portfolio size N and the length of the time series Y are varied in the comparative static analysis. The parameter values of the DGP are shown in Table 1.

The range of asset correlations is mainly motivated by the parameter values in the risk weight functions for wholesale credit exposures in Basel II and previous empirical results.[19]

The default probabilities cover rating grades from *BBB* to *B+*. As the accuracy of an estimation from default rates is expected to decline with lower PDs due to the scarcity of default events, borrowers with higher ratings than *BBB* should only be considered if this estimation method still performs well in this credit category.

The minimum of 5 years for the length of the time series of default rates is inspired by the regulatory minimum requirements in Basel II for the estimation of PDs. The maximum of 40 years is close to the maximum length that is currently available in the industry, e. g. in rating agency databases.

[17] See Gordy (2000).

[18] For further reference see Appendix C of Gordy (2000).

[19] See Basel Committee on Banking Supervision (2005), para 272 and for previous empirical results Section 1.

Table 1
Parameter Values of the Data Generating Process

This table shows the 24 parameter sets of asset correlation, length of time series and default probability which were used for the data generating processes.

		Values	Number of values
Asset correlation	ρ	10% and 25%	2
Length of time series	Y	5, 10, 20 and 40 years	4
Probability of default	PD	0.5%, 1.2% and 2.03%	3
			24 combinations

For each of the possible twenty-four different parameter combinations, 10,000 simulations are run. The parameters of the asset value process which are the same for all simulation runs are $V_0 = 1,000$, $\sigma = 0.3$, $\mu = 0.12$ and $r = 0.03$, which is constant for all maturities. The debt value D is a free parameter and calibrated to the desired PD by means of equation (5). The portfolio size N is set to 5,000 borrowers, which is enough for a meaningful estimation of default rates while still being realistic for portfolios of real banks.

Asset returns are simulated for weekly time intervals. Since defaults are rare events, the generated default rates are calculated on a yearly basis. In order to ensure consistency between default rates and stock prices, the yearly default rates are generated from the same asset returns. For this purpose, the weekly random asset returns $W_{i,t+\Delta t} - W_{i,t}$ are summed up in every year and divided by $\sqrt{52}$ to obtain the standardized yearly asset returns. The default rates are computed by counting how often negative yearly asset returns reach or exceed the distance-to-default, given by equation (6).

It turns out that the performance of an estimation from stock prices does not further improve if weekly stock prices are simulated for more than fifty firms and over more than two years. Therefore, we employ for the correlation estimation from market prices only a subset of fifty firms and two years of market prices, notwithstanding that we use the full sample of 5,000 firms and 40 years for the generation of default rates.

Finally, we allow for stochastic asset correlations in the DGP in order to account for model risk. This model error affects only the DGP but is not accounted for in the parameter estimation. This “robustness check” requires specifying the mean reversion parameter κ and the volatility σ_ρ in equation (8). The parameters of the stochastic asset correlation process are set to $\rho \in \{0.1, 0.25\}$, $\kappa = 1$ and $\sigma_\rho = 0.085$. The values of the mean ρ are given by the constant asset correlations used before.

The parameters κ and σ_ρ are set such that the asymptotic 90% confidence interval of the stochastic correlation is given by $\rho \pm 10\%$.[20] The value of one for κ corresponds with a half-life time of 0.7 years.[21]

The following two indicators are used to measure the estimation performance of the two asset correlation estimators based on stock prices and also for the three estimators based on default rates in each of the 24 parameter settings:

Bias $= \frac{1}{S}\sum_{s=1}^{S}\hat{\rho}_s - \rho,$

Root mean squared error: RMSE $= \sqrt{\frac{1}{S}\sum_{s=1}^{S}(\hat{\rho}_s - \rho)^2}.$

The root mean squared error (RMSE) is a hybrid measure which brings together the bias and the standard deviation of the estimator.

The performance analysis of the asset correlation estimators is structured as follows: The estimators based on default rates are analysed in subsection (5.2), followed by the estimators which require stock prices in subsection (5.3). In subsection (5.4) the robustness of our results is explored in the presence of a model error introduced by stochastic asset correlations.

5.2. Performance Results of Estimators Based on Default Rates

The performance measures bias and RMSE of estimations from default rates are listed in Table 2, given an asset correlation of 10%. The PD varies between 0.5% and 2.03% and the sample length for yearly default rates varies between 5 and 40 years. For every PD and sample length, bias and RMSE are given for the AML and the AMM estimator.

Considering first a sample length of 40 years, the bias is less than a percentage point with a varying sign, dependent on the PD and the estimation method. For shorter sample lengths the bias becomes increasingly negative, indicating that the estimator is biased downwards in small samples. For the shortest time series of only 5 years, the bias increases up to three percentage points (for the AMM estimator) which is quite substantial given the true asset correlation is only 10%. The results indicate that the correlation estimators based on default rates are substantially downward biased in small samples.

The RMSE also increases if the sample length is gradually reduced from its maximum value of 40 years. An RMSE that is roughly 50% of the true asset correlation value

[20]The confidence level is obtained from the asymptotic distribution of ρ_t for $t \to \infty$ which is given by $\mathcal{N}\left(\rho, \frac{\sigma_\rho^2}{2\kappa}\right)$.

[21]The half-life time is given by the formula $-\frac{ln(0.5)}{\kappa}$.

Table 2
Bias and RMSE of the AML and AMM Estimator With a True Asset Correlation of 10%

This table shows the bias and RMSE of asset correlation estimates for the Asymptotic Maximum Likelihood (AML) method and the Asymptotic Method of Moments (AMM), separately for three default probabilities (PD) and four sample lengths Y of yearly default rates.

	$PD = 0.5\%$		$PD = 1.2\%$		$PD = 2.03\%$	
	AML	AMM	AML	AMM	AML	AMM
Bias						
$Y = 5$	-.0137	-.0290	-.0167	-.0246	-.0191	-.0214
$Y = 10$	-.0005	-.0179	-.0047	-.0144	-.0077	-.0126
$Y = 20$	.0048	-.0103	.0003	-.0081	-.0016	-.0064
$Y = 40$	.0072	-.0055	.0034	-.0037	-.0011	-.0029
RMSE						
$Y = 5$	.0554	.0512	.0566	.0533	.0554	.0548
$Y = 10$	.0414	.0439	.0417	.0450	.0407	.0452
$Y = 20$	.03	.0374	.0301	.037	.0295	.0364
$Y = 40$	.0224	.0309	.0221	.0291	.0215	.0281

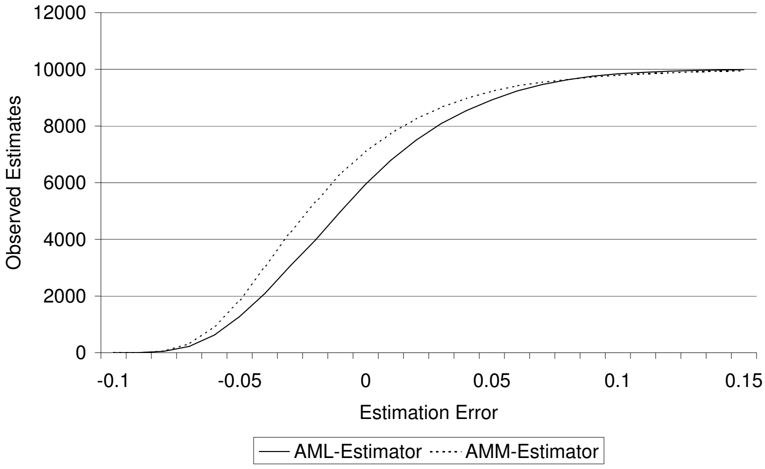
for the highest PD of 2.03% reveals the limitations if asset correlations need to be estimated from default rates.

Comparing the two estimation methods, the AMM estimates are more strongly biased downwards than the AML estimates. The lower downward bias of the AML method becomes more visible for shorter sample lengths and for lower PDs. For the RMSE as performance measure, the differences are ambiguous. Although for sample lengths of 10–40 years the AML estimator has the lower RMSE, the difference is smaller in relative terms than in the case of the bias. Furthermore, in the 5-years case, the AMM method performs slightly better in terms of RMSE than the AML estimator but the difference is arguably immaterial in practice.

Figure 1 shows the distribution function of the estimation errors for the AML and the AMM estimator, given an asset correlation of 10%. It reveals that the downward bias is stronger with the AMM estimator. Furthermore, the distribution of this estimator is much less symmetric and positively skewed. The last finding indicates that erroneously high correlation estimates occur more often with the AMM rather than the AML method. The superior performance of the AML estimator, in particular its lower downward bias, is plausible as it makes better use of the model structure.

Figure 1. Distribution of the Estimation Errors of the AML and the AMM Method

This figure shows the interpolated distribution function of errors in asset correlation estimates, both for the Asymptotic Maximum Likelihood (AML) method and the Asymptotic Method of Moments (AMM). The DGP is characterized by an asset correlation of 0.1, a time series of 10 yearly default rates, 5,000 homogenous borrowers and a PD of 1.2%.



In order to measure the impact of the level of the true correlation parameter, Figure 2 shows the sample density distribution of the estimation error of the AML estimator, given asset correlations of 10% and 25%. Both density distributions clearly show that the bias strongly depends on the value of the correlation parameter. For the higher correlation of 25%, the mean bias is stronger and estimates are much more scattered than for a correlation of 10%. These results signal a strong dependence of the estimation performance on the true asset correlation.

Table 3 describes the performance of both estimators, given an asset correlation of 25%. Compared with the results in Table 2 for a correlation of 10%, the downward bias strongly increases. Even for the longest sample length of 40 years the bias rises to 5 percentage points, also depending on the estimation method. For shorter time series the negative bias increases in absolute terms as expected and it is higher for lower PDs than for higher PDs. For the shortest time series of 5 years it can increase to 13 percentage points or around 50% of the true asset correlation. Again, the bias also depends on the estimation method, with the AML method coming out as superior in all parameter constellations.

The results for the bias also hold qualitatively if the RMSE is considered as performance indicator. The higher correlation increases the number of defaults, which could raise the expectation that the performance improves over Table 2. However, this is not the case as, for example for 5 years, the RMSE is still roughly 50% of the true correlation value.

Summarizing, we find that estimates from default rates are typically downward biased. This negative bias increases with shorter sample lengths, high correlations, lower PDs, and if the AMM-estimator is used instead of the AML-estimator. The level of the true correlation parameter also has a strong impact on the estimation performance. Increasing the asset correlation from 10% to 25% reveals not only a stronger downward bias, but also that the estimates are more dispersed. This finding confirms the need to consider not only the bias but also the RMSE when evaluating the small sample properties of the estimators.

5.3. Performance Results of Estimators Based on Stock Prices

As described in Section 3, we employ a direct and an indirect estimation method to infer asset correlations from market prices. The direct method uses equity returns whereas the indirect method requires first inferred asset returns from which asset correlations are estimated in a second step.

The two performance measures for the direct method, bias and RMSE, are shown in Table 4. We consider again three PDs and two asset correlation values. Furthermore, asset correlations are estimated by the mean of pairwise sample correlations of equity

Figure 2. Histogram of Estimation Errors if the AML Method is Applied to Default Rates, Given Asset Correlations of 0.1 and 0.25

This figure shows histograms of errors in asset correlation estimates for the Asymptotic Maximum Likelihood (AML) method, given asset correlations in the DGP of 0.1 in the first panel and 0.25 in the second panel. The DGP is characterized by a time series of 10 yearly default rates, 5,000 homogenous borrowers and a PD of 1.2%.

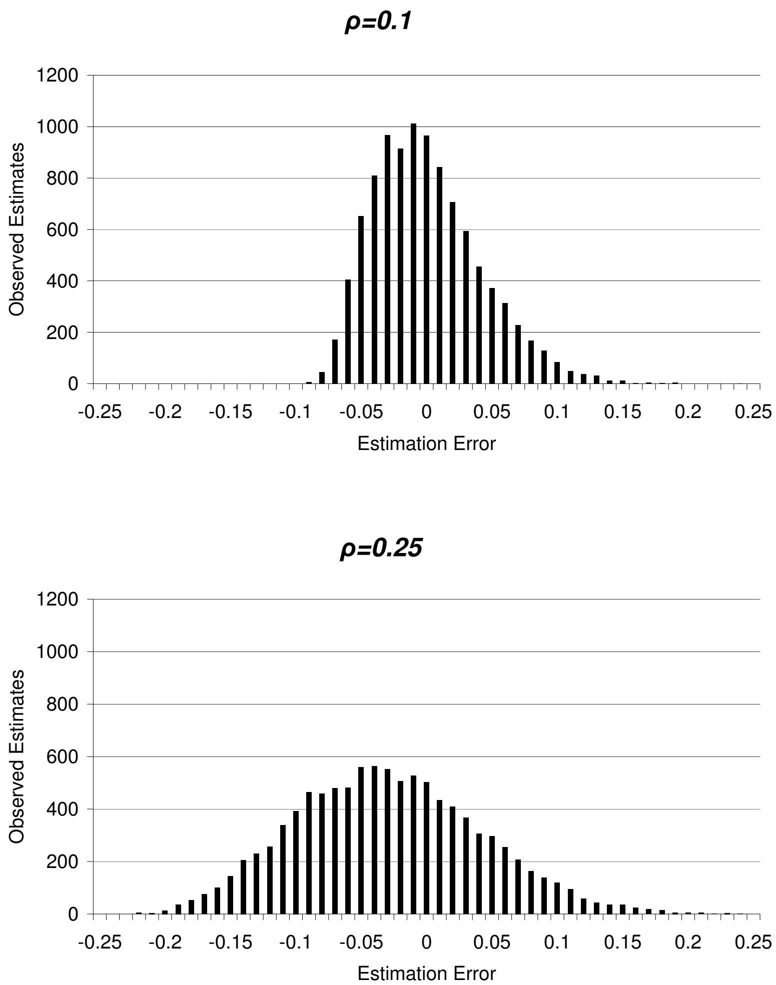


Table 3
Bias and RMSE of the AML- and AMM-Estimator With a True Asset Correlation of 25%

This table shows the bias and RMSE of asset correlation estimates for the Asymptotic Maximum Likelihood (AML) method and the Asymptotic Method of Moments (AMM), separately for three default probabilities (PD) and four sample lengths Y of yearly default rates.

	$PD = 0.5\%$		$PD = 1.2\%$		$PD = 2.03\%$	
	AML	AMM	AML	AMM	AML	AMM
Bias						
$Y = 5$	-.0891	-.1287	-.0623	-.1089	-.0528	-.0937
$Y = 10$	-.0683	-.0964	-.0345	-.0751	-.0276	-.0645
$Y = 20$	-.0564	-.0682	-.0228	-.0498	-.0142	-.0402
$Y = 40$	-.0504	-.0435	-.0168	-.0316	-.0067	-.0225
RMSE						
$Y = 5$	.1236	.1464	.1147	.1373	.1150	.1333
$Y = 10$	.0929	.1220	.0790	.1139	.0807	.1119
$Y = 20$	.0729	.1013	.0552	.0951	.0564	.0932
$Y = 40$	.0605	.0856	.0399	.0796	.0394	.0748

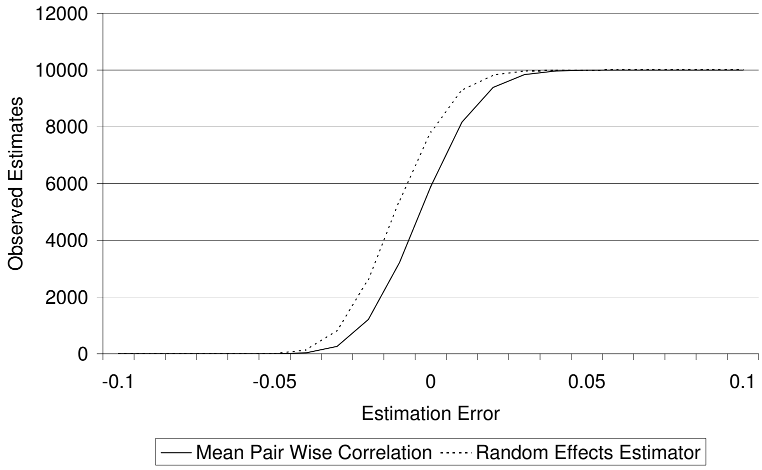
Table 4
Bias and RMSE of Asset Correlation Estimates as Pairwise Stock Price Correlations and Based on the Random Effects Model

This table shows the bias and RMSE of asset correlation estimates from stock returns. They are estimated by pairwise sample correlations (PW) and a random effects model (RE) based on six DGPs with three default probabilities (PD) and two asset correlation values ρ. The generated samples consist of two years of weakly stock returns of 50 firms.

$N = 50$		$PD = 0.005$		$PD = 0.012$		$PD = 0.0203$	
$T = 2y$		PW	RE	PW	RE	PW	RE
$\rho = 0.1$	Bias	-.0024	-.0094	-.0028	-.0106	-.0031	-.0113
	RMSE	.0150	.0171	.0149	.0175	.0150	.0180
$\rho = 0.25$	Bias	-.0059	-.0201	-.0059	-.0218	-.0065	-.0236
	RMSE	.0283	.0338	.0285	.0349	.0282	.0356

Figure 3. Distribution of Estimation Errors Based on Stock Prices

This figure shows the interpolated distribution function of errors in asset correlation estimates, both for pairwise sample correlations and a random effects model. The DGP is characterized by an asset correlation of 0.1, a time series of 2 years of weekly stock returns of 50 firms and a PD of 1.2%.



returns and from the random effects model. Comparing these two methods first, we find that the sample correlations are biased and have a lower RMSE for all three PDs and both asset correlation values. This observation is confirmed by Figure 3 which presents the cumulative distribution of estimation errors.

Although the mean sample correlation of equity returns is still downward biased, bias and RMSE are by far smaller than in the case of using default rates for the estimation of asset correlations. Consider, for example, the case of an asset correlation of 25% and a PD of 0.5%. According to Table 3, even with the longest sample length of 40 yearly default rates, the RMSE of the AML estimator is still 0.06% or double the RMSE of the correlation estimates based on equity returns in Table 4. The superior performance of the estimation from market prices also depends on the parameters of the DGP. It becomes *ceteris paribus* more pronounced with higher PDs and higher asset correlations.

Whereas Table 4 presents the estimators' performance for the direct method, Table 5

Table 5
Bias and RMSE of the Asset Correlation Estimates Based on Asset Returns.

This table shows the bias and RMSE of asset correlation estimates from asset returns. They are estimated by pairwise sample correlations (PW) and a random effects model (RE) based on six DGPs with three default probabilities (PD) and two asset correlation values ρ. The generated samples consist of 2 years of weakly stock returns of 50 firms.

$N = 50$		$PD = 0.005$		$PD = 0.012$		$PD = 0.0203$	
$T = 2y$		PW	RE	PW	RE	PW	RE
$\rho = 0.1$	Bias	-.0013	-.0005	-.0015	-.0006	-.0016	-.0007
	RMSE	.0148	.0149	.0148	.0148	.0149	.0149
$\rho = 0.25$	Bias	-.0030	-.0011	-.0037	-.0017	-.0031	-.0010
	RMSE	.0277	.0278	.0278	.0279	.0278	.0279

gives the corresponding performance indicators for the indirect estimation method, based on inferred asset returns. Comparing first the results of the sample correlation estimates with those of the random effects model, we find that the latter method is superior in terms of bias but that both methods are nearly indistinguishable in terms of RMSE. Given the small absolute values of the downward bias for both methods, the difference in the bias is immaterial.

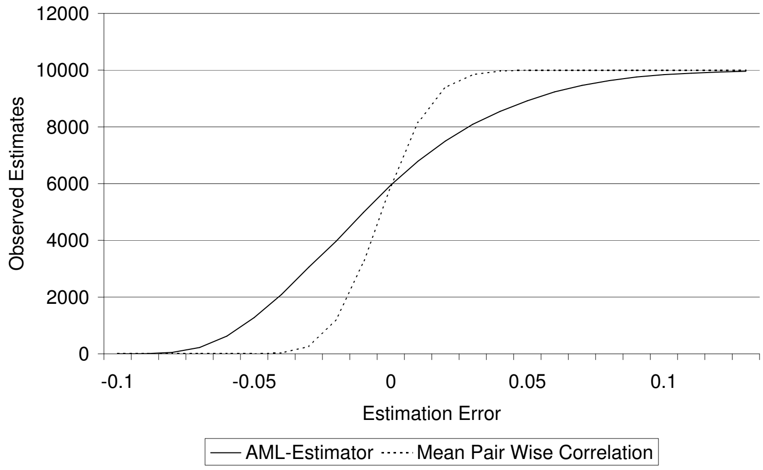
The similar performance of both indirect estimation methods based on asset returns contrasts with the results for the asset correlation estimation from equity returns, given in Table 5, where the sample correlations emerged as the superior estimation method.

Considering that the indirect method of using inferred asset returns instead of equity returns is the theoretically better founded method, its superior performance is plausible. The random effects model applied to the asset returns exploits the model structure of the DGP better than the other methods. Therefore, it should be expected to be at least as good as the other methods. If equity returns are used instead of asset returns the input data to the estimation are already misspecified. The consequence is that estimators are more robust against model errors if they pose less assumptions about the underlying model as, for example, in this case the mean sample correlation.

Finally, Figure 4 compares the distribution of estimation errors of the AML method based on default rates with the estimation based on the sample correlation of asset returns. It shows the substantially higher downward bias and also the fatter tails of

Figure 4. Distribution of Estimation Errors Based on Default Rates and From Asset Returns

This figure shows the interpolated distribution function of errors in asset correlation estimates, for the pairwise sample correlations and the AML estimator. The DGP is characterized by an asset correlation of 0.1, a time series of 10 years of default rates (2 years of weekly stock returns) for 5,000 firms (50 firms) and a PD of 1.2%.



the estimation errors, produced by the AML method.

Summarizing, we find that both, the direct estimation from equity returns and the indirect estimation from asset returns are superior to an estimation of asset correlations from default rates, both in terms of the bias and the RMSE. This holds even if a relatively short sample of two years of weekly equity returns is compared with a sample comprising 40 years of yearly default rate observations. Comparing the direct and the indirect estimation method we find that the indirect estimation is superior. The better performance depends also on the true parameters of the DGP. With increasing PDs and higher asset correlations, the out–performance of the direct estimation method becomes more visible.

These results hold in the absence of a model error, i. e. if the estimation is based on data generated from the model of the DGP. In the following section we explore, how robust they are against a model error, more specifically if asset correlations are no longer constant over time but follow an Ornstein-Uhlenbeck process.

5.4. Estimation Performance Under Stochastic Correlations

There is substantial evidence in the literature that asset correlations fluctuate over time.[22] If correlations follow a mean-reverting process with a sufficiently short half-life, the randomness may have a stronger effect on correlation estimates from stock prices as they rely on a much higher data frequency compared with correlation estimates from default rates. If this hypothesis proves to be true, the result from the previous section that correlation estimation from default rates is strictly preferable may no longer hold.

It can be argued that accounting only for the model error of a mean-reverting process of the asset correlations unduly benefits the estimation from default rates relative to the estimation from market prices. Since the required time series of default rates are typically long, estimation from default rates is arguably much more susceptible to structural brakes, introduced for example by changes in the legal framework, which are not considered. Given that the estimation from market prices has emerged as clearly superior in the case of a correctly specified model, it seems, however, justified to focus on a model error that affects mainly the already superior estimation method. In this case the results can be considered also as a robustness check for the superiority of the estimation from market prices.

Table 6 presents the correlation estimates, based on default rates, employing the AML and the AMM method. Since the asset correlation is stochastic, we assume an average correlation of 10% for the DGP to facilitate a comparison with results in the previous section for a constant asset correlation. For five and 10 years the results

[22] See, for example, Longin and Solnik (1995), Ang and Chen (2002) or Duellmann et al. (2007).

Table 6
Bias and RMSE of the AML and AMM Estimator With a Mean Asset Correlation of 10% and Stochastic Asset Correlations

This table shows the bias and RMSE of asset correlation estimates for the Asymptotic Maximum Likelihood (AML) method and the Asymptotic Method of Moments (AMM), separately for three default probabilities (PD) and four sample lengths Y of yearly default rate observations. The asset correlations of the DGP follow an Ornstein-Uhlenbeck process with mean 0.1.

	$PD = 0.5\%$		$PD = 1.2\%$		$PD = 2.03\%$	
	AML	AMM	AML	AMM	AML	AMM
Bias						
$Y = 5$	-.0128	-.0312	-.0145	-.0269	-.0161	-.0226
$Y = 10$	-.0005	-.0205	-.0005	-.0156	-.0036	-.0129
$Y = 20$	.0057	-.0123	.0016	-.0080	.0041	-.0053
$Y = 40$	.0095	-.0052	-.0021	-.0020	.0075	-.0009
RMSE						
$Y = 5$	.0600	.0551	.0644	.0579	.0649	.0615
$Y = 10$	.0454	.0497	.0504	.0524	.0502	.0547
$Y = 20$	.0340	.0445	.0379	.0462	.0382	.0470
$Y = 40$	.0260	.0388	.0288	.0400	.0287	.0380

are ambiguous but for 20 and 40 years the bias is higher in the case of stochastic correlations. The RMSE is always higher in this case but the difference is below one percentage point. The sensitivity of the estimation performance to PD and to the number of years is similar to the previous results in Table 2. The better performance if asset correlations are estimated from equity returns instead of from default rates is confirmed by Figure 5 for the case of stochastic correlations.

Table 7 corresponds with Table 6 but with an average asset correlation of 25% in the DGP. Although both bias and RMSE increase, the sensitivity to PD, the length of the time series and the estimation method (AMM or AML) is similar.

Summarizing, introducing mean-reverting asset correlations increases the RMSE if the estimation is based on default rates but not substantially. This result supports the hypothesis that a mean-reverting asset correlation has only a minor impact on the estimation performance. Estimation performance, measured by RMSE, substantially improves for longer time series, particularly for low correlations.

Table 8 shows bias and RMSE if pairwise asset correlations are estimated from stock

Figure 5. Distribution of Estimation Errors for Stochastic Asset Correlations Estimated From Default Rates and by Equity Return Correlations

This figure shows the interpolated distribution function of errors in asset correlations which were estimated from default rates with the Asymptotic Maximum Likelihood (AML) method and from stock returns by pairwise sample correlations. The DGP is characterized by an asset correlation of 0.1, a time series of 10 years of default rates (2 years of weekly stock returns) for 5,000 firms (50 firms) and a PD of 1.2%.

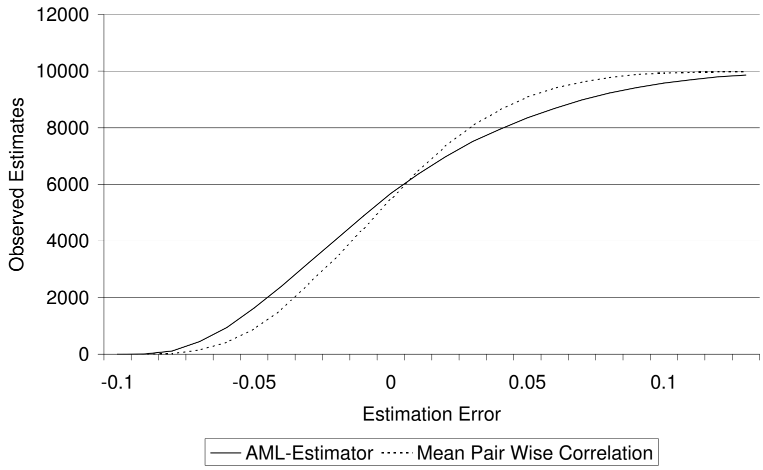


Table 7
Bias and RMSE of the AML and AMM Estimator With a Mean Asset Correlation of 25% and if Asset Correlations are Stochastic

This table shows the bias and RMSE of asset correlation estimates for the Asymptotic Maximum Likelihood (AML) method and the Asymptotic Method of Moments (AMM), separately for three default probabilities (PD) and four sample lengths Y of yearly default rate observations. The asset correlations of the DGP follow an Ornstein-Uhlenbeck process with mean 0.25.

	$PD = 0.005$		$PD = 0.012$		$PD = 0.0203$	
	AML	AMM	AML	AMM	AML	AMM
Bias						
$Y = 5$	-.0895	-.1293	-.0612	-.1093	-.0545	-.0970
$Y = 10$	-.0674	-.0962	-.0353	-.0767	-.0265	-.0603
$Y = 20$	-.0560	-.0672	-.0219	-.0495	-.0120	-.0403
$Y = 40$	-.0514	-.0454	-.0157	-.0305	-.0052	-0.0225
RMSE						
$Y = 5$	.1248	.1477	.1155	.1388	.1177	.1351
$Y = 10$	.0931	.1235	.0804	.1178	.0831	.1151
$Y = 20$	.0728	.1042	.0562	.0998	.0579	.0963
$Y = 40$	.0612	.0888	.0402	.0839	.0411	.0808

Table 8
Bias and RMSE of Estimates From Pairwise Stock Prices if Correlations are Stochastic

This table shows the bias and RMSE of asset correlation estimates from stock returns. They are estimated by pairwise sample correlations (PW) based on six DGPs with three default probabilities (PD) and two asset correlation values ρ which follow Ornstein-Uhlenbeck processes with mean 0.1 and 0.25.

$N = 50$ $T = 2y$		$PD = 0.005$ PW	$PD = 0.012$ PW	$PD = 0.0203$ PW
$\rho = 0.1$	Bias	-.0016	-.0014	-.0020
	RMSE	.0371	.0370	.0370
$\rho = 0.25$	Bias	-.0046	-.0060	-.0066
	RMSE	.0465	.0455	.0463

returns. Although the indirect estimation based on asset returns instead of stock returns is better founded in theory, the results from the previous section indicate that the differences between both cases are minor. Given the substantially lower computational burden, we apply only the direct method that uses stock prices as input. We do not present the results for the random effects model with stock price data as this model already proved to be inferior when applied to stock returns in the case of a constant correlation.

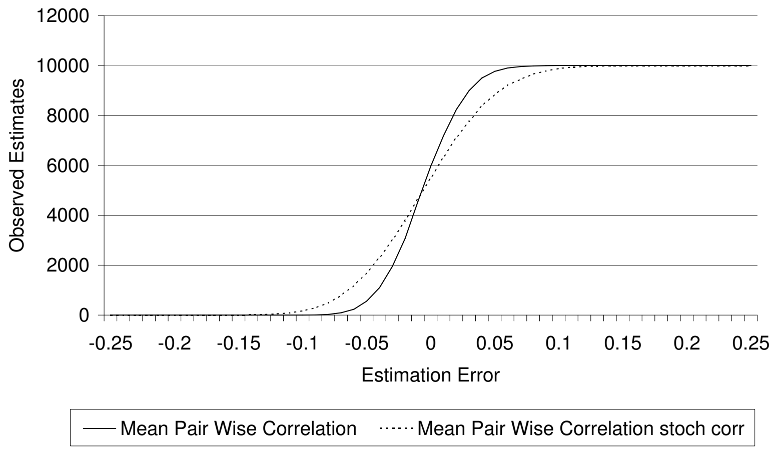
The numbers in Table 8 show a clear deterioration in the estimation performance, measured by RMSE, compared with Table 4. For an asset correlation of 0.25, a sample length of 20 years and a PD of 1.2%, the RMSE increases, for example, from 0.0285 to 0.0455. The bias instead stays nearly unaffected.

Figure 6 compares the estimation errors if the asset correlation is estimated by pairwise equity return correlations both without and with stochastic correlation. It confirms the finding of a higher dispersion of errors if the asset correlation follows a stochastic process.

Summarizing, the observed stronger increase in the RMSE for the estimation from market prices compared with the estimation from default rates confirms our expectation. Our results indicate that choosing between the use of market prices or default rates as data basis of the correlation estimation requires taking into account a trade-off: The closer the Merton model describes real world processes, the better estimates based on market prices perform. If the model is instead miss-specified, for example as it is agnostic to the stochastic character of asset correlations, the superi-

Figure 6. Distribution of Estimation Errors Both Without and With Stochastic Correlations.

This figure shows the interpolated distribution function of errors in asset correlations which were estimated from stock returns by pairwise sample correlations. The DGP is characterized by an asset correlation of 0.25, a time series of 10 years for 5,000 firms and a PD of 1.2%.



ority over an estimation from default rates diminishes. The reason is the estimation from default rates requires less assumptions on model structure which renders it more robust against model miss-specifications.

6. Summary and Conclusions

Linear factor models, based on the classic Merton (1974) model, have become a cornerstone of credit risk modelling in the literature as well as in industry practice. In this model framework, default dependencies are typically captured by asset correlations. These key parameters of a model are usually estimated either from time series of stock prices or default rates. In this paper we explore to which extent differences in small sample properties of the respective estimators are responsible for the substantial diversity in empirical estimates of asset correlations. For this purpose, we carry out a comprehensive simulation study in which the time series of default rates and stock prices of realistic length are generated from the same model, i. e. the same DGP. Furthermore, we introduce a model error in the form of stochastic, mean-reverting asset correlations. We compare the performance of the estimators from default rates with those from stock prices, using the bias and the RMSE as benchmarks. Our main findings are:

- Estimates from default rates are typically downward biased. This negative bias increases with shorter sample lengths, high correlations, lower PDs, and if the AMM-estimator is used instead of the AML-estimator.
- The level of the true correlation parameter has a strong impact on the estimation performance. Increasing the asset correlation from 10% to 25% increases not only the downward bias, but also produces more scattered estimates, particularly in the tails.
- Both, the direct estimation of asset correlations from equity returns and the indirect estimation from asset returns are superior to an estimation from default rates, both in terms of bias and RMSE. This holds even if a relatively short sample of two years of weekly equity returns is compared with a sample comprising 40 years of yearly default rate observations.
- Comparing the direct and the indirect estimation method we find that the indirect estimation based on inferred asset returns is superior. The better performance depends also on the true parameters of the DGP. With increasing PDs and higher asset correlations, the out–performance of the direct estimation method becomes more visible.
- If the constant asset correlation is replaced in the DGP by an Ornstein-Uhlenbeck process, the superiority of an estimation from equity prices instead of default rates diminishes but does not disappear.

These findings have implications for the interpretation of empirical studies of asset correlations and for future risk modelling:

1. Our results indicate that different small sample properties may have contributed to a large extent to the differences in correlation estimates which emerged in previous empirical studies.
2. If time series of market prices of equity and default rates are available, it is generally recommendable to estimate asset correlations from market prices.
3. We have observed relatively high RMSEs even in a clinical study in which the model is correctly specified which is unlikely in practical applications. This finding strongly advocates that care must be taken if correlation estimates are applied in credit risk modelling. The simulation setup used in this paper presents a way to quantify the estimation error.

Introducing a model error through stochastic, mean-reverting correlations is certainly only one of many plausible alternatives. Other causes of model errors which are also plausible given results from other empirical work, are, for example, a t-distribution of asset returns following Mashal et al. (2003) or a cross-sectional diversity of asset correlations, inspired by Duellmann et al. (2007). A more comprehensive analysis of the effect of various model errors is left for further work.

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