Determining and Forecasting High-Frequency Value at Risk by Using Lévy Processes

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Abstract

A new approach for using Lévy processes to compute value at risk (VaR) using high-frequency data is presented in this paper. The approach is a parametric model using an ARMA(1,1)-GARCH(1,1) model where the tail events are modeled using fractional Lévy stable noise and Lévy stable distribution. Using high-frequency data for the German DAX Index, the VaR estimates from this approach are compared to those of a standard nonparametric estimation method that captures the empirical distribution function, and with models where tail events are modeled using Gaussian distribution and fractional Gaussian noise. The results suggest that the proposed parametric approach yields superior predictive performance.

Key Words: Fractional Gaussian noise, Fractional Lévy stable noise, High-frequency data, Lévy stable distribution, Value at Risk

JEL Classification: C15, C46, C52, G15

1. Introduction

A commonly used methodology for estimating market risk that has been endorsed by regulators and financial industry advisory groups is value at risk (VaR). Financial institutions with significant trading and investment volumes have adopted the VaR methodology in their risk management operations; corporations have used VaR for risk reporting. In general, VaR estimations can aid in decisions involving capital resource allocation, setting position limits, and performance evaluation.[1]

The standard VaR computation (e.g., delta-normal VaR) requires that the underlying return generating processes for the assets of interest be normally distributed, where the moments are time invariant and can be estimated with historical data. Neftci (2000) points out that extreme events are structurally different from the return-generating process under normal market conditions. Höchstötter et al. (2005) and Rachev et al. (2005b, 2007b) make the same argument, focusing on the stylized fact that returns are heavy tailed. Brooks et al. (2005) argue that heavy tailedness might lead to an underprediction of both the size of extreme market movements and the frequency with which they occur. Khindanova et al. (2001) propose a methodology for computing VaR based on the stable distribution.

Despite the increased use of the VaR methodology, it does have well-known drawbacks. VaR is not a coherent risk measure and does not give any indication of the risk beyond the quantile.[2] Beder (1995) has empirically demonstrated how different VaR models can lead to dramatically different VaR estimates. Moreover, when employing the VaR methodology, it is possible for an investor, unintentionally or not, to decrease portfolio VaR while simultaneously increasing the expected losses beyond the VaR (i.e., by increasing the "tail risk" of the portfolio).[3] There are superior measures to VaR for measuring market risk. Expected tail loss (or expected shortfall), for example, is a coherent risk measure that overcomes the conceptual deficiencies of VaR.[4] Even with these well-known limitations, however, VaR remains the most popular measure of market risk employed by risk managers. Dowd (2002) identifies two characteristics of VaR that make it appealing to risk managers. First, VaR provides a common consistent measure of risk across different positions and risk factors. Second, it takes account of the correlation between different risk factors. Dowd also offers an explanation for the popularity of VaR given its well-documented limitations and the superiority of risk measures such as expected tail loss. First, it is a simple measure of expected tail risk. Second, VaR is often needed to estimate the expected tail loss if there is no formula to calculate expected tail loss directly.

With the availability of intra-daily price data (i.e., high-frequency data), researchers and practitioners have focused more attention on market microstructure issues to understand and help formulate strategies for the timing of trades. Besides heavy tailedness, high-frequency data have several stylized facts. For example, Sun et al (2007a, 2007c) have found that long-range dependence and volatility clustering are major characteristics of high-frequency data. Long-range dependence or long memory denotes the property of a time series to exhibit persistent behavior (i.e., a significant dependence between very

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[1] Rachev et al. (2005) provide a review of adoption of VaR for measuring market risk. A more technical discussion of market risk can be found in Khindanova and Rachev (2000), Khindanova et al. (2001), and Gamrowski and Rachev (1996).

[2] See Artzner et al. (1999).

[3] See Martin et al. (2003) and the references therein.

[4] Ssee, for example, Acerbi and Tasche (2002) and Rachev et al. (2005a).

distant observations and a pole in the neighborhood of the zero frequency of its spectrum.[5] Long-range dependence time series typically exhibit self-similarity. The stochastic processes with self-similarity are invariant in distribution with respect to changes of time and space scale.[6]

In this paper, we propose an approach for calculating VaR with high-frequency data. The approach utilizes the ARMA(1,1)-GARCH(1,1) model with Lévy stable processes for computing VaR. The empirical evidence we present suggests that this approach outperforms three other parametric models investigated. Our findings are consistent with the empirical results reported in Sun et al. (2007a) that an ARMA(1,1)-GARCH(1,1) model with Lévy stable noise residuals exhibits superior performance in modeling high-frequency stock returns.

We have organized the paper as follows. In Section 2, we introduce the methodology for estimating and evaluating parametric and nonparametric VaR. In Section 3, we specify the three Lévy family models investigated in our study (the Lévy stable distribution, fractional Gaussian noise, and Lévy fractional stable noise) utilized in modeling the residuals distribution for computing parametric VaR with the help of the ARMA(1,1)-GARCH(1,1) model. The study's data and empirical methodology are described in Section 4. In Section 5 we compare the performance of our VaR models based on high-frequency data at 1-minute level for the German DAX index. We summarize our conclusions in Section 6.

2. Value at Risk

In mathematical terms, VaR is defined as follows. Given $\alpha \in (0, 1]$, R is a random gain or loss of an investment over a certain period. VaR of a random variable R at level of α is the absolute value of the worst loss not to be exceeded with a probability of at least α, more formally, if α-quantile of R is $q_\alpha(R) = \inf\{r \in \Re : P[R \leq r] \geq \alpha\}$, the VaR at confidence level α of R is $VaR_\alpha(R) = q_\alpha(-R)$.

2.1 Non-parametric Approach of VaR Estimation

VaR is in fact the quantile of loss distribution for an asset. Consequently, the estimation of VaR is indeed the estimation of the loss distribution. The kernel estimator is the basic methodology employed to estimate the density (see Silverman (1986)). If random variable X has density $f(x)$, then

$$f(x) = \lim_{a \to 0} \frac{1}{2a} P(x - a < X < x + a) \tag{1}$$

By counting the proportion of sampling observations falling in the interval of $(x-a, x+a)$, the probability $P(x - a < X < x + a)$ can be estimated for any given a. Defining the kernel function K for

$$\int_{-\infty}^{\infty} K(x)d(x) = 1 \tag{2}$$

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[5]Baillie (1996) provides a survey of the major econometric research on long-range dependence processes, fractional integration, and applications in economics and finance. Doukhan et al. (2003) and Robinson (2003) provide a comprehensive review of the studies on long-range dependence. Bhansali and Kokoszka (2006) review recent research on long-range dependence time series. Recent theoretical and empirical research on long-range dependence in economics and finance is provided by Rangarajan and Ding (2006) and Teyssiére and Kirman (2006). Sun et al. (2007c) provide a review of long-range dependence research based on using intra-daily data.

[6]See Samorodnisky and Taqqu (1994) and Doukhan et al. (2003).

in which $K(x)$ usually but not always is regarded as a symmetric probability density function (e.g., normal density), the kernel estimator is defined by

$$\tilde{f}(x) = \frac{1}{na} \sum_{i=1}^{n} K\left(\frac{x - X_i}{a}\right) \tag{3}$$

where a is the window width and n is sample size. The kernel estimator can be reviewed as a sum of bumps placed at the observations X_i. Kernel function $K(x)$ determines the shape of the bumps and window width a determines the width of the bumps.

For the purpose of evaluating the quality of the estimation in Section 5, we will use the mean square error (MSE) and define it as:

$$MSE_x(\tilde{f}) = E\left(\tilde{f}(x) - f(x)\right)^2 = \left(E\tilde{f}(x) - f(x)\right)^2 + var\left(\tilde{f}(x)\right) \tag{4}$$

To measure the global closeness of fit of $\tilde{f}(x)$ to $f(x)$ by integrating the MSE over x, the mean integrated square error (MISE) is defined as following,

$$MISE_x(\tilde{f}) = E\int \left(\tilde{f}(x) - f(x)\right)^2 dx = \int \left(E\tilde{f}(x) - f(x)\right)^2 dx + \int var\left(\tilde{f}(x)\right) dx \tag{5}$$

Given a symmetric kernel function K, $\int tK(t)dt = 0$ and $\int t^2 K(t)dt = k_2 \neq 0$, Silverman (1986) shows that the approximation of MISE is:

$$\frac{1}{4}a^4 k_2^2 \int f''(x)^2 dx + \frac{1}{na} \int K(t)^2 dt$$

It is clear that the bias in the estimation of $f(x)$ depends on the window width. The optimal window width a_{opt} can be chosen by minimizing the MISE. Silverman (1986) shows that

$$a_{opt} = n^{-1/5} k_2^{-2/5} \left(\int f''(x)^2 dx\right)^{-1/5} \left(\int K(t)^2 dt\right)^{1/5} \tag{6}$$

and the optimal solution is given by the Epanechnikov kernel $K_E(x)$:

$$K_E(x) = \begin{cases} \frac{3}{4\sqrt{5}}(1 - \frac{x^2}{5}), & -\sqrt{5} \leq x \leq \sqrt{5} \\ 0, & \text{else} \end{cases} \tag{7}$$

A slight drawback suffered by the kernel estimator is the inefficiency in dealing with long-tailed distributions. Since across the whole sample the window width is fixed, a good degree of smoothing over the center of the distribution will often leave spurious noise in the tail (see Silverman (1986) and Dowd (2005)). Silverman (1986) offers several solutions such as the nearest neighbor method and variable kernel method. For the nearest neighbor method, the window width placed on an observation depends on the distance between that observation and its nearest neighbors. For the variable kernel estimator, the density $f(x)$ is estimated as follows:

$$\tilde{f}(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{ah_{i,k}} K\left(\frac{x - X_i}{ah_{i,k}}\right) \tag{8}$$

where $h_{i,k}$ is the distance between X_i and the kth nearest of the other data points. The window width of the kernel placed on X_i is proportional to $h_{i,k}$, therefore flatter kernels will be placed on more sparse data.

2.2 Parametric Approach of VaR Estimation

The parametric approach for VaR estimation is based on the assumption that the financial returns R_t are a function of two components μ_t and ε_t (i.e., $R_t = f(\mu_t, \varepsilon_t)$). R_t can be regarded as a function of ε_t conditional on a given μ_t; typically this function takes a simple linear form $R_t = \mu_t + \varepsilon_t = \mu_t + \sigma_t u_t$. Usually μ_t is referred to as the location component and σ_t the scale component. u_t is an independent and identically distributed (i.i.d.) random variable that follows a probability density function f_u. VaR based on information up to time t is then

$$\tilde{VaR}_t := q_\alpha(R_t) = -\tilde{\mu}_t + \tilde{\sigma}\, q_\alpha(u) \tag{9}$$

where $q_\alpha(u)$ is the α-quantile implied by f_u.

Unconditional parametric approaches set μ_t and σ_t as constants, therefore the returns R_t are i.i.d random variables with density $\sigma^{-1} f_u(\sigma^{-1}(R_t - \mu))$. Conditional parametric approaches set location component and scale component as functions not constants. The typical time-varying conditional location setting is the ARMA(r, m) processes. That is, the conditional mean equation is:

$$\mu_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i\, R_{t-i} + \sum_{j=1}^{m} \beta_j \varepsilon_{t-j} \tag{10}$$

The typical time-varying conditional variance setting is GARCH(p,q) processes given by

$$\sigma_t^2 \;=\; \kappa + \sum_{i=1}^{p} \gamma_i\, \sigma_{t-i}^2 + \sum_{j=1}^{q} \theta_j\, \varepsilon_{t-j}^2 \tag{11}$$

Different distributional assumptions for the innovation distribution f_u can be made. In the empirical analysis below, distributional assumptions analyzed for the parametric approaches are the normal distribution, fractional Gaussian noise, fractional Lévy stable noise, and Lévy stable distribution.

2.3 Evaluation of VaR Estimators

Backtesting is the usual method to evaluate the VaR estimators and its forecasting quality. It can be performed for in-sample estimation evaluation and for out-of-sample interval forecasting evaluation. The backtesting is based on the indicator function I_t which is defined as $I_t(\alpha) = I(r_t < -VaR_t(\alpha))$. The indicator function shows violations of the quantiles of the loss distribution. Then the process $\{I_t\}_{t \in T}$ is a process of i.i.d Bernoulli variables with violation probability $1 - \alpha$. Christoffersen (1998) shows that evaluating the accuracy of VaR can be reduced to checking whether (1) the number of violations is correct on average and (2) the pattern of violations is consistent with i.i.d processes. In another word, an accurate VaR measure should satisfy both the unconditional coverage property and independent property. The unconditional coverage property means that the probability of realization of a loss in excess of the estimated $VaR_t(\alpha)$ must be exactly $\alpha\%$ (i.e., $P(I_t(\alpha) = 1) = \alpha$). The independent property means that previous VaR violations do not presage future VaR violations.

Kupiec (1995) proposes a frequency of failures test that checks how many time an estimated VaR is violated in a given time period. If the observed frequency of failures of the estimated VaR differs significantly from $\alpha \times 100\%$, the underlying risk measure is less reliable. The shortcoming of the backtesting proposed by Kupiec is that it fails to focus on the independence property. In order to

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detect violations of the independence property of an estimated VaR measure, say, $I_t(\alpha)$, Christoffersen (1998) suggests the Markov test. This test detects whether the likelihood of a VaR violation depends on another VaR violation occurring in the previous time period. The assumption underlying this test is that VaR is adequate to capture the risk factors and previous VaR violations cannot cause a future VaR violation; that is, the chance of violating the current period's VaR should not depend on whether the previous period's VaR was violated and the chance of violating current period's VaR should not influence next period's violation. Campbell (2005) provides a review of backtests examining the adequacy of VaR models.

3. Lévy Processes with Specifications

Lévy processes[7] have become increasingly popular in mathematical finance because they can describe the observed behavior of financial markets in a more accurate way than other processes typically used such as the normal distribution. They capture jumps, heavy-tails, and skewness observed in the market for asset price processes. Moreover, Lévy processes provide the appropriate option pricing framework to model implied volatilities across strike prices and across maturities with respect to the "risk-neutral" assumption.

When investors select stocks and put them together to form a portfolio, they use the normal distribution to calculate risk by first calculating the beta, a measure of a particular stock's volatility in relation to the overall market, for every investment in the portfolio. Unfortunately, the normal distribution is not adequate enough to capture market characteristics (see, for example, Rachev and Mittnik (2000)). Some recurring themes in the pattern of stock returns such as volatility clustering have been observed. So how might portfolio managers better measure risk if stock returns followed a pattern that is not a normal distribution? Mandelbrot makes a few suggestions for measuring market risk based on the tools of fractal geometry. Among the more alluring are the "H" (Hurst) index, which is an indication of the "persistence" or trend that affects a stock's return. Roughly speaking, a high "H" value could indicate crowd behaviour of stock returns, while a low "H" value may indicate a more random "classic" market force (see Mandelbrot (1997, 2005) and Sun et al. (2007a)). The Lévy stable distribution and Lévy fractional stable processes provide a good solution to capture such characteristics when the market exhibits characteristics such as volatility clustering, heavier tails than the normal distribution, and persistence of stock returns.

Moreover, fractal processes have a close relationship to the fractal market hypothesis (FMH), which states that (1) a market consists of many investors with different investment horizons and (2) the information set that is important to each investment horizon is different. As long as the market maintains this fractal structure, with no characteristic time scale, the market remains stable. When the market's investment horizon becomes uniform, the market becomes unstable because everyone is trading based upon the same information set (see Peters (1989, 1994)). The roughness induced by the fractal market hypothesis can be modeled by fractal processes (see Mandelbrot (1997, 2005)).

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[7]We review the definition of Lévy processes as well as one specific form (the Lévy fractional stable motion) and two extensions of infinitely divisible distributions (the Lévy stable distribution and fractional Brownian motion) that we use in the appendix to this paper. Further details can be found in Sato (1999).

3.1 Lévy processsses

Suppose $\phi(u)$ is the characteristic function of a distribution. If for every positive interger n, $\phi(u)$ is also the n-th power of a characteristic function, this distribution is said to be infinitely divisible. A stochastic process $X = (X_t)_{t\geq 0}$ can be defined for every such an infinitely divisible distribution. For this stochastic process $X = (X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathrm{P})$ to be called a Lévy process, the following five conditions (see Sato (1999)) have to be satisfied:

1. $X_0 = 0$ a.s.

2. X has independent increments: Given $0 < t_1 < t_2 < \cdots < t_n$, the random variables $X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.

3. X has stationary increment: For $t \geq 0$, the distribution of $X_{t+s} - X_s$ does not depend on $s \geq 0$.

4. X is stochastically continuous: $\forall\, t \geq 0$ and $\varepsilon > 0$, $\lim_{s\to t} P[(X_s - X_t) > \varepsilon] = 0$.

5. X is right continuous and has left limits (càdlàg).

The cumulant characteristic function $\psi(u) = \log \phi(u)$ must satisfy the Lévy-Khintchine formula given as follows:

$$\psi(u) = i\gamma u - \frac{\sigma^2}{2}u^2 + \int_{-\infty}^{+\infty}\left(\exp(iux) - 1 - iux\mathbf{1}_{\{|x|<1\}}\right)v(dx)$$

where $\gamma \in R, \sigma^2 \geq 0$ and v is a measure on $R\backslash\{0\}$ with

$$\int_{-\infty}^{+\infty} \inf\{1, x^2\}v(dx) = \int_{-\infty}^{+\infty}(1 \wedge x^2)v(dx) < \infty.$$

As to this case, the infinitely divisible distribution has a Lévy triplet $[\gamma, \sigma^2, v(dx)]$ and v is called the Lévy measure of X (see Sato (1999) for more general reference).

3.2 Lévy Stable Distribution

The Lévy stable distribution (sometimes referred to as α-stable distribution) has four parameters for complete description: an index of stability $\alpha \in (0, 2]$ (also called the tail index), a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\gamma > 0$, and a location parameter $\zeta \in \Re$. There is unfortunately no closed-form expression for the density function and distribution function of a Lévy stable distribution. Rachev and Mittnik (2000) give the definition for the Lévy stable distribution: A random variable X is said to have a Lévy stable distribution if there are parameters $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\gamma \geq 0$ and ζ real such that its characteristic function has the following form:

$$E\exp(i\theta X) = \begin{cases} \exp\{-\gamma^\alpha|\theta|^\alpha(1 - i\beta(\sin\theta)\tan\frac{\pi\alpha}{2}) + i\zeta\theta\}, & if \quad \alpha \neq 1 \\ \exp\{-\gamma|\theta|(1 + i\beta\frac{2}{\pi}(\sin\theta)\ln|\theta|) + i\zeta\theta\}, & if \quad \alpha = 1 \end{cases} \tag{12}$$

and,

$$\mathrm{sign}\theta = \begin{cases} 1, & if \quad \theta > 0 \\ 0, & if \quad \theta = 0 \\ -1, & if \quad \theta < 0 \end{cases} \tag{13}$$

For $0 < \alpha < 1$ and $\beta = 1$ or $\beta = -1$, the stable density is only for a half line.

3.3 Fractional Brownian Motion

For a given $H \in (0,1)$, there is basically a single Gaussian H-sssi [8] process, namely fractional Brownian motion (fBm), first introduced by Kolmogorov (1940). Mandelbrot and Wallis (1968) and Taqqu (2003) define fBm as a Gaussian H-sssi process $\{B_H(t)\}_{t \in R}$ with $0 < H < 1$. Mandelbrot and van Ness (1968) define the stochastic representation

$$B_H(t) := \frac{1}{\Gamma(H + \frac{1}{2})} \left(\int_{-\infty}^{0} [(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}] dB(s) + \int_{0}^{t} (t-s)^{H-\frac{1}{2}} dB(s) \right), \quad (14)$$

where $\Gamma(\cdot)$ represents the Gamma function:

$$\Gamma(a) := \int_{0}^{\infty} x^{a-1} e^{-x} dx,$$

and $0 < H < 1$ is the Hurst parameter. The integrator B is ordinary Brownian motion. The principal difference between fractional Brownian motion and ordinary Brownian motion is that the increments in Brownian motion are independent while in fractional Brownian motion they are dependent. For fractional Brownian motion, Samorodnitsky and Taqqu (1994) define its increments $\{Y_j, j \in Z\}$ as fractional Gaussian noise (fGn), which is, for $j = 0, \pm 1, \pm 2, ..., Y_j = B_H(j-1) - B_H(j)$.

3.4 Lévy Stable Motion

While fractional Brownian motion can capture the effect of long-range dependence, it has less power to capture heavy tailedness. The existence of abrupt discontinuities in financial data, combined with the empirical observation of sample excess kurtosis and unstable variance, confirms the stable Paretian hypothesis identified by Mandelbrot (1963, 1983). It is natural to introduce the stable Paretian distribution in self-similar processes in order to capture both long-range dependence and heavy tailedness. Samorodinitsky and Taqqu (1994) discuss the α-stable H-sssi processes $\{X(t), t \in R\}$ with $0 < \alpha < 2$. If $0 < \alpha < 1$, the exponent of self-similarity is $H \in (0, 1/\alpha]$ and if $1 < \alpha < 2$, the exponent of self-similarity is $H \in (0,1)$. In addition, Cohen and Samorodnitsky (2006) show that with exponent $H' = 1 + H(1/\alpha - 1)$, process $\{X(t), t \in R\}$ is a well-defined symmetric α-stable ($S\alpha S$) process. It has stationary increments and is self-similar. They show that (1) for $0 < \alpha < 1$, a family of H'-sssi $S\alpha S$ processes with $H' \in (1, 1/\alpha)$ is obtained, (2) for $1 < \alpha < 2$, a family of H'-sssi $S\alpha S$ processes with $H' \in (1/\alpha, 1)$ is obtained, and (3) for $\alpha = 1$, a family of 1-sssi $S\alpha S$ processes is obtained.

There are many extensions of fractional Brownian motion to the Lévy stable distribution. The most commonly used is linear fractional Lévy motion (also called linear fractional stable motion), $\{L_{\alpha,H}(a, b; t), t \in (-\infty, \infty)\}$, which Samorodinitsky and Taqqu (1994) define as

$$L_{\alpha,H}(a, b; t) := \int_{-\infty}^{\infty} f_{\alpha,H}(a, b; t, x) M(dx), \quad (15)$$

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[8]The abbreviation of "sssi" means self-similar stationary increments, if the exponent of self-similarity H is to be emphasized, then "H-sssi" is adopted. Lamperti (1962) first introduced the semi-stable processes (which we today refer to as self-similar processes). Let T be either R, $R_+ = \{t : t \geq 0\}$ or $\{t : t > 0\}$. The real-valued process $\{X(t), t \in T\}$ has stationary increments if $X(t + a) - X(a)$ has the same finite-dimensional distriuions for all $a \geq 0$ and $t \geq 0$. Then the real-valued process $\{X(t), t \in T\}$ is self-similar with exponent of self-similarity H for any $a > 0$, and $d \geq 1$, $t_1, t_2, ..., t_d \in T$, satisfying: $(X(at_1), X(at_2), ..., X(at_d)) \stackrel{d}{=} (a^H X(t_1), a^H X(t_2), ... a^H X(t_d))$.

where

$$f_{\alpha,H}(a,b;t,x) := a\left((t-x)_+^{H-\frac{1}{\alpha}} - (-x)_+^{H-\frac{1}{\alpha}}\right) + b\left((t-x)_-^{H-\frac{1}{\alpha}} - (-x)_-^{H-\frac{1}{\alpha}}\right), \tag{16}$$

and a, b are real constants. $|a| + |b| > 1$, $0 < \alpha < 2$, $0 < H < 1$, $H \neq 1/\alpha$, and M is an α-stable random measure on R with Lebesgue control measure and skewness intensity $\beta(x)$, $x \in (-\infty, \infty)$ satisfying: $\beta(\cdot) = 0$ if $\alpha = 1$. They define linear fractional stable noises expressed by $Y(t)$, and $Y(t) = X_t - X_{t-1}$,

$$\begin{aligned} Y(t) &= L_{\alpha,H}(a,b;t) - L_{\alpha,H}(a,b;t-1) \tag{17}\\ &= \int_R \left(a\left[(t-x)_+^{H-\frac{1}{\alpha}} - (t-1-x)_+^{H-\frac{1}{\alpha}}\right]\right.\\ &\quad + \left. b\left[(t-x)_-^{H-\frac{1}{\alpha}} - (t-1-x)_-^{H-\frac{1}{\alpha}}\right]\right)M(dx), \end{aligned}$$

where $L_{\alpha,H}(a,b;t)$ is a linear fractional stable motion defined by equation (21), and M is a stable random measure with Lebesgue control measure given $0 < \alpha < 2$. Samorodinitsky and Taqqu (1994) show that the kernel $f_{\alpha,H}(a,b;t,x)$ is d-self-similar with $d = H - 1/\alpha$ when $L_{\alpha,H}(a,b;t)$ is $1/\alpha$-self-similar. This implies $H = d + 1/\alpha$ (see Taqqu and Teverovsky (1998) and Weron et al. (2005)).[9] In this paper, if there is no special indication, the fractional stable noise (fsn) is generated from a linear fractional Lévy motion.

As mentioned by Rachev and Mittnik (2000), a crucial restriction in the Mandelbrot and Fama α-stable model is the assumption that the returns are i.i.d random variables. This restriction can be relaxed by considering the more realistic model of self-similar processes. Let underlying asset Y_k at time t_k have the form

$$Y_k = \sum_{j \in Z} c_j X_{k-j} \tag{18}$$

for $Z = \{0, \pm 1, \pm 2, ...\}$, where

$$c_j = \begin{cases} j^{-\beta-1} & if \quad j > 0 \\ 0 & if \quad j = 0 \\ -|j|^{-\beta-1} & if \quad j < 0 \end{cases} \tag{19}$$

for some $\beta \in (\frac{1}{\alpha} - 1, \frac{1}{\alpha})$ and $\alpha \in (0, 2)$; $X_j, j \in Z$ is a sequence of i.i.d. random variables in the domain of attraction of a strictly stable random variable with index α whose character function has the form of

$$E(e^{i\theta X_k}) = \exp\{-|\theta|^\alpha(A_1) + iA_2 \sin \theta\} \tag{20}$$

for some $A_1 > 0$, $A_2 \in R$, with $|A_1^{-1}A_2| \leq \tan(\alpha\pi/2)$. From the representation of Y_k, it follows that the trading duration at t_k depends on past durations and affects future durations. The nature of dependence of Y_k is determined by the parameters α and β which will be estimated. As for the usual α-stable approximation of i.i.d random variables in the domain of attraction of an α-stable distribution, the fractional stable process is defined by Maejima (1983): for $t \in (0, 1)$,

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[9]Some properties of these processes have been discussed in Mandelbrot and Van Ness (1968), Maejima (1983), Maejima and Rachev (1987), Rachev and Mittnik (2000), Rachev and Samorodnitsky (2001), Samorodnitsky (1994, 1996, 1998), Samorodnitsky and Taqqu (1994), and Cohen and Samorodnitsky (2006).

$$\Delta_n(t) \equiv |\beta|n^{-H}\left(\sum_{k=1}^{[nt]} Y_k + (nt - [nt])Y_{[nt]+1}\right) \tag{21}$$

where $[nt]$ is the integer part of nt; summation $\sum_{k=1}^{0}$ is defined to be 0 and $H = 1/\alpha - \beta$. There exists two independent stable processes $\{Z_+(t), t \geq 0\}$ and $\{Z_-(t), t \geq 0\}$ both having character function

$$E(e^{izZ_\pm(t)}) = \exp\{-t|z|^\alpha(A_1) + iA_2 \sin z\} \tag{22}$$

Then the fractional stable process is given by

$$\Delta(t) = \int_{-\infty}^{+\infty} \left(|t - s|^{H-\frac{1}{\alpha}} - |s|^{H-\frac{1}{\alpha}}\right) dZ(s) \tag{23}$$

where $\Delta(0) = 0$ and $Z(s) = Z_+(s)I[s \geq 0] - Z_-(0 - s)I[s < 0]$.

By using wavelet transformation and discrete linear filter transformation, Stoev et al. (2002) propose estimators of linear fractional stable motion. Stoev and Taqqu (2004) simulate linear fractional stable motion and FARIMA model based on Fast Fourier Transform (FFT) methods. Sun et al. (2007a) empirically compare the performance of fractional Lévy stable noise in modeling high-frequency data and their results support its superiority.

4. Data

To test the relative performance of the models we presented in this paper, we use high-frequency data of the Deutsche Aktien Xchange (DAX) index from January 2 to September 30, 2006 that were aggregated to the 1-minute frequency level.[10] The aggregation algorithm is based on the linear interpolation introduced by Wasserfallen and Zimmermann (1995). That is, given an inhomogeneous series with times t_i and values $\varphi_i = \varphi(t_i)$, the index i identifies the irregularly spaced sequence. The target homogeneous time series is given at times $t_0 + j\Delta t$ with fixed time interval Δt starting at t_0. The index j identifies the regularly spaced sequence. The time $t_0 + j\Delta t$ is bounded by two times t_i of the irregularly spaced series, $I = \max(i | t_i \leq t_0 + j\Delta t)$ and $t_I \leq t_0 + j\Delta t > t_{I+1}$. Data are interpolated between t_I and t_{I+1}. The linear interpolation shows that

$$\varphi(t_0 + j\Delta t) = \varphi_I + \frac{t_0 + j\Delta t - t_I}{t_{I+1} - t_I}(\varphi_{I+1} - \varphi_I).$$

Dacorogna et al. (2001) point out that linear interpolation relies on the forward point of time and Müller et al. (1990) suggests that linear interpolation is an appropriate method for stochastic processes with i.i.d. increments.

Empirical evidence has identified the seasonality in high-frequency data. In order to remove such disturbance, several methods of data adjusting have been adopted in modeling. Engle and Russell (1998) and other researchers adopt several methods to adjust the seasonal effect in data. In our study,

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[10]The DAX index is a stock market index whose components include 30 blue chip German stocks that are traded on the Frankfurt Stock Exchange. Starting in 2006, the DAX index is calculated every second. In our original dataset, the DAX index is sampled at the one-second level.

seasonality is treated as one type of self-similarity which can be captured by factional processes we employed. Consequently, it is not necessary to adjust for the seasonal effect in the data.

In previous studies that have studied the computing of VaR, low-frequency data typically have been used. Because stock indexes change their composition quite often over time, it is difficult to find the impact of these changes in composition when analyzing the return history of stock indexes using low-frequency data. Dacorogna et al. (2001) call this phenomenon the "breakdown of the permanence hypothesis". In order to overcome this problem, we use high-frequency data in our study. In addtion, because more and more day trading strategies are being employed by practitioners, measuring the risk in a short time interval is being sought. Therefore, using high-frequency data to compute VaR has practical importance.

Employing high-frequency data has several advantages compared with low-frequency data. First, with a very large amount of observations, high-frequency data offers a higher level of statistical significance. Second, high-frequency data are gathered at a low level of aggregation, thereby capturing the heterogeneity of players in financial markets. These players should be properly modeled in order to make valid inferences about market movements. Low-frequency data, say daily or weekly data, aggregate the heterogeneity in a smoothing way. As a result, many of the movements in the same direction are strengthened and those in the opposite direction cancelled in the process of aggregation. The aggregated series generally show smoother style than their components. The relationships between the observations in these aggregated series often exhibit greater smoothness than their components. For example, a curve exhibiting a one-week market movement based on daily return data might be a line with a couple of nodes. The smooth line segment masks the intra-daily fluctuation of the market. But high-frequency data can reflect such intra-daily fluctuations and the intra-daily factors that influence the risks can be taken into account. Third, using high-frequency data in computing VaR in an equity market can consider both microstructure risk effects and macroeconomic risk factors. This is because information contained in high-frequency data can be separated into a higher frequency part (i.e., the intra-daily fluctuation) and a lower frequency part (i.e., low-frequency smoothness). The information provided by the higher frequency part mirrors the microstructure effect of the equity markets and the information in the lower frequency part shows the smoothed trend that is usually influenced by macroeconomic factors in these markets.

Standard econometric techniques are based on homogeneous time series analysis. If a researcher uses analytic methods of homogeneous time series for inhomogeneous time series, the reliability of the results will be doubtful. Aggregating inhomogeneous tick-by-tick data to the equally spaced (homogeneous) time series is required. Engle and Russell (1998) argue that for aggregating tick-by-tick data to a fixed time interval, if a short time interval is chosen, there will be many intervals in which there is no new information, and if choosing a wide interval, micro-structure features might be missing. Aït-Sahalia (2005) suggests keeping the data at the ultimate frequency level. In our empirical study, intra-daily data (which we refer to as high-frequency data in this paper) at the 1-minute level is aggregated from tick-by-tick data to compute VaR for the models investigated.

5. Empirical study

An ARMA(1,1)-GARCH(1,1) model with different residuals (i.e., the normal distribution, the Lévy stable distribution, the Lévy fractional stable noise, and the fraction Gaussian noise) is employed to compute VaR using the high frequency DAX data described in Section 4. We will refer to the ARMA(1,1)-GARCH(1,1) model as simply the "AG model". The methodology for estimation and simulation of the AG model with different residuals is introduced in Sun et al. (2007a). We performed two experiments. In the first experiment, we calculate 95%-VaR values and 99%-VaR values for the entire data sample. In the second experiment, we split the dataset into two subsets: an in-sample (training) set and an out-of-sample (forecasting) set. The purpose of this second experiment is mainly to check the prediction power of parametric VaR values computed from the in-sample set.

We compute the in-sample 95% VaR and 99% VaR with a horizon of six months. Our dataset contains nine months of data. In order to ensure randomness of the data from which VaR is computed, after each computation, we shift the next starting point of the training dataset for VaR computation two weeks afterwards. Table 1 shows the results of the VaR value computed six times by the above-mentioned models for the in-sample estimation. The empirical VaR is computed using the nonparametric Kernel estimator. Table 2 shows for the different parametric AG models with different residuals (i.e., normal distribution, stable distribution, fractional stable noise, and fraction Gaussian noise) comparing with the empirical VaR values. The results in this table indicate that the 95% VaR computed from AG model with fractional stable noise has minimal absolute distance to the empirical value. However, the AG model with standard normal residuals has minimal absolute distance to the empirical VaR value in computing the 99% VaR than the other alternatives. This result suggests that although the AG model with fractional Lévy stable residuals provides a good modeling mechanism to capture the stylized facts observed from high-frequency data (see Sun et al. (2007a)), it is focusing on capturing long-term dependence of extreme events. In this case, the VaR value turns out to overestimate the VaR value if α is set too small.

In order to test the predictive power of the VaR value computed for each model, in our second experiment we test the following six predictions using different size of training and prediction datasets:

1. Training period is 6 months (236,160 data points) and one-step-ahead forecast for 6 months, 3 months, 1 month, 1 week, 1 day and 1 hour;

2. Training period is 3 months (124,800 data points) and one-step-ahead forecast for 3 months, 1 month, 1 week, 1 day and 1 hour;

3. Training period is 1 month (40,320 data points) and one-step-ahead forecast for 1 month, 1 week, 1 day and 1 hour;

4. Training period is 1 week (9,600 data points) and one-step-ahead forecast for 1 week, 1 day and 1 hour;

5. Training period is 1 day (1,920 data points) and one-step-ahead forecast for 1 day and 1 hour;

6. Training period is 1 hour(240 data points) and one-step-ahead forecast for 1 hour.

Tables 3 and 4 show the results for computing 95% VaR and 99% VaR, respectively.

Table 5 shows the admissible VaR violations and violation frequencies based on the Kupiec test. Considering the admissible VaR violations and violation frequencies shown in Table 3 for the prediction of 95% VaR, the AG model with fractional stable noise performs better than the other alternatives. For the prediction of 99% VaR, although all computed VaR values turns out to be conservative, the AG model with standard normal distribution has better performance among others, since they have violations close to the admissible VaR violations.

Backtesting is the typical method for evaluating VaR estimators and their forecasting performance. It can be performed for in-sample estimation evaluation and for out-of-sample interval forecasting evaluation. The backtesting is based on the indicator function I_t which is defined as $I_t(\alpha) = I(r_t < -VaR_t(\alpha))$. The indicator function shows violations of the quantiles of the loss distribution. Then the process $\{I_t\}_{t \in T}$ is a process of i.i.d Bernoulli variables with violation probability $1 - \alpha$. Christoffersen (1998) shows that evaluating the accuracy of VaR can be reduced to checking whether the number of violations is correct on average and the pattern of violations is consistent with i.i.d processes (see Section 2.3). In another words, an accurate VaR measure should satisfy both the unconditional coverage property and independent property. The unconditional coverage property means that the probability of realization of a loss in excess of the estimated $VaR_t(\alpha)$ must be exactly $\alpha\%$ (i.e., $P(I_t(\alpha) = 1) = \alpha$). The independent property means that previous VaR violations do not presage future VaR violations. Table 3 reports the p value for the Christoffersen test, where the null hypothesis is that the pattern of violations is consistently independent.

By comparing the admissible VaR violations with the violation in our out-of-sample forecasting experiment, we find that the parametric VaR values computed from the AG models provide a relatively conservative risk measure for forecasting if the α is set too small for the training dataset used to compute in-sample VaR. The reason is that given the model used for parametric VaR calculation is well specified, the VaR values calculated from the in-sample dataset in which the data illustrate high volatility or significant regime switching will provide a conservative prediction (over-prediction) for the out-of-sample (forecasting) dataset. If the out-of-sample (forecasting) dataset share similar trends as those in the in-sample (training) dataset, the VaR value can provide adequate prediction with respect to the α level specified. If the in-sample dataset illustrate different trends in the out-of-sample dataset, particularly when regime switching is observed for the in-sample dataset to the out-of-sample dataset, the predictive power of the VaR value for the latter dataset will be reduced.

Table 6 reports the p-value for Christoffersen test. The results of this test suggest that the pattern of violation of our parametric VaR in the out-of-sample forecasting is consistently independent. We can observe that the VaR value calculated by the AG model with Lévy stable and fractional Lévy noise performs better than the alternative models because the p-values for rejecting the null hypothsis are less than that of the alternative models.

6. Conclusion

There is considerable interest in the computing of VaR for market risk management. Most models follow the Gaussian distribution despite the overwhelming empirical evidence that fails to support the

hypothesis that financial asset returns can be characterized as Gaussian random walks. There are a number of arguments against both the Gaussian assumption and random walk assumption. One of the most compelling is that there exist fractals in financial markets. In this paper, we propose a new approach for computing VaR. We test this approach using the DAX index at one minute frequency level with parametric models which capture stylized facts observed in high-frequency data, i.e., ARMA(1,1)-GARCH(1,1) model with Lévy-type of residuals.

In our empirical analysis, we investigate both in-sample (training) performance and out-of-sample (forecasting) performance based on the Kupiec violation test and Christoffersen independent test. Our empirical results show that the VaR calculated from the underlying models (i.e., AG model with residuals of Lévy stable distribution and AG model with residuals of fractional Lévy noise) performs better than VaR calculated based on the alternative models (i.e., AG model with residuals of normal distribution and AG model with fractional Gaussian noise).

The empirical evidence based on out-of-sample forecasting suggests that VaR calculated by the underlying models turns out to be conservative with respect to violation frequencies. However, the prediction power of parametric VaR is limited by the training dataset and our models focus on capturing consistent extreme events. In order to make the VaR value less conservative, the underlying data generating processes should have tempered tails in their distributions.

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Table 1: VaR values calculated by Kernel estimator (empirical) and ARMA(1,1)-GARCH(1,1) with different residuals (i.e., normal, stable, fractional stable noise, and fractional Gaussian noise).

		95%					99%		
	Empirical	Normal	stable	FSN	FGN	Empirical	Normal	stable	FSN	FGN
6 M	1.4018	1.8682	1.3874	1.3929	1.8727	2.6607	2.6382	3.7350	3.6972	2.6419
3 M	1.5218	2.0601	1.5090	1.4778	2.0544	2.8501	2.9267	3.9461	3.8371	2.9178
1 M	1.5402	2.3234	1.5562	1.6189	2.3158	2.9148	3.2939	4.3943	4.0179	3.2580
1 W	1.4901	1.8864	1.3882	1.4224	1.9109	2.8036	2.6445	4.4273	3.5853	2.6159
1 D	1.3861	1.7735	1.5159	1.4412	1.8549	2.4768	2.6127	4.5385	4.4081	2.8099
1 H	1.6819	1.5163	1.5085	1.7600	1.9079	2.7741	2.6503	5.6604	6.4754	2.9549

Table 2: Difference between VaR values calculated by Kernel estimator (empirical) and ARMA(1,1)-GARCH(1,1) with different residuals (i.e., normal, stable, fractional stable noise, and fractional Gaussian noise).

		95%				99%		
	Normal	stable	FSN	FGN	Normal	stable	FSN	FGN
6 M	-0.4044	0.0144	0.0089	-0.4709	0.0225	-1.0743	-1.0365	0.0188
3 M	-0.5383	0.0128	0.0440	-0.5326	-0.0766	-1.0960	-0.9870	-0.0677
1 M	-0.7832	-0.0160	-0.0787	-0.7756	-0.3791	-1.4795	-1.1031	-0.3432
1 W	-0.3963	0.1019	0.0677	-0.4208	0.1591	-1.6237	-0.7817	0.1877
1 D	-0.3874	-0.1298	-0.0551	-0.4688	-0.1359	-2.0617	-1.9313	-0.3331
1 H	0.1656	0.1734	-0.0781	-0.2260	0.1238	-2.8863	-3.7013	-0.1808
$|\sum|$	2.6752	0.4483	0.3325	2.8947	0.8970	10.2215	9.5409	1.1313

Table 3: Violations of VaR (95%) computed from ARMA(1,1)-GARCH(1,1) with different residuals.

	6 months (236160)		3 months (124800)		1 month (40320)		1 week (9600)		1 day (1920)		1 hour (240)	
	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN
1 hour (240)	13	25	12	25	4	11	13	21	8	16	8	6
	0.0542	*0.1042*	*0.0500*	*0.1042*	*0.0167*	*0.0458*	*0.0542*	*0.0875*	*0.0333*	*0.0667*	*0.0333*	*0.0250*
1 day (1920)	35	71	49	105	33	73	47	110	59	95		
	0.0182	*0.0370*	*0.0255*	*0.0547*	*0.0172*	*0.0380*	*0.0245*	*0.0573*	*0.0307*	*0.0495*		
1 week (9600)	117	246	185	425	187	458	260	545				
	0.0122	*0.0256*	*0.0193*	*0.0443*	*0.0195*	*0.0477*	*0.0271*	*0.0568*				
1 month (40320)	545	1124	686	1602	661	1624						
	0.0135	*0.0279*	*0.0170*	*0.0397*	*0.0164*	*0.0403*						
3 months (124800)	1572	3165	1810	4222								
	0.0126	*0.0254*	*0.0145*	*0.0338*								
6 months (236160)	2744	5538										
	0.0116	*0.0235*										

	6 months (236160)		3 months (124800)		1 month (40320)		1 week (9600)		1 day (1920)		1 hour (240)	
	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN
1 hour (240)	25	14	25	14	13	4	18	13	15	8	6	6
	0.1042	*0.0583*	*0.1042*	*0.0583*	*0.0542*	*0.0167*	*0.0750*	*0.0542*	*0.0625*	*0.0333*	*0.0250*	*0.0250*
1 day (1920)	71	37	98	50	90	33	96	47	93	56		
	0.0370	*0.0193*	*0.0510*	*0.0260*	*0.0469*	*0.0172*	*0.0500*	*0.0245*	*0.0484*	*0.0292*		
1 week (9600)	244	113	408	182	542	180	484	260				
	0.0254	*0.0118*	*0.0425*	*0.0190*	*0.0565*	*0.0188*	*0.0504*	*0.0271*				
1 month (40320)	1146	556	1608	684	1955	647						
	0.0284	*0.0138*	*0.0399*	*0.0170*	*0.0485*	*0.0160*						
3 months (124800)	3259	1588	4221	1801								
	0.0261	*0.0127*	*0.0338*	*0.0144*								
6 months (236160)	5728	2781										
	0.0243	*0.0118*										

Table 4: Violations of VaR (99%) computed from ARMA(1,1)-GARCH(1,1) with different residuals.

	6 months (236160)		3 months (124800)		1 month (40320)		1 week (9600)		1 day (1920)		1 hour (240)	
	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN
1 hour (240)	2	0	5	2	1	1	9	2	4	1	3	0
	0.0083	0.0000	0.0208	0.0083	0.0042	0.0042	0.0375	0.0083	0.0167	0.0042	0.0125	0.0000
1 day (1920)	13	4	17	10	12	7	24	5	22	4		
	0.0068	0.0021	0.0089	0.0052	0.0063	0.0036	0.0125	0.0026	0.0115	0.0021		
1 week (9600)	40	10	62	25	69	36	114	30				
	0.0042	0.0010	0.0065	0.0026	0.0072	0.0038	0.0119	0.0031				
1 month (40320)	198	48	241	93	257	120						
	0.0049	0.0012	0.0060	0.0023	0.0064	0.0030						
3 months (124800)	593	188	637	268								
	0.0048	0.0015	0.0051	0.0021								
6 months (236160)	1055	354										
	0.0045	0.0015										

	6 months (236160)		3 months (124800)		1 month (40320)		1 week (9600)		1 day (1920)		1 hour (240)	
	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN
1 hour (240)	0	2	2	5	1	1	1	9	1	2	0	2
	0.0000	0.0089	0.0083	0.0208	0.0042	0.0042	0.0042	0.0375	0.0042	0.0083	0.0000	0.0083
1 day (1920)	4	13	11	18	6	11	2	25	4	20		
	0.0021	0.0068	0.0057	0.0094	0.0031	0.0057	0.0010	0.0130	0.0021	0.0104		
1 week (9600)	11	40	24	65	35	69	21	112				
	0.0011	0.0042	0.0025	0.0068	0.0036	0.0072	0.0022	0.0117				
1 month (40320)	58	197	90	241	133	248						
	0.0014	0.0049	0.0022	0.0060	0.0028	0.0062						
3 months (124800)	213	591	260	651								
	0.0017	0.0047	0.0021	0.0052								
6 months (236160)	388	1044										
	0.0016	0.0044										

Table 5: Admissible VaR violations and violation frequencies

		$T =$	violation $\alpha = 0.05$	violation frequency $\alpha = 0.05$	violation $\alpha = 0.01$	violation frequency $\alpha = 0.01$
	1 hour	240	[6,18]	[0.0250,0.0750]	[4,20]	[0.0166,0.0833]
	1 day	1920	[80,120]	[0.0417,0.0583]	[74,118]	[0.0385,0.0614]
95% VaR	1 week	9600	[445,515]	[0.0463,0.0536]	[430,530]	[0.0447,0.0552]
	1 month	40320	[1944,2088]	[0.0482,0.0517]	[1915,2118]	[0.0474,0.0525]
	3 months	124800	[6114,6366]	[0.0489,0.0510]	[6061,6419]	[0.0485,0.0514]
	6 months	236160	[11634,11982]	[0.0492,0.0507]	[11562,12054]	[0.0489,0.0510]
	1 hour	240	[0,5]	[0.0000,0.0208]	[0,6]	[0.0000,0.0250]
	1 day	1920	[12,26]	[0.0062,0.0135]	[9,29]	[0.0046,0.0151]
99%VaR	1 week	9600	[80,112]	[0.0083,0.0117]	[73,119]	[0.0076,0.0123]
	1 month	40320	[370,436]	[0.0092,0.0108]	[357,450]	[0.0088,0.0111]
	3 months	124800	[1190,1306]	[0.0095,0.0105]	[1167,1330]	[0.0093,0.0106]
	6 months	236160	[2282,2441]	[0.0097,0.0103]	[2249,2474]	[0.0095,0.0104]

Table 6: The p-value of Christoffersen test for in-sample VaR computed from ARMA(1,1)-GARCH(1,1) with different residuals. The p-value of Christoffersen test for 99%-VaR is shown by italic fonds.

	6 months (236160)		3 months (124800)		1 month (40320)		1 week (9600)		1 day (1920)		1 hour (240)	
	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN	Normal	FSN
1 hour	0.0331	0.0083	0.0520	0.0375	0.0367	0.0165	0.0458	0.0291	0.0416	0.0083	0.0421	0.0020
(240)	*0.0250*	*0.0000*	*0.0183*	*0.001*	*0.0041*	*0.0001*	*0.00375*	*0.0001*	*0.0125*	*0.0025*	*0.0041*	*0.0000*
1 day	0.0388	0.0094	0.0161	0.0197	0.0354	0.0192	0.0401	0.0223	0.0234	0.0021		
(1920)	*0.0063*	*0.0021*	*0.0125*	*0.0036*	*0.0093*	*0.0021*	*0.0098*	*0.0005*	*0.03416*	*0.00416*		
1 week	0.0152	0.0013	0.0104	0.0113	0.0157	0.0160	0.0304	0.0025				
(9600)	*0.0354*	*0.0145*	*0.0813*	*0.0022*	*0.0146*	*0.0028*	*0.0137*	*0.0017*				
1 month	0.0544	0.0008	0.0667	0.0074	0.0415	0.0404						
(40320)	*0.0231*	*0.0011*	*0.0054*	*0.0012*	*0.0085*	*0.0019*						
3 months	0.0080	0.0017	0.0051	0.0007								
(124800)	*0.0031*	*0.0010*	*0.0054*	*0.0012*								
6 months	0.0207	0.0152										
(236160)	*0.0129*	*0.0083*										

	6 months (236160)		3 months (124800)		1 month (40320)		1 week (9600)		1 day (1920)		1 hour (240)	
	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN	Stable	FGN
1 hour	0.0041	0.0233	0.0125	0.0366	0.0458	0.0691	0.0000	0.0066	0.0083	0.0533	0.0000	0.0004
(240)	*0.0000*	*0.0250*	*0.0000*	*0.0167*	*0.0000*	*0.0042*	*0.0416*	*0.0520*	*0.0083*	*0.0125*	*0.0000*	*0.0143*
1 day	0.0014	0.0093	0.0014	0.0187	0.0348	0.0392	0.0182	0.0223	0.0036	0.0171		
(1920)	*0.0016*	*0.0063*	*0.0031*	*0.0313*	*0.0021*	*0.0093*	*0.0005*	*0.0094*	*0.0020*	*0.0119*		
1 week	0.0169	0.0384	0.0180	0.0215	0.0182	0.0271	0.0270	0.0304				
(9600)	*0.0012*	*0.0035*	*0.0021*	*0.0082*	*0.0020*	*0.0114*	*0.0015*	*0.0124*				
1 month	0.0154	0.0280	0.0070	0.0141	0.0181	0.0215						
(40320)	*0.0010*	*0.0031*	*0.0012*	*0.0054*	*0.0012*	*0.0084*						
3 months	0.0154	0.0180	0.0092	0.0099								
(124800)	*0.0083*	*0.0231*	*0.0009*	*0.0035*								
6 months	0.0036	0.0071										
(236160)	*0.0007*	*0.0029*										