Private Equity Funds: Valuation, Systematic Risk and Illiquidity*

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Abstract

This paper is concerned with the question how the value and systematic risk of private equity funds evolve over time and how they are affected by illiquidity. First, we develop a continuous-time approach modeling the cash flow dynamics of private equity funds. Second, intertemporal asset pricing considerations are applied to derive the dynamics of the value, expected return and systematic risk of private equity funds over time under liquidity and illiquidity. Third, the closed-form solutions obtained are used to calibrate the model to a comprehensive sample of mature European private equity funds. Most importantly, the analysis of the calibrated model shows that: (i) the average private equity fund has a risk-adjusted excess value on the order of 24.80 percent relative to $1 committed; (ii) (upper boundary) illiquidity costs are around 2.49 percent of committed capital p.a. and illiquidity discounts of the fund values are increasing functions in the remaining lifetimes of the funds; (iii) expected fund returns are non-stationary as systematic fund risk and illiquidity premiums vary with distinct patterns over time.

Keywords:
Private equity, venture capital, continuous-time stochastic modeling, illiquidity, risk-neutral valuation, systematic risk, expected returns.

JEL Classification Code: G24, G12

1 Introduction

Investments in private equity have become an increasingly significant portion of institutional portfolios as investors seek diversification benefits relative to traditional stock and bond investments. Despite the increasing importance of the private equity asset class, we have only a limited understanding of the economics of private equity funds – the typical vehicle through which private equity investments are made. Particularly, three questions are mostly unresolved in the current private equity literature: (i) What is the value of a private equity fund and how does it develop over time? (ii) How does a fund's expected return and systematic risk change over time? (iii) How does illiquidity affect fund values and expected returns?

These questions are difficult to evaluate without models that explicitly tie these variables of interest to the cash flow dynamics of private equity funds. In this paper we provide such a model and use it to develop answers to the above-mentioned questions. Private equity funds differ from other managed funds because of their particular bounded life cycle. When the fund starts, the investors make an initial capital commitment. The fund manager then gradually draws down the committed capital into investments. Finally, returns and proceeds are distributed as the investments are realized and the fund is eventually liquidated as the final investment horizon is reached. Modeling private equity funds therefore requires two stages: modeling capital drawdowns and modeling capital repayments (or capital distributions) of the funds. This paper develops a new continuous-time approach modeling these two components. Specifically, a mean-reverting square-root process is applied to model the rate at which capital is drawn over time. Capital distributions are assumed to follow an arithmetic Brownian Motion with a time-dependent drift component that incorporates the typical time-pattern of the repayments of private equity funds.

We use our model to explore the dynamics of private equity funds in three ways. First, by applying equilibrium intertemporal asset pricing considerations, we endogenously infer the value of a fund through time as the difference between the present value of all outstanding future distributions and the present value of all outstanding future capital drawdowns. The closed-form expressions we obtain permit us to illustrate how the dynamic evolution of the value of a fund within our model is related to its cash flow dynamics and to other economic variables, such as the riskless rate of return or the correlation of the cash flows to the return of the market portfolio. Private equity funds are also characterized by illiquidity as there is no liquid and organized market where they can be traded. We capture this feature by a simple model extension that allows us to demonstrate the effect of illiquidity on fund values over time. This shows, for example, that the impact of illiquidity on fund values increases with the remaining lifetime of the fund. This result is in line with economic intuition.

Second, we employ our model to explore the dynamics of the expected return and systematic risk of private equity funds through time. We start by deriving analytical expressions for the conditional expected return and systematic risk of private equity funds. These expressions then allow us to evaluate directly how, within the model, these variables depend on the underlying economic characteristics of a fund and how they change over time. In particular, we find that the beta coefficient of the fund returns is simply the value weighted average of the betas of a fund's capital distributions and drawdowns. This is an important result, as it implies that the beta coefficient of the fund returns – and hence also the expected fund return – will, in general, be time-dependent. This result is foreshadowed in the finance literature by the works of Brennan (1973), Myers and Turnbull (1977) and Turnbull (1977) on the systematic risk of firms. However, we are the first to acknowledge the existence and importance of this effect for private equity funds. In addition, we also show that incorporating illiquidity into the analysis induces a second time-variable component into expected fund returns.

Third, the closed-form expressions we obtain permit us to calibrate the model to real fund data and analyze its empirical implications in detail. For the purpose of our empirical analysis, we use a comprehensive data set of European private equity funds that has been provided by Thomson Venture Economics (TVE). We first calibrate the model to the cash flow data of 203 mature funds and show that its fits historical data nicely. The calibrated model is then used to explore a number of interesting observations, some of which have not been documented in the private equity literature before. Regarding the value and illiquidity costs, several important observations are in order: (i) We find that private equity funds create excess value on a risk-adjusted basis. The results show that the risk-adjusted excess value (net-of-fees) of an average private equity fund in our sample is on the order of 24.80 percent relative to \$1 committed. That is, \$1 committed to a private equity funds is worth 1.2480 in present value terms. These excess values hold for both venture and buyout funds, though, in our sample buyout funds create slightly more value. (ii) By interpreting these excess values as compensation required by investors for illiquidity of the funds, we can implicitly derive illiquidity costs of the funds. Overall, the results suggest that (upper boundary) illiquidity costs are on the order of 2.49 percent of committed capital p.a. Interestingly, the results also imply that buyout funds have higher (upper boundary) illiquidity costs than venture funds. One possible explanation for this is that investors of buyout funds require higher compensation for illiquidity because of the larger size of the investments of these funds. (iii) We document how fund values and illiquidity discounts of the funds evolve over time. In particular, this shows how illiquidity discounts of private equity funds decrease over time.

The most important implications of our empirical analysis with respect to expected return and systematic risk of private equity funds are as follows: (i) We find that expected returns and systematic risk of private equity funds decrease over the lifetime of the funds. From an economic standpoint, this result follows as the structure of stepwise capital drawdowns of private equity funds acts like a financial leverage that increases the beta coefficients of the funds as long as the committed capital has not been completely drawn. (ii) The results show that venture funds have higher beta coefficients and therefore generate higher ex-ante expected returns than buyout funds. Specifically, the results suggest that the beta risk of venture funds is higher than the market and that the beta risk of buyout funds is substantially lower than the market for all times during the lifetime considered. (iii) We also document how illiquidity of private equity funds affects expected returns. As expected, the results show that illiquidity increases expected funds returns. This holds, in particular, at the start and at the end of a fund's lifetime, where increases in expected returns are highest. This follows as relative illiquidity costs are highest when fund values are low.

This paper is related to two branches of the private equity literature. First, it shares with a number of recent papers, notably Takahashi and Alexander (2002) and Malherbe (2004, 2005), the goal of developing models for the value and cash flow dynamics of private equity funds. Takahashi and Alexander (2002) carried out the first attempt to develop a model for the value and cash flow dynamics of a private equity fund. However, their model is fully deterministic and thus fails to reproduce the erratic nature of real private equity cash flows. Furthermore, as a deterministic model it does not allow calculating any risk parameters. Malherbe (2004, 2005) developed a continuous-time version of the deterministic model of Takahashi and Alexander (2002) and introduced stochastic components into the model. In specific, Malherbe (2004, 2005) uses a standard lognormal specification for the dynamics of the investment value. Squared Bessel processes are utilized for the dynamics of the rates of drawdowns and repayments over time. With this stochastic multi-factor model, Malherbe (2004, 2005) is able to reproduce the erratic nature of private equity cash flows. However, the model relies on the specification of the dynamics of the unobservable value of the fund's assets over time, where model parameters have to be estimated from the disclosed net asset values of the fund management. This can cause severe problems as the reported net asset values of the fund management might suffer from stale pricing problems and thus might not reflect the true market valuations.[1] The model developed here does not suffer from this drawback. This follows as we endogenously derive the value of a fund by using equilibrium intertemporal asset pricing considerations. The dynamics of our model are thus solely based on observable cash flows data, which seems to be a more promising stream for future research in the area private equity fund modeling. Furthermore, these models do not focus on how fund values and expected fund returns evolve over time and they do not analyze the impact of illiquidity on these variables. Both these features are central to our model.

Second, we relate to the literature on risk and return of private equity investments. Examples of this area of research include, among others, Cochrane (2005), Kaplan and Schoar (2005), Diller and Kaserer (2009), Ljungquist and Richardson (2003a,b), Metrick and Yasuda (2009), Moskowitz and Vissing-Jorgenson (2002), Peng (2001a,b) and Phalippou and Gottschalg (2009). These articles have in common that they deal with the risk and return characteristics of the private equity industry, either on a fund, individual deal or aggregate industry level. Our model contributes to this strand of literature by developing implications for the dynamics of expected returns and systematic risk of private equity funds through time. As pointed out above, we can show that the systematic risk of private equity funds will, in general, be time-dependent. This aspect that has not been pointed out in the private equity literature before. It is, for example, important with respect to portfolio optimization. If a constant systematic risk and expected return of a private equity fund over its lifetime are assumed, this could result in non-optimal investment decisions.

Finally, our paper is also related to the literature of liquidity risk. This strand of literature was pioneered by Amihud and Mendelson (1986) that show that investors require a higher return for investments that are more liquid than to otherwise similar assets that are liquid. More recent articles in this area include, for example, Amihud (2002), Pastor and Stambaugh (2003) and Acharya and Pedersen (2005). These articles almost focus exclusively on the effects of illiquidity on traded assets. In contrast, compensation for illiquidity of private equity funds is still a largely unresolved area of research. One exception is Ljungquist and Richardson (2003a) that estimate the risk-adjusted excess value of the typical private equity fund is on the order of

[1] Note that Malherbe (2004, 2005) tries to account for the inaccurate valuation of the fund management by incorporating an estimation error in his model. However, the basic problem remains under his model specification.

24 percent relative to the present value of the invested capital and interpret this as compensation for holding an illiquid investment. Our empirical results extend the evidence of significant compensation for illiquidity to private equity fund investors. In addition, we are the first to show how illiquidity affects fund values and expected returns over time.

The rest of the paper is organized as follows. In the next section we set forth the notation, assumptions, and structure of the model. Section 3 shows how the value of a private equity fund can be derived by using a risk-neutral valuation approach. Section 4 presents our expressions for the expected return and systematic risk of a private equity fund. In Section 5 we present the results of the model calibration and discuss the empirical implications of our model. The paper concludes with Section 6. Additional proofs and derivations are contained in Appendixes A and B. Details of our estimation methodology are outlined in Appendix C.

2 The Model

This section develops our new model for the cash flow dynamics of private equity funds. We start with a brief description that lays out the typical construction of private equity funds. This gives the motivation for our subsequent model that is composed of two independent components. In developing the stochastic model, we explicitly choose to work in a continuous-time framework. The reason for this approach is modeling convenience, which allows us to obtain analytical results that would be unavailable in discrete-time. We assume that all random variables introduced in the following are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and that all random variables indexed by t are measurable with respect to the filtration $\{\mathcal{F}_t\}$, representing the information commonly available to investors. After presenting our new model, a theoretical model analysis illustrates the influence of the various model parameters on the drawing and distribution process.

2.1 Institutional Framework

Investments in private equity are typically intermediated through private equity funds. Thereby, a private equity fund denotes a pooled investment vehicle whose purpose is to negotiate purchases of common and preferred stocks, subordinated debt, convertible securities, warrants, futures and other securities of companies that are usually unlisted. As the vast majority of private equity funds, the fund to be modeled here is organized as a limited partnership in which the private equity firm serves as the general partner (GP). The bulk of the capital invested in private equity funds is typically provided by institutional investors, such as endowments, pension funds, insurance companies, and banks. These investors, called limited partners (LPs), commit to provide a certain amount of capital to the private equity fund – the committed capital denoted as C. The GP then has an agreed time period in which to invest this committed capital – usually on the order of five years. This time period is commonly referred to as the commitment period of the fund and will be denoted by T_c in the following. In general, when a GP identifies an investment opportunity, it "calls" money from its LPs up to the amount committed, and it can do so at any time during the prespecified commitment period. That is, we assume that capital calls of the fund occur unscheduled over the commitment period T_c, where the exact timing does only depend on the investment decisions of the GPs.

However, total capital calls over the commitment period T_c can never exceed the total committed capital C. The capital calls are also called drawdowns or takedowns. As those drawdowns occur, the available cash is immediately invested in managed assets and the portfolio begins to accumulate. When an investment is liquidated, the GP distributes the proceeds to its LPs either in marketable securities or in cash. The GP also has an agreed time period in which to return capital to the LPs – usually on the order of ten to fourteen years. This time period is also called the total legal lifetime of the fund and will be referred to by T_l in the following, where obviously $T_l \geq T_c$ must hold. In total, the private equity fund to be modeled is essentially a typical closed-end fund with a finite lifetime.[2]

Following the construction of the private equity fund outlined above, our stochastic model of the cash flow dynamics consists of two components that are modeled independently: the stochastic model for the drawdowns of the committed capital and the stochastic model of the distribution of dividends and proceeds.

2.2 Capital Drawdowns

We begin by assuming that the fund to be modeled has a total initial committed capital given by C, as defined above. Cumulated capital drawdowns from the *LPs* up to some time t during the commitment period T_c are denoted by D_t, undrawn committed capital up to time t by U_t. When the fund is set up, at time $t = 0$, $D_0 = 0$ and $U_0 = C$ are given by definition. Furthermore, at any time $t \in [0, T_c]$, the simple identity

$$D_t = C - U_t \tag{2.1}$$

must hold. In the following, we assume capital to be drawn over time at some non-negative rate from the remaining undrawn committed capital $U_t = C - D_t$.

Assumption 2.1 *Capital drawdowns over the commitment period T_c occur in a continuous-time setting. The dynamics of the cumulated drawdowns D_t can be described by the ordinary differential equation (ODE)*

$$dD_t = \delta_t U_t 1_{\{0 \leq t \leq T_c\}} dt, \tag{2.2}$$

where $\delta_t \geq 0$ denotes the rate of contribution, or simply the fund's drawdown rate at time t and $1_{\{0 \leq t \leq T_c\}}$ is an indicator function.

In most cases, capital drawdowns of private equity funds are heavily concentrated in the first few years or even quarters of a fund's life. After high initial investment activity, drawdowns of private equity funds are carried out at a declining rate, as fewer new investments are made, and follow-on investments are spread out over a number of years. This typical time-pattern of the capital drawdowns is well reflected in the structure of equation (2.2). Under the specification (2.2), cumulated capital drawdowns D_t are given by

$$D_t = C - C \ \exp\left(- \int_0^{t \leq T_c} \delta_u du \right) \tag{2.3}$$

[2] For a more thorough introduction on the subject of private equity funds, for example, refer to Gompers and Lerner (1999), Lerner (2001) or to the recent survey article of Phalippou (2007).

and instantaneous capital drawdowns $d_t = dD_t/dt$, i.e. the (annualized) capital drawdowns that occur over an infinitesimally short time interval from t to $t + dt$, are equal to

$$d_t = \delta_t C \exp\left(-\int_0^{t \leq T_c} \delta_u du\right) 1_{\{0 \leq t \leq T_c\}}. \tag{2.4}$$

Equation (2.4) shows that the initially very high capital drawdowns d_t at the start of the fund converge to zero for $t = T_c \to +\infty$. This follows as the undrawn amounts, $U_t = C \exp\left(-\int_0^{t \leq T_c} \delta_u du\right)$, decay exponentially over time. A condition that leads to the realistic feature that capital drawdowns are highly concentrated in the early times of a fund's life under this specification. Furthermore, equation (2.3) shows that the cumulated drawdowns D_t can never exceed the total amount of capital C that was initially committed to the fund under this model setup, i.e., $D_t \leq C$ for all $t \in [0, T_c]$. At the same time the model also allows for a certain fraction of the committed capital C not to be drawn, as the commitment period T_c acts as a cut-off point for capital drawdowns.

Usually, the capital drawdowns of real world private equity funds show an erratic feature, as investment opportunities do not arise constantly over the commitment period T_c. A stochastic component can easily be introduced into the model by defining a continuous-time stochastic process for the drawdown rate δ_t.

Assumption 2.2 *We model the drawdown rate by a stochastic process $\{\delta_t, 0 \leq t \leq T_c\}$, adapted to the stochastic base $(\Omega, \mathcal{F}, \mathbb{P})$ introduced above. The mathematical specification under the objective probability measure $\mathbb{P}$ is given by*

$$d\delta_t = \kappa(\theta - \delta_t)dt + \sigma_\delta \sqrt{\delta_t} dB_{\delta,t}, \tag{2.5}$$

where $\theta > 0$ is the long-run mean of the drawdown rate, $\kappa > 0$ governs the rate of reversion to this mean and $\sigma_\delta > 0$ reflects the volatility of the drawdown rate; $B_{\delta,t}$ is a standard Brownian motion.

This process is known in the financial literature as a square-root diffusion.[3] The drawdown rate behavior implied by the structure of this process has the following two relevant properties: (i) Negative values of the drawdown rate are precluded under this specification.[4] This is a necessary condition, as we model capital distributions and capital drawdowns separately and must, therefore, restrict capital drawdowns to be strictly non-negative at any time t during the commitment period T_c. (ii) Furthermore, the mean-reverting structure of the process reflects the fact that we assume the drawdown rate to fluctuate randomly around some mean level θ over time.

Under the specification of the square-root diffusion (2.5), the conditional expected cumulated and instantaneous capital drawdowns can be inferred. Given that

[3] It was initially proposed by Cox et al. (1985) as a model of the short rate, generally referred to as the CIR model. Apart from interest rate modeling, this process also has other financial applications. For example, Heston (1993) proposed an option pricing in which the volatility of asset returns follows a square-root diffusion. In addition, the process (2.5) is sometimes used to model a stochastic intensity for a jump process in, for example, modeling default probabilities.

[4] If $\kappa, \theta > 0$, then δ_t will never be negative; if $2\kappa\theta \geq \sigma_\delta^2$, then δ_t remains strictly positive for all t, almost surely. See Cox et al. (1985), p. 391.

$E_s[\cdot]$ denotes the expectations operator conditional on the information set available at time s, expected cumulated drawdowns at some time $t \geq s$ are given by

$$
\begin{aligned}
E_s[D_t] &= C - U_s\ E_s\left[\exp\left(-\int_s^t \delta_u du\right)\right] \\
&= C - U_s \exp[A(s,t) - B(s,t)\delta_s],
\end{aligned} \tag{2.6}
$$

where $A(s,t)$ and $B(s,t)$ are deterministic functions that are given by:[5]

$$
\begin{aligned}
A(s,t) &\equiv \frac{2\kappa\theta}{\sigma_\delta^2} \ln\left[\frac{2fe^{[(\kappa+f)(t-s)]/2}}{(\kappa+f)(e^{f(t-s)}-1)+2f}\right], \\
B(s,t) &\equiv \frac{2(e^{f(t-s)}-1)}{(f+\kappa)(e^{f(t-s)}-1)+2f}, \\
f &\equiv \left(\kappa^2 + 2\sigma_\delta^2\right)^{1/2}.
\end{aligned} \tag{2.7}
$$

The expected instantaneous capital drawdowns, $E_s[d_t] = E_s[dD_t/dt]$, are given by

$$
\begin{aligned}
E_s[dD_t/dt] &= \frac{d}{dt}E_s[D_t] = \\
&= -U_s[A'(s,t) - B'(s,t)\delta_s]\exp[A(s,t) - B(s,t)\delta_s],
\end{aligned} \tag{2.8}
$$

where $A'(s,t) = \partial A(s,t)/\partial t$ and $B'(s,t) = \partial B(s,t)/\partial t$.

We are now equipped with the first component of our model. The following section turns to the modeling of the capital distributions of private equity funds

2.3 Capital Distributions

As capital drawdowns occur, the available capital is immediately invested in managed assets and the portfolio of the fund begins to accumulate. As the underlying investments of the fund are gradually exited, cash or marketable securities are received and finally returns and proceeds are distributed to the LPs of the fund. We assume that cumulated capital distributions up to some time $t \in [0, T_l]$ during the legal lifetime T_l of the fund are denoted by P_t and $p_t = dP_t/dt$ denotes the instantaneous capital distributions, i.e., the (annualized) capital distributions that occur over infinitesimally short time interval from t to $t + dt$.

We model distributions and drawdowns separately and, therefore, must also restrict instantaneous capital distributions p_t to be strictly non-negative at any time $t \in [0, T_l]$. The second constraint that needs to be imposed on the distributions model is the addition of a stochastic component that allows a certain degree of irregularity in the cash outflows of private equity funds. An appropriate assumption that meets both requirements is that the logarithm of instantaneous capital distributions, $\ln p_t$, follows an arithmetic Brownian motion.

Assumption 2.3 *Capital distributions over the legal lifetime T_l of the fund occur in continuous-time. Under the objective probability measure $\mathbb{P}$, the logarithm of the instantaneous capital distributions, $\ln p_t$, follows an arithmetic Brownian motion of the form*

$$
d\ln p_t = \mu_t dt + \sigma_P dB_{P,t}, \tag{2.9}
$$

[5] See Cox et al. (1985), p.393.

where μ_t denotes the time dependent drift and σ_P the constant volatility of the stochastic process; $B_{P,t}$ is a second standard Brownian motion, which – for simplicity – is assumed to be uncorrelated with $B_{\delta,t}$, i.e., $d\langle B_{P,t}B_{\delta,t}\rangle = 0$.[6]

From (2.9), it follows that the instantaneous capital distribution p_t must exhibit a lognormal distribution. Therefore, the process (2.9) has the relevant property that it precludes instantaneous capital distributions p_t from becoming negative at any time $t \in [0, T_l]$, and is therefore an economically reasonable assumption. For an initial value p_s, the solution to the stochastic differential equation (2.9) is given by

$$p_t = p_s \exp\left[\int_s^t \mu_u du + \sigma_P(B_{P,t} - B_{P,s})\right], \qquad t \geq s. \tag{2.10}$$

Taking the expectation $E_s[\cdot]$ of (2.10), conditional on the available information at time $s \leq t$, yields

$$E_s[p_t] = p_s \exp\left[\int_s^t \mu_u du + \frac{1}{2}\sigma_P^2(t-s))\right]. \tag{2.11}$$

The dynamics of (2.10) and (2.11) both depend the specification of the integral over the time-dependent drift μ_t. The question posed now is to find a reasonable and parsimonious yet realistic way to model this parameter.

Defining an appropriate function for μ_t is not an easy task, as this parameter must incorporate the typical time pattern of the capital distributions of a private equity fund. In the early years of a fund, capital distributions tend to be of minimal size as investments have not had the time to be harvested. The middle years of a fund, on average, tend to display the highest distributions as more and more investments can be exited. Finally, later years are marked by a steady decline in capital distributions as fewer investments are left to be harvested. We model this behavior by first defining a fund multiple. If C denotes the committed capital of the fund, the fund multiple M_t at some time t is given by $M_t = P_t/C$, i.e., the cumulated capital distributions P_t are scaled by C. This variable will follow a continuous-time stochastic process as the multiple can also be expressed as $M_t = \int_0^t p_u du/C$. When the fund is set up, i.e. at time $t = 0$, $M_0 = 0$ holds by definition. As more and more investments of the fund are exited, the multiple increases over time. We assume that its expectation converges towards some long-run mean m. In specific, our modeling assumption can be stated as follows:

Assumption 2.4 *Let $\mathcal{M}_t^s = E_s^{\mathbb{P}}[M_t]$ denote the conditional expectation of the multiple at time t, given the available information at time $s \leq t$. We assume that the dynamics of $\mathcal{M}_t^s$ can be described by the ordinary differential equation (ODE)*

$$d\mathcal{M}_t^s = \alpha_t(m - \mathcal{M}_t^s)dt, \tag{2.12}$$

where m is the long-run mean of the expectation and $\alpha_t = \alpha t$ governs the speed of reversion to this mean.

Solving for $\mathcal{M}_t^s$ by using the initial condition $\mathcal{M}_s^s = M_s$ yields

$$\mathcal{M}_t^s = m - (m - M_s)\exp\left[-\frac{1}{2}\alpha(t^2 - s^2)\right]. \tag{2.13}$$

[6]This simplifying assumption can easily be relaxed to incorporate a positive or negative correlation coefficient ρ between the two processes.

With the condition, $p_t = (dM_t/dt)C$, the expected instantaneous capital distributions $E_s[p_t] = (d\mathcal{M}_t^s/dt)C$ turn out to be

$$E_s\left[p_t\right] = \alpha\ t(m\ C - P_s)\exp\left[-\frac{1}{2}\alpha(t^2 - s^2)\right]. \tag{2.14}$$

With equations (2.11) and (2.14) we are now equipped with two equations for the expected instantaneous capital distributions. Setting (2.11) equal to (2.14), we can solve for the integral $\int_s^t \mu_u du$. Substituting the result back into equation (2.10), the stochastic process for the instantaneous capital distributions at some time $t \geq s$ is given by

$$p_t = \alpha t(mC - P_s)\exp\left\{-\frac{1}{2}[\alpha(t^2 - s^2) + \sigma_P^2(t-s)] + \sigma_P\epsilon_t\sqrt{t-s}\right\}, \tag{2.15}$$

where $\epsilon_t\sqrt{t-s} = (B_{P,t} - B_{P,s})$ and $\epsilon_t \sim N(0,1)$. The structure of this process implies that high capital distributions in the past decrease average future capital distributions. This follows as the expression in the first bracket, $(mC - P_s)$, decreases with increasing levels of cumulated capital distributions P_s up to time s. This assumption can easily be relaxed by making the long run multiple m also dependent on the available information at time s. The stochastic process (2.15) can directly be used as a Monte-Carlo engine to generate sample paths of the capital distributions of a fund. In the next section, we illustrate the dynamics of both model components and analyze their sensitivity to changes in the main model parameters.

2.4 Model Analysis

The purpose of this section is to illustrate the model dynamics and to evaluate the model's ability to reproduce qualitatively some important features that can be observed from the drawdown and distribution patterns of real world private equity funds. First, we examine the influence of the various model parameters on the dynamics of the drawing and distribution process. Second, the model dynamics are illustrated by considering two hypothetical funds.

Our model consists of two independent components that are governed by different model parameters. The only parameter that enters both model components is the committed capital C. This variable does not affect the timing of the capital drawdowns or distributions. Rather, it serves as a scaling factor to influence the magnitude of the overall expected cumulative drawdowns and distributions. As far as the capital drawdowns are concerned, the main model parameter governing the timing of the drawing process is the long-run mean drawdown rate θ. Increasing θ accelerates expected drawdowns over time. Thus, higher values of θ, on average, increase the capital drawn at the start of the fund and decreases the capital drawn in later phases of the fund's lifetime – a behavior which is in line with intuition. Compared to the impact of θ, the influence of the mean reversion coefficient κ and the volatility σ_δ on the expected drawing process are only small. In general, the effect of both parameters is about the same relative magnitude. However, the direction may differ in sign. Increases in σ_δ tend to slightly decelerate expected drawdowns, whereas increases in κ tend to slightly accelerate them. In contrast, σ_δ is the main model parameter governing the volatility of the capital drawdowns. The higher σ_δ, the more erratic the capital drawdowns will be over time. In addition, the volatility of the capital drawdowns is also influenced by the mean reversion coefficient κ.

Thereby, high values of κ tend to decrease the volatility of the capital drawdowns, as a high levels of mean reversion "kill" some volatility of the drawdown rate.

The timing and magnitude of the capital distributions is determined by three main model parameters. The coefficient m is the long-run multiple of the fund, i.e., m times the committed capital C determines the total amount of capital that is expected to be returned to the investors over the fund's lifetime. The higher m, the more capital per dollar committed is expected to be paid out. The coefficient σ_P governs the volatility of the capital distributions. Higher values of σ_P, hence, lead to more erratic capital distributions over time. Finally, α governs the speed at which capital is distributed over the fund's lifetime. To make this parameter easier to interpret, it can simply be related to the expected amortization period of a fund. Let t_A denotes the expected amortization period of the fund, i.e., the expected time needed until the cumulated capital distributions are equal to or exceed the committed capital C of the fund, then it follows from equation (2.13) that

$$E_0[M_{t_A}] \equiv 1 = m \left[1 - \exp\left(-\frac{1}{2} \cdot \alpha \cdot t_A^2\right)\right] \tag{2.16}$$

must hold. Solving for α gives

$$\alpha = \frac{2 \ln \frac{m}{m-1}}{t_A^2}. \tag{2.17}$$

That is, α is inversely related to the expected amortization period t_A of the fund. Consequently, higher values of α lead to shorter expected amortization periods.

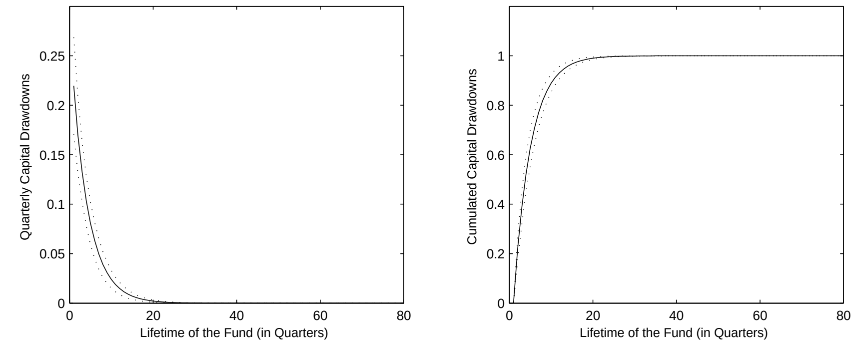
Table 1: Model Parameters for the Capital Drawdowns and Distributions of Two Different Funds

Table 1 shows the sample model parameters for the capital drawdowns and distributions of two different funds. The committed capital C of both funds is standardized to 1. In addition, the starting values of the drawdown rates δ_0 are set equal their corresponding long-run means θ.

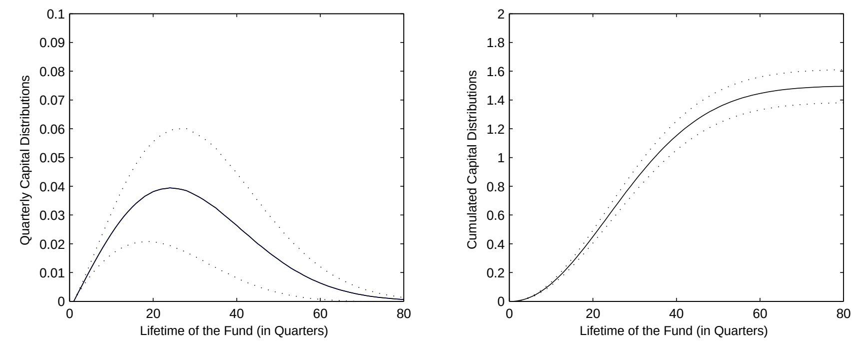
Model	Drawdowns			Distributions		
	κ	θ	σ_δ	m	α	σ_P
Fund 1	2.00	1.00	0.50	1.50	0.03	0.20
Fund 2	0.50	0.50	0.70	1.50	0.06	0.30

To illustrate the model dynamics, Figures 1 and 2 compare the expected cash flows (drawdowns, distributions and net fund cash flows) and standard deviations for two different hypothetical funds.[7] As the different sets of parameter values in Table 1 reveal, both hypothetical funds are assumed to have the same long-run multiple m and a committed capital C that is standardized to 1. That is, both funds are assumed to have cumulated capital drawdowns equal to 1 over their lifetime and expected cumulated capital distributions equal to 1.5. However, they differ in the timing and volatility of the capital drawdowns and distributions, as indicated by the different values of the other model parameters. For the first fund it is assumed

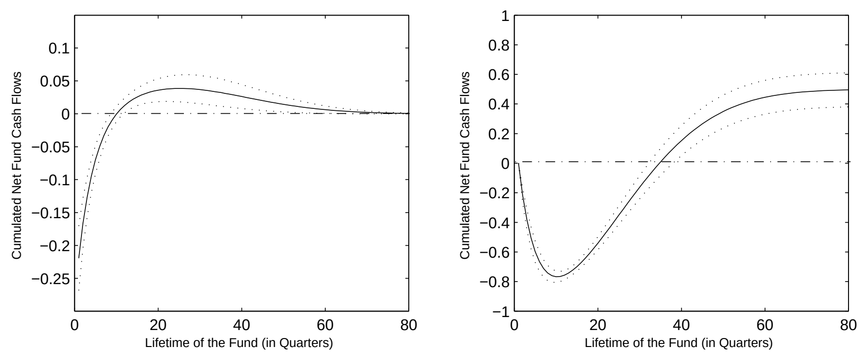
[7] The corresponding expectations are obtained from equations (2.6) and (2.8) for the capital drawdowns and from equations (2.13) and (2.14) for the capital distributions. In addition, note that unconditional expectations are shown, i.e., we set $s = 0$.



(a) Expected Quarterly Capital Drawdowns (Left) and Cumulated Capital Drawdowns (Right)

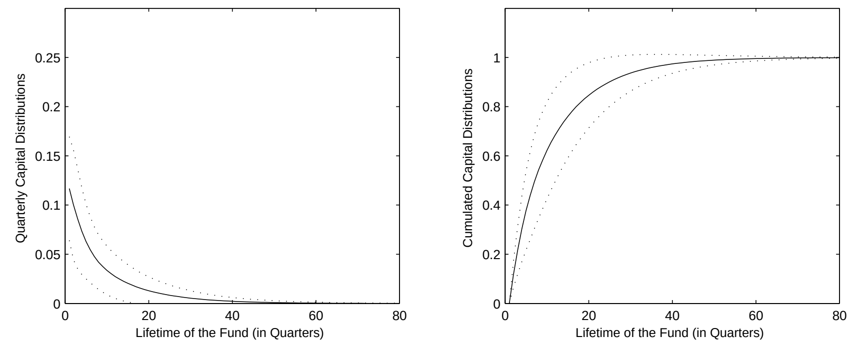


(b) Expected Quarterly Capital Distributions (Left) and Cumulated Capital Distributions (Right)

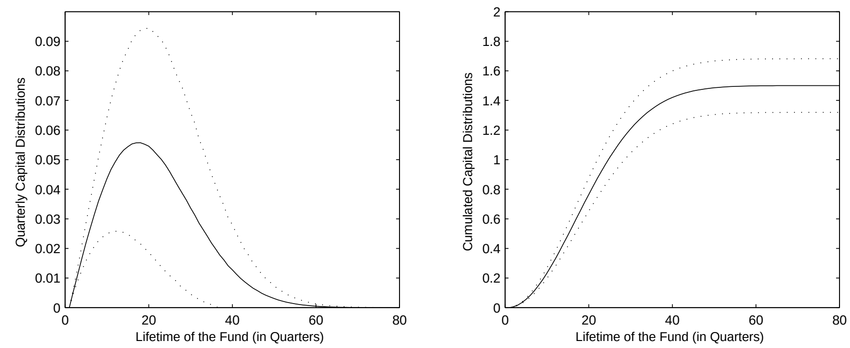


(c) Expected Quarterly Net Fund Cash Flows (Left) and Cumulated Net Fund Cash Flows (Right)

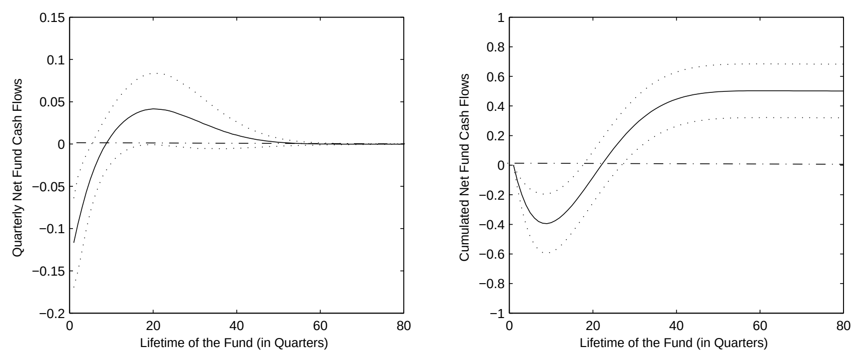
Figure 1: Model Expectations for Fund 1 (Solid Lines represent Expectations; Dotted Lines represent Expectations $\pm$ Standard Deviations)



(a) Expected Quarterly Capital Drawdowns (Left) and Cumulated Capital Drawdowns (Right)



(b) Expected Quarterly Capital Distributions (Left) and Cumulated Capital Distributions (Right)



(c) Expected Quarterly Net Fund Cash Flows (Left) and Cumulated Net Fund Cash Flows (Right)

Figure 2: Model Expectations for Fund 2 (Solid Lines represent Expectations; Dotted Lines represent Expectations $\pm$ Standard Deviations)

that drawdowns occur rapid in the beginning, whereas capital distributions take place late. Conversely, for the second fund it is assumed that drawdowns occur more progressive and that distributions take place sooner. This is mainly achieved through a lower value of α and a higher value of θ for Fund 1. The effect can be inferred by comparing the different model expectations in Figures 1 and 2. For this reason, both funds also have different expected amortization periods. From equation (2.17), the expected amortization periods of Funds 1 and 2 are given by 8.6 years and 6.1 years, respectively. In addition, the cash flows of the two funds are also assumed to have different volatilities. Thereby, the capital drawdowns and distributions of Fund 2 exhibit higher variability as can be observed by comparing the standard deviations displayed in Figures 1 and 2.

It is important to acknowledge that the basic patterns of the model graphs of the capital drawdowns, distributions and net cash flows in Figures 1 and 2 conform to reasonable expectations of private equity fund behavior. In particular, the cash flow streams that the model can generate will naturally exhibit a lag between the capital drawdowns and distributions, thus reproducing the typical development cycle of a fund and leading to the private equity characteristic J-shaped curve for the cumulated net cash flows that can be observed on the right of Figures 1 (c) and 2 (c). Furthermore, it is important to stress that our model is flexible enough to generate the potentially many different patterns of capital drawdowns and distributions. By altering the main model parameters, both timing and magnitude of the fund cash flows can be controlled in the model. Finally, our model captures well the erratic nature of real world private equity fund cash flows. This is illustrated in Figures 1 and 2 by the standard deviations shown. The results show that our model incorporates the economically reasonable feature that volatility of the fund cash flows varies over time. Specifically, the volatility of the drawdowns (distributions) is high in times when average drawdowns (distributions) are high, and low otherwise.

So far, we our analysis was focussed on the cash flows of private equity funds. Modeling private equity funds also requires a third ingredient: the valuation of private equity funds, which is considered in the following section.

3 Valuation

In this section we derive the value of a private equity fund. The valuation of private equity funds is complicated by the fact that the state variables underlying the valuation, the cash flow processes of private equity funds, do not represent traded assets. Therefore, we are dealing with an incomplete market setting and a preference-free pricing, which based on arbitrage considerations alone, is not feasible. For this reason, we have to impose additional assumptions on preferences of the investors in the economy. We first outline the assumptions underlying our valuation. The valuation results are presented and illustrated thereafter. Our basic valuation framework assumes that private equity funds are traded in a liquid market that is free of arbitrage. We then relax this assumption by a model extension that incorporates illiquidity of private equity funds into the valuation.

3.1 Assumptions

Our subsequent valuation framework for private equity funds is based on four major assumptions that are outlined and discussed in the following:

Assumption 3.1 *Absence of Arbitrage: Assume that the private equity funds considered here are traded in a liquid, frictionless market that is free of arbitrage.*

This assumption seems to be in contradiction to reality where no organized and liquid markets for private equity funds exist. However, even if no market exists our results can be used as an upper price boundary for private equity funds. In addition, we relax this assumption in following by incorporating illiquidity of private equity funds into the valuation.

In a complete market setting, the price of any new financial claim is uniquely determined by the requirement of Assumption 3.1. This follows because in a complete market setting, every new financial claim can be perfectly replicated by a portfolio of traded securities and thus pricing by arbitrage considerations alone is feasible. Since the capital distributions and drawdowns are not assumed to be spanned by the assets in the economy, the risk factors underlying our model cannot be eliminated by arbitrage considerations. Therefore, we are dealing in an incomplete market setting. In an incomplete market the requirement of no arbitrage alone is no longer sufficient to determine a unique price for a financial claim. The reason for this is that several different price systems may exist, all of which are consistent with the absence of arbitrage. In general, prices of financial claims result from balancing supply and demand among agents who trade to optimize their lifetime investment and consumption. Thus, unique prices of financial claim in incomplete markets emerge as a consequence of investor's preferences and not just as a constraint to preclude arbitrage. Therefore, our valuation requires an additional assumption on investor's preferences in the economy.

Assumption 3.2 *Investor Preferences: All investors have (identical) time-additive von Neumann-Morgenstern utility functions of the logarithmic form defined over the rate of consumption of a single consumption good.*

Using this assumption, Merton (1973) has shown that the equilibrium expected security returns will satisfy the specialized version of the intertemporal capital asset pricing model:[8]

$$\mu_i - r_f = \sigma_{iW}, \tag{3.1}$$

where μ_i is the expected instantaneous rate of return on some asset i, σ_{iW} is the covariance of the return on asset i with the return of the market portfolio W, and r_f is the instantaneously riskless interest rate. By μ_W and σ_W we denote the expected return and standard deviation of the market portfolio that are both assumed to be constant. It is this general equilibrium model that we employ in the following to derive a valuation model for private equity funds.

Using this general equilibrium model for valuation of private equity funds requires two additional assumptions on the correlation structure of the risk factors underlying our model with the returns on the market portfolio W.

Assumption 3.3 *Correlation of Capital Drawdowns: The drawdown rate δ_t is uncorrelated with returns of the market portfolio W, i.e., $\rho_{\delta W} = 0$.*

[8] This specialization arises from the assumption of logarithmic utility, which permits us to omit all additional terms relating to stochastic shifts in the investment opportunity set that would arise in (3.1) under a more general specification of investor's preferences. See Merton (1973) and Brennan and Schwartz (1982) for a detailed derivation and discussion.

For simplicity, it is assumed that changes in the drawdown rates of private equity funds are uncorrelated with returns of the market portfolio. This is not an unreasonable assumption and basically means that the drawdown rate carries zero systematic risk.[9] Stated differently, we assume that all changes in the capital drawdowns of private equity funds constitute idiosyncratic risk that is not priced in the model economy.

Assumption 3.4 *Correlation of Capital Distributions: There is a constant correlation, $\rho_{PW} \in [0,1]$, between changes in capital distributions and returns of the market portfolio W.*

In contrast to the capital drawdowns, we allow for systematic risk of changes in capital distributions by assuming an arbitrary correlation $\rho_{PW} \in [0,1]$ between changes in capital distributions and returns of the market portfolio.

It is these four assumptions that we employ for our following valuation of private equity funds.

3.2 Liquid Case

Using our stochastic models for the capital drawdowns and distributions, we can now derive the value of a fund over its lifetime. The value V_t^F of a private equity fund at time $t \in [0, T_l]$ is defined as the discounted value of the future cash flows, including all capital distributions and drawdowns, of the fund. From Assumption 3.1, the arbitrage free value of a private equity fund can be stated as

$$\begin{aligned} V_t^F &= E_t^{\mathbb{Q}}\left[\int_t^{T_l} e^{-r_f(\tau - t)} dP_\tau\right] - E_t^{\mathbb{Q}}\left[\int_t^{T_l} e^{-r_f(\tau - t)} dD_\tau\right] 1_{\{t \leq T_c\}} \\ &= V_t^P - V_t^D. \end{aligned} \tag{3.2}$$

That is, the market value V_t^F is simply given by the difference between the present value of all capital distributions V_t^P and the present value of all capital drawdowns V_t^D. To assure that discounting by the riskless rate r_f is valid in equation (3.2), all expectations are now defined under risk-neutral (or equivalent martingale) measure $\mathbb{Q}$. Valuing private equity funds therefore involves two steps. First, the risk sources underlying our model have to be transformed under the equivalent martingale measure $\mathbb{Q}$. Second, the conditional expectations in (3.2) have to be determined under this transformed probability measure.

Applying Girsanov's Theorem, it follows that the underlying stochastic processes for the capital drawdowns and distributions under the $\mathbb{Q}$-measure are given by

$$d\delta_t = [\kappa\ (\theta - \delta_t) - \lambda_\delta \sigma_\delta \sqrt{\delta_t}]\ dt + \sigma_\delta\ \sqrt{\delta_t}\ dB_{\delta,t}^{\mathbb{Q}}, \tag{3.3}$$

$$d \ln p_t = (\mu_t - \lambda_P \sigma_P) dt + \sigma_P dB_{P,t}^{\mathbb{Q}}, \tag{3.4}$$

where $B_{\delta,t}^{\mathbb{Q}}$ and $B_{P,t}^{\mathbb{Q}}$ are $\mathbb{Q}$-Brownian motions;[10] λ_δ and λ_P are market prices of risk, defined by

$$\lambda_\delta \equiv \frac{\mu(\delta_t, t) - r_f}{\sigma(\delta_t, t)}, \qquad \lambda_P \equiv \frac{\mu(\ln p_t, t) - r_f}{\sigma(\ln p_t, t)}. \tag{3.5}$$

[9] Some empirical evidence for this assumption is provided by Ljungquist and Richardson (2003a,b). Ljungquist and Richardson (2003a,b) show that the rate at which private equity funds draw down capital is not correlated with conditions in the public equity markets.

[10] For a detailed derivation and discussion of Girsanov's Theorem see, for example, Duffie (2001).

We can derive these two market prices of risk by using the intertemporal CAPM that arises from Assumption 3.2. Additionally imposing Assumption 3.3, we have $\lambda_\delta = 0$. Similarly, with the additional Assumption 3.4, it follows $\lambda_P = \sigma_{PW}/\sigma_P$, where $\sigma_{PW} = \sigma_P \sigma_W \rho_{PW}$ and σ_W is the constant standard deviation of the market portfolio returns. Inserting these results into equations (3.3) and (3.4) finally gives the two processes under the risk-neutral measure $\mathbb{Q}$.

Solving the conditional expectations in (3.2) by using these transformed processes gives the following theorem for the arbitrage free value of a private equity fund.

Theorem 3.1 *The arbitrage free value of a private equity fund at any time $t \in [0, T_l]$ during its finite lifetime T_l can be stated as*

$$V_t^F = \alpha\ (m\ C - P_t) \int_t^{T_l} e^{-r_f(\tau - t)} D(t,\tau)\ d\tau \\ + U_t \int_t^{T_l} e^{-r_f(\tau - t)} C(t,\tau) d\tau 1_{\{t \le T_c\}}, \tag{3.6}$$

where $C(t,\tau)$ and $D(t,\tau)$ are deterministic functions given by

$$C(t,\tau) = (A'(t,\tau) - B'(t,\tau)\delta_t) \exp[A(t,\tau) - B(t,\tau)\delta_t],$$
$$D(t,\tau) = \exp\left[\ln\tau - \frac{1}{2}\alpha(\tau^2 - t^2) - \sigma_{PW}(\tau - t)\right],$$

and $A(t,\tau)$, $B(t,\tau)$ are as stated in condition (2.7).

PROOF: see Appendix A.

Note that except for the two integrals that can easily be evaluated by using numerical techniques, Theorem 3.1 provides an analytically tractable solution for the arbitrage free value of a private equity fund.

Figure 3 illustrates the dynamics of the theoretical market value over a fund's lifetime for varying values of the correlation coefficient ρ_{PW}.[11] The model inputs used are given as follows: $C = 1$, $T_c = T_l = 20$, $r_f = 0.05$, $\kappa = 0.5$, $\theta = 0.5$, $\sigma_\delta = 0.1$, $\delta_0 = 0.01$, $m = 1.5$, $\alpha = 0.025$ and $\sigma_P = 0.5$. In addition, the standard deviation of the market portfolio return is assumed to be $\sigma_W = 0.2$. It is important here to stress that the basic time patterns of the fund values shown in Figure 3 conform to reasonable expectations of private equity fund behavior. Specifically, the value of a fund increases over time as the investment portfolio is build up and decreases towards the end as fewer and fewer investments are left to be harvested. In the context of our model, this characteristic behavior follows mainly from the fact that capital drawdown occur, on average, earlier than the capital distributions (see also Figures 1 and 2 above). Hence, the present value of outstanding capital drawdowns decreases faster over the fund's lifetime than the present value of outstanding capital distributions. Figure 3 also illustrates the effect of the correlation coefficient ρ_{PW} on the fund values. A positive correlation

[11] Note that for all $t > 0$, Figure 3 shows the unconditional expectations of the fund values, that is $V_t^F = E[V_t^F]$ for all $t > 0$. This is done here for illustrative purposes. Otherwise, the fund values would certainly be stochastic processes over time as the value of a fund at some time t in Theorem 3.1 does also depend on the cumulated capital distributions P_t and the undrawn amounts U_t at time t.

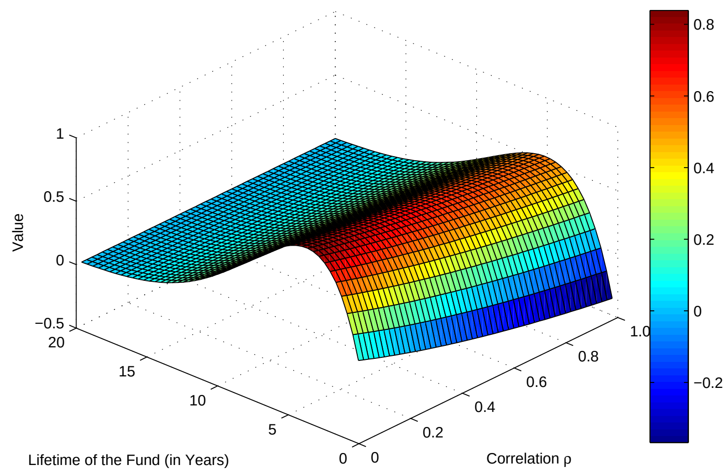


Figure 3: Market Value over the Fund's Lifetime for Varying Values of the Correlation Coefficient ρ_{PW}

ρ_{PW} implies that high capital distribution especially occur in states of the world in which the return of the market portfolio, and consequently aggregate wealth, is also high. This is an unfavorable condition for a rational investor seeking attractive risk-diversification benefits. Therefore, Figure 3 shows that the fund values slightly decrease with rising values of ρ_{PW}.[12]

3.3 Illiquid Case

In the preceding section, we have derived a valuation formula under the assumption that private equity funds are traded in a frictionless market that is free of arbitrage. This assumption seems to be in contradiction to reality where no organized and liquid markets for private equity funds exist. The impact of illiquidity on asset prices has been the subject of numerous empirical and theoretical studies.[13] In general, both the theory and the empirical evidence suggest that investors attach a lower price to assets that are more illiquid than to otherwise similar assets that are liquid. Thus, if private equity investors value liquidity, they will discount the value of a private equity fund for illiquidity. Using this line of argument, we define the fund value under illiquidity $V_t^{F,ill}$ as

$$V_t^{F,ill} = V_t^F - C_t^{ill}, \tag{3.7}$$

where C_t^{ill} is the illiquidity discount of the fund at time t and V_t^F is the arbitrage free value of a liquid fund as defined above. Following Amihud and Mendelson (1986), C_t^{ill} is the present value of all illiquidity costs of the fund during its remaining

[12] One can easily infer that this relationship would be reversed in the case that $\rho_{PW} < 0$.

[13] See, for example, Amihud and Mendelson (1986), Amihud (2002), Pastor and Stambaugh (2003) and Acharya and Pedersen (2005). These studies almost exclusively focus on illiquidity in stock markets. The only studies that we are aware of that deal with illiquidity in the context of private equity are Das et al. (2003), Lerner and Schoar (2004) and Franzoni et al. (2009).

lifetime $T_l - t$. From risk-neutral valuation arguments, we can then define

$$V_t^{F,ill} = V_t^F - E_t^{\mathbb{Q}} \left[\int_t^{T_l} e^{-r_f(\tau - t)} c_\tau^{ill} \, d\tau \right], \tag{3.8}$$

with instantaneous illiquidity costs c_t^{ill}. If instantaneous illiquidity costs c_t^{ill} are, for simplicity, assumed to be constant over time, i.e. $c_t^{ill} = c^{ill}$, it follows

$$V_t^{F,ill} = V_t^F - c^{ill} \frac{(1 - e^{-r_f(T_l - t)})}{r_f}. \tag{3.9}$$

We can calculate V_t^F from our valuation equation derived above. In contrast, the value $V_t^{F,ill}$ of a "real" illiquid fund is generally unobservable over time. However, we know that investors enter private equity funds at $t = 0$ without paying an explicit price.[14] Therefore, $V_0^{F,ill} = 0$ must hold. Using this equality, we can implicitly derive instantaneous illiquidity costs c^{ill} from the identity

$$V_0^{F,ill} = V_0^F - c^{ill} \frac{(1 - e^{-r_f T_l})}{r_f} = 0. \tag{3.10}$$

Solving for c^{ill} yields

$$c^{ill} = \frac{V_0^F r_f}{(1 - e^{-r_f T_l})}. \tag{3.11}$$

Substituting (3.11) into (3.9) gives the following theorem for the value of an illiquid fund.

Theorem 3.2 *The value of an illiquid private equity fund at any time $t \in [0, T_l]$ during its finite lifetime T_l can be stated as*

$$V_t^{F,ill} = V_t^F - w_t^{ill} V_0^F, \tag{3.12}$$

where

$$w_t^{ill} = \frac{(1 - e^{-r_f(T_l - t)})}{(1 - e^{-r_f T_l})}$$

holds and V_t^F denotes the value of the corresponding liquid fund as given in Theorem 3.1.[15]

The value $V_t^{F,ill}$ of an illiquid fund at time t is the difference between the value V_t^F and V_0^F of the corresponding liquid fund. Thereby, the value V_0^F is weighted by the factor w_t^{ill}. This weighting factor is a increasing function in the fund's remaining lifetime $(T_l - t)$. That is, the longer the remaining lifetime of the fund, the more

[14]This holds, at least, when ignoring all direct or indirect transaction costs, such as search costs for the investor.

[15]Note that the value V_t^F might get (slightly) negative at the end of a fund's lifetime under this specification. This happens here if the present value of the constant illiquidity cost c^{ill} at some time t is higher than the value of the liquid fund V_t^F. One might argue here that a rational investor will never sell a private fund if the costs are higher than its current value. Therefore, it is better to define the value $V_t^{F,ill}$ as

$$V_t^{F,ill} = \max \left[V_t^F - w_t^{ill} V_0^F ; 0 \right],$$

for all $t \in [0, T_l]$.

illiquidity affects its value.[16] A result that is fully in line with economic intuition. At $t = T_l$, it must hold that $w_t^{ill} = 0$. From this, it follows that the values of the illiquid and liquid fund at the end of the fund's lifetime are equal and are given by $V_{T_l}^{F,ill} = V_{T_l}^{F} = 0$ per definition.

Figure 4 illustrates our results by comparing the values of an illiquid fund with the values of the corresponding liquid fund over time.[17] It shows that the value of the illiquid fund is always below the value of the liquid fund. Over time, the difference between the values decreases as the illiquidity discount of the fund is a decreasing function of the fund's remaining lifetime.

Perhaps the strongest assumption for this model extension is that illiquidity costs are constant over time. A more general modeling framework would have to account for time-varying, possibly stochastic illiquidity costs. While our model extension to account for illiquidity is mostly suggestive, it is helpful since it shows how illiquidity affects fund values over time. Specifically, the approach also allows us to explicitly calculate the illiquidity costs c^{ill} of a fund, which would be an onerous task under a more complex model setting.

Regarding the illiquidity costs, two additional points have to acknowledged. First, it should be noted that instantaneous illiquidity costs c^{ill} here should be understood as expected liquidation costs of a fund (per unit time), equal to the product of the probability that an (average) investor has to liquidate his fund investment in the time interval $[t, t + dt]$ by the costs of selling the fund.[18] Second, in a narrow interpretation, the costs of selling represent transaction costs such as fees for an intermediary that sells the fund on the secondary market. More broadly, however, the costs of selling the fund could also represent other real costs, for instance, those arising from delay and search when selling the fund.

We now turn to the implications of our model with respect to the expected return and systematic risk of private equity funds.

4 Expected Return and Systematic Risk

The objective of this section is to derive expressions for the expected return and systematic risk of a private equity fund. This shows how returns are related to underlying economic characteristics of a fund's cash flows. Similar to the preceding section, we first examine expected returns and systematic risk under the simplifying assumption that private equity funds are traded on a frictionless market that is free of arbitrage. In a second step, we relax this assumption to show how illiquidity affect expected fund returns in equilibrium.

4.1 Liquid Case

A fund's gross return is given by its cash flows plus changes in value, divided by its current value. The conditional expectation of instantaneous fund returns can be

[16]This result is qualitatively similar to many other theoretical studies that also point out that the illiquidity discounts depend on the investor's holding periods. See, for example, Amihud and Mendelson (1986).

[17]The model inputs used for Figure 4 are given as follows: $C = 1$, $T_c = T_l = 20$, $r_f = 0.05$, $\kappa = 0.5$, $\theta = 0.5$, $\sigma_\delta = 0.1$, $\delta_0 = 0.01$, $m = 1.5$, $\alpha = 0.025$, $\sigma_P = 0.5$ and $\sigma_{PW} = 0$. In addition, note that Figure 4 shows the unconditional expectations of the fund values over time for illustrative purposes.

[18]See Amihud and Mendelson (1986) for a similar definition of illiquidity costs.

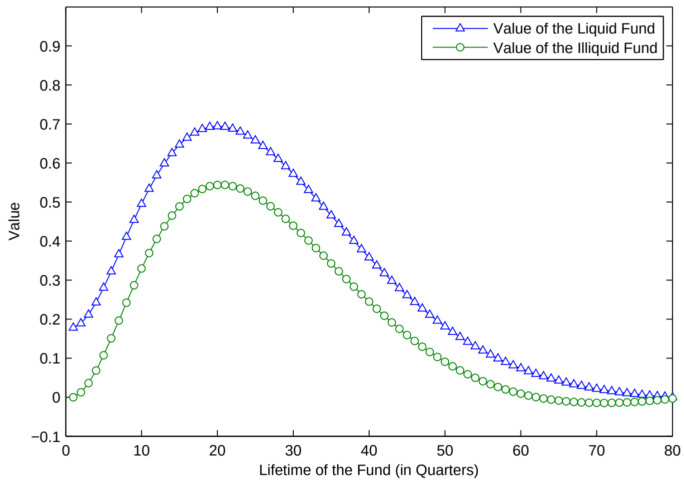


Figure 4: Comparing the Value of an Illiquid Fund to the Corresponding Liquid Fund

stated as[19]

$$E_t\left[R_t^F\right] = \frac{E_t\left[\frac{dV_t^F}{dt}\right] + E_t\left[\frac{dP_t}{dt}\right] - E_t\left[\frac{dD_t}{dt}\right]}{V_t^F}. \tag{4.1}$$

To evaluate condition (4.1), we must obtain expressions for the conditional expectation of a fund's instantaneous price changes, i.e. $E_t[dV_t^F/dt]$. This quantity, in turn, is determined by the difference of the expected instantaneous changes in present values of the capital distributions and capital drawdowns, as $E_t[dV_t^F/dt] = E_t[dV_t^P/dt] - E_t[dV_t^D/dt]$ holds. Under the condition that Assumptions 3.1 to 3.4 hold, Appendix B shows that these two quantities can be represented as

$$E_t\left[\frac{dV_t^D}{dt}\right] = -E_t\left[\frac{dD_t}{dt}\right] + r_f\ V_t^D \tag{4.2}$$

and

$$E_t\left[\frac{dV_t^P}{dt}\right] = -E_t\left[\frac{dP_t}{dt}\right] + (r_f + \sigma_{PW})V_t^P. \tag{4.3}$$

Substituting (4.2) and (4.3) into (4.1), the expected instantaneous fund return is

$$E_t\left[R_t^F\right] = r_f + \sigma_{PW}\frac{V_t^P}{V_t^P - V_t^D}. \tag{4.4}$$

That is, the expected return of a private equity fund is given by the riskless rate of return r_f plus the risk premium $\sigma_{PW}V_t^P/(V_t^P - V_t^D)$. This risk premium depends on two components. First, this risk premium is determined by the covariance σ_{PW}. The economic intuition behind this follows from standard asset pricing arguments. A positive covariance σ_{PW} implies that high returns of the market portfolio are associated with high capital distributions. That is, as the market portfolio increases in

[19]See Appendix B for a detailed derivation of this relationship.

value, the probability of large capital distributions increases. This is an unfavorable condition for an investor, as the highest capital distributions occur in states where marginal utility is already low. Therefore, the ex-ante expected return will exceed the riskless rate of return r_f in this case. Conversely, for $\sigma_{PW} < 0$ the opposite will be true. Second, the risk premium is also determined by the ratio $V_t^P/(V_t^P - V_t^D)$. This second term is particularly interesting because it implies that the expected returns of a fund will vary over time. The reason for this is that the systematic risk of a fund changes over time. To make this result more obvious, we can view the expected fund returns in (4.4) from a traditional "beta" perspective. It follows

$$E_t\left[R_t^F\right] = r_f + \beta_{F,t}(\mu_W - r_f), \tag{4.5}$$

where $\beta_{F,t}$ is the beta coefficient of the fund returns at time t and μ_W is the expected return of the market portfolio. Setting (4.4) equal to (4.5), the beta coefficient of a private equity fund turns out to be

$$\beta_{F,t} = \beta_P \frac{V_t^P}{V_t^P - V_t^D}, \tag{4.6}$$

where $\beta_P = \sigma_{PW}/\sigma_W^2$ is the constant beta coefficient of the capital distributions of the fund.[20] From specification (4.6), we can infer that beta coefficient of a fund varies over time as the present values V_t^P and V_t^D change stochastically over time. The result (4.6) follows partly from the fact that we have assumed the capital drawdowns to be uncorrelated with the return on the market portfolio. Therefore, a beta coefficient of the capital drawdowns does not enter into equation (4.6). Under a more general setting, where the beta coefficient of the capital drawdowns $\beta_D \neq 0$, the beta of the fund returns can be represented as

$$\beta_{F,t} = \beta_P \frac{V_t^P}{V_t^P - V_t^D} - \beta_D \frac{V_t^D}{V_t^P - V_t^D}. \tag{4.7}$$

The beta coefficient of the fund returns is then of course simply the value weighted average of the betas of the fund's capital distributions and drawdowns. In general, this result implies that the beta coefficient of the fund is non-stationary whenever $\beta_P \neq \beta_D$ holds, i.e., capital distributions and drawdowns carry different levels of systematic risks. This follows again because over time, stochastic changes in the values V_t^P and V_t^D affect the weighting factors $w_t^P = V_t^P/(V_t^P - V_t^D)$ and $w_t^D = V_t^D/(V_t^P - V_t^D)$. To our best knowledge, we are the first to acknowledge the existence and importance of this effect for private equity funds.[21]

Our model sheds light on the determinants of the systematic fund risk and provides insight into some of the deficiencies of recent empirical studies on the expected return and systematic risk of private equity funds. Typically, the expected return and systematic risk of a private equity fund are estimated by a single constant number for the whole lifetime of the vehicle. Using (4.7), it is possible to examine

[20] Note that according to the general equilibrium model (3.1), it must hold that the market risk premium can be represented as $\mu_W - r_f = \sigma_W^2$.

[21] This basic result is foreshadowed in the finance literature by the works of Brennan (1973), Myers and Turnbull (1977) and Turnbull (1977) on the systematic risk of firms. For example, Brennan (1973) defines the beta coefficient of a firm as "...the market value weighted average of the betas of all the firm's expected cash flows" and concludes that this specification will generally imply that "...the beta coefficient of the firm is non-stationary ..." (Brennan (1973), p. 671.)

some of the deficiencies of this traditional procedure. Aggregating systematic risk of a fund into a single number implicitly assumes that all fund cash flows are equally risky and that the risk arises from a common economic source. However, if these conditions are not met, this will imply a misspecification, as the systematic risk of a fund will, in general, be changing stochastically over time. Using the beta representation form, the structure of the fund returns is given by

$$E_t\left[R_t^F\right] = r_f + w_t^P \beta_P(\mu_W - r_f) - w_t^D \beta_D(\mu_W - r_f). \tag{4.8}$$

The variables on the right hand side of (4.8) are of the form of an elasticity multiplied by the expected excess return on the market. Both weighting factors, w_t^P and w_t^D, change over time and thus a constant expected return will not, in general, provide an adequate description of a private equity fund. This result is also important with respect to portfolio optimization. If a constant systematic risk and expected return of a private equity fund over its lifetime are assumed, this could also result in non-optimal investment decisions.

4.2 Illiquid Case

This section derives a liquidity-adjusted version of the expected fund return. We start by noting that the conditional expectation of a fund's net return in an economy with constant instantaneous illiquidity costs, c^{ill}, can be stated as

$$E_t^{ill}\left[R_t^F - \frac{c^{ill}}{V_t^F}\right] = \frac{E_t\left[\frac{dV_t^F}{dt}\right] + E_t\left[\frac{dP_t}{dt}\right] - E_t\left[\frac{dD_t}{dt}\right]}{V_t^F}. \tag{4.9}$$

Then, from a similar derivation as applied in the preceding section, expected fund returns in the beta representation form are given by

$$E_t^{ill}\left[R_t^F\right] = r_f + \frac{c^{ill}}{V_t^F} + \beta_{F,t}(\mu_W - r_f), \tag{4.10}$$

where the fund's beta, $\beta_{F,t}$, is again as given by (4.6), or more generally by (4.7). This condition states that the required excess return is now the sum of the (expected) relative illiquidity costs, c^{ill}/V_t^F, plus the fund's beta times the risk premium.[22] Intuitively, the positive association between expected returns and relative illiquidity costs reflects the compensation required by investors for the lack of an organized and liquid market where private equity funds can be sold at any instant in time.

It is also important here to acknowledge that incorporating illiquidity costs here induces a second time variable component into the expected fund returns. For constant instantaneous illiquidity costs c^{ill}, relative illiquidity costs c^{ill}/V_t^F vary over time as the fund value V_t^F follows a distinct, albeit stochastic time pattern. In general, the value of the fund V_t^F increases over time as the investments portfolio is build up and decreases towards the end as fewer and fewer investments are left to be harvested. Therefore, the ratio c^{ill}/V_t^F will be highest in the beginning and at the end of a fund's lifetime when its value is low.[23] This effect particularly increases expected fund returns at the start and towards the end of its lifetime.

[22] This statement is qualitatively similar to the relationship found theoretically and empirically by Amihud and Mendelson (1986) for conventional stocks markets.

[23] Note that this result will continue to hold in a more general setting with time-varying illiquidity costs.

So far, our analysis was based on theoretical reasoning. The next section turns to the application of the model and examines its empirical implications.

5 Empirical Evidence

This purpose of this section is to present the empirical results of our paper. We show how our model can be calibrated to private equity fund data and discuss its empirical implications in various directions. We start by introducing the private equity fund data and our estimation methodology used for empirical analysis. The empirical results are presented thereafter.

5.1 Data Description

We use a dataset of European private equity funds that has been provided by Thomson Venture Economics (TVE). It should be noted that TVE uses the term private equity to describe the universe of all venture investing, buyout investing and mezzanine investing.[24] We have been provided with various information related to the exact timing and size of cash flows, residual net asset values (NAV), fund size, vintage year, fund type, fund stage and liquidation status for a total of 777 funds over the period from January 1, 1980 through June 30, 2003.[25] All cash flows and reported NAVs are net of management fees and carried interest.

Table 2: Descriptive Statistics

Tabelle 2 provides a descriptive overview on the fund data provided by Thomson Venture Economics (TVE). The complete data set includes 777 European private equity funds. In accordance with TVE we use the following stage definitions: Venture capital funds (VC) represent the universe of venture investing. It does not include buyout investing, mezzanine investing, fund of fund investing or secondaries. Angel investors or business angels are also not included. Buyout funds (BO) represent the universe of buyout investing and mezzanine investing.

	All	Liquidated Funds	Extended Sample
Number of Funds			
VC			
absolute	456	47	102
relative	58.69%	49.47%	50.25%
BO			
absolute	321	48	101
relative	41.31%	50.53%	49.75%
Total			
absolute	777	95	203
relative	100.00%	100.00%	100.00%

Before presenting our empirical results, we have to deal with a problem caused by the limited number of liquidated funds included in our data set. The purpose

[24] Fund of fund investing and secondaries are also included in this broadest term. TVE is not using the term to include angel investors or business angels, real estate investments or other investing scenarios outside of the public market. For a detailed overview on the TVE dataset and a discussion of its potential biases see Kaplan and Schoar (2005).

[25] The initial database also contained 14 funds of funds which we excluded from our data set for the analysis as we focus on core private equity funds.

of our study requires the knowledge of the full cash flow history of the analyzed funds. In principle, this is only possible for those funds that have already been fully liquidated at the end of our observation period. Table 2 shows that this reduces our data set to a total number of only 95 funds. So, only a small subset of the full data can be used for analysis. Furthermore, given that the age of the liquidated funds in our sample is about 13 years, one can infer that restricting the analysis to liquidated funds could also limit our results as more recently founded funds would be systematically left out.

In order to mitigate this problem, we follow an approach of Diller and Kaserer (2009) that increase their data universe by adding funds that have small net asset values compared to their realized cash flows at the end of the observation period. In specific, we add non-liquidated funds to our sample if their residual value is not higher than 10 percent of the undiscounted sum of the absolute value of all previously accrued cash flows. In such cases, treating the current net asset value at the end of the observation period as a final cash flow will have a minor impact on our results. All funds that are not liquidated by 30 June 2003 and satisfy this condition are added to the liquidated funds to form an extended data sample. As one can see from Table 2, the extended sample consists of a total of 203 funds and comprises 102 venture capital funds (50.25 percent) and 101 buyout funds (49.75 percent). We base our subsequent model estimation and empirical analysis on this extended sample of private equity funds.

5.2 Estimation Methodology

The application of stochastic models frequently encounters the difficulty that structural parameters underlying the model are unobservable. In our model, the committed capital C, the commitment period T_c and the fund's legal lifetime T_l are fixed by contractual arrangements, whereas the other structural model parameters cannot be observed directly and need to be estimated. The procedure employed for the following model calibration is an explicit parameter estimation based on historical private equity fund cash flow data. The immediate practical difficulty here arises from the fact that the observation period of private equity fund data is generally only quarterly or monthly at most. This low observation frequency means that only limited time-series data of fund cash flows is available for estimation. It is well known that standard maximum likelihood estimation (MLE) methods for continuous-time stochastic processes do not work considerably well when the total number of observations is very low or time steps are too large.

To mitigate these problems we estimate the model parameters by using the concept of *Conditional Least Squares* (CLS). The concept of conditional least squares, which is a general approach for estimating the parameters involved in a continuous-time stochastic process $\{X(t)\}$, was given a thorough treatment by Klimko and Nelson (1978).[26] The general intuition behind the CLS method is to estimate the parameters from discrete-time observations of the stochastic process X_t, $t = 1, 2, \ldots, n$, such that the sum of squares

$$\sum_{t=1}^{n} (X_t - E_{t-1}[X_t])^2 \tag{5.1}$$

[26] An application of the CLS method to the CIR process is given by Overbeck and Rydén (1997).

is minimized, where $E_{t-1}[X_t]$ is the conditional expectation given the information set generated by the observations $X_1, \ldots, X_{t-1}$. This basic idea can be slightly adapted to our particular estimation problem. As we have time-series as well as cross-sectional data of the cash flows of our sample funds, a natural idea is to replace the single observations X_t in relation (5.1) by the sample average values $\bar{X}_t$.[27] See Appendix C for a detailed description of the estimation methodology.

5.3 Estimation Results

In implementing our estimation procedure, we use the quarterly cash flow data of the 203 sample private equity funds. To make funds of different investment size comparable, all capital drawdowns and distributions are first expressed as a percentage of the corresponding total committed capital. Model parameters are then estimated by using the time-series and cross-sectional data of the normalized fund cash flows. Therefore, estimated model parameters represent the dynamics of an average fund from our sample.

Table 3: Drawdown and Distribution Process Parameters

Table 3 shows the estimated (annualized) model parameters for the capital drawdowns and distributions of the 203 sample funds. Standard errors of the estimates are given in parentheses. Standard errors of the estimated θ, κ and α coefficients are derived by a bootstrap simulation. In addition, note that we set $\delta_0 = 0$ for all (sub-)samples.

Model	Drawdowns			Distributions		
	κ	θ	σ_δ	m	α	σ_P
Total	7.3259	0.4691	4.7015	1.8462	0.0284	1.4152
	(5.8762)	(0.1043)	-	(0.1355)	(0.0007)	-
VC	13.3111	0.4641	5.2591	2.0768	0.0230	1.4667
	(6.4396)	(0.0869)	-	(0.2254)	(0.0010)	-
BO	4.9806	0.4797	4.5696	1.6133	0.0379	1.1966
	(2.9514)	(0.1142)	-	(0.0833)	(0.0006)	-

Table 3 shows the model parameters we estimated for the three (sub-)sample of all (Total), venture (VC) and buyout (BO) funds. As for the capital drawdowns, the estimated annualized long-run mean drawdown rate θ of all $N = 203$ sample funds amounts to 0.47 p.a. This implies that in the long-run approximately 11.75 percent of the remaining committed capital is drawn on average in each quarter of a fund's lifetime. In addition, the exceptionally high value reported for the volatility σ_δ indicates that the sample private equity funds draw down their capital at a very fluctuating pace over time. When comparing the parameter values for venture and buyout funds, it seems that venture and buyout funds on average draw down capital at a qualitatively similar pace as the coefficients θ are almost equal among these two sub-samples. However, it can also be inferred that the venture funds draw down capital at a more fluctuating pace than their buyout counterparts. This conclusion

[27] In general, this technique especially draws appeal from the fact that it is very easy to implement and works well with a low number of observations. Of course, other statistical techniques could be utilized, tested, and compared against this approach as well.

is supported by the higher value of the volatility σ_δ for the venture funds.[28] As far as the capital distributions are concerned, the long-run multiple m for all $N = 203$ sample funds is estimated to equal 1.85. That is, on average, the funds in our sample distribute 1.85 times their committed capital over the total lifetime. The reported α coefficient further implies that all sample funds have an average amortization period of 7.41 years or around 89 months.[29] The sub-sample of venture funds returned substantially more capital to the investors than the corresponding sample buyout funds. An additional comparison of the α coefficients reveals that the average buyout fund tends to pay out its capital much faster than the average venture fund.[30] This result corresponds to the common economic notion that venture funds invest in young and technology-oriented start-ups, whereas buyout funds invest in mature and established companies. Growth companies typically do not generate significant cash flows during their first years in business. Furthermore, it usually takes longer until these investments can successfully be existed, for example, by an IPO or trade-sale to a strategic investor. These conceptual differences in venture and buyout funds can also help to explain the differences in the standard deviations σ_P given in Table 3. Buyout funds distribute their capital at a more constant pace and less erratic than venture funds. Therefore, the volatility σ_P is lower for the buyout funds in our data sample.

Table 4: Other Structural Model Parameters

Table 4 shows the estimated annualized model parameters for the riskless rate of return (r_f), the expected return (μ_W) and standard deviation (σ_W) of the market portfolio, and the correlation (ρ_{PW}) between changes in log capital distributions and market returns.

				Correlation		
	Interest Rate	Market Returns		Total	VC	BO
Parameter	r_f	μ_W	σ_W	ρ_{PW}	ρ_{PW}	ρ_{PW}
Value	0.0587	0.1072	0.1507	0.0662	0.1190	0.0129

To apply our model for the purpose of valuation and to calculate the expected return of private equity funds, four additional parameter values have to be estimated: the riskless rate of interest r_f; the expected return μ_W and standard deviation σ_W of the market portfolio; and the correlation ρ_{PW} between changes in log capital distributions and market returns. Table 4 summarizes our choices of parameter values for these variables. We set $r_f = 0.0587$, the sample mean of the annualized monthly money market rates for three-month funds reported by Frankfurt banks over the period from January 1, 1980 through June 30, 2003.[31] The parameters μ_W and σ_W are estimated from continuously compounded monthly returns of the MSCI World Index over the same observation period, and are stated annualized. The correlation ρ_{PW} between changes in log capital distributions and the market

[28]However, note that some of the higher volatility of the venture funds is absorbed by the higher mean reversion coefficient κ that tends to "kill" some of the volatility.

[29]The expected amortization period can directly be inferred by solving equation (2.17) for the amortization period t_A.

[30]Note that these differences in α are also statistically significant for both data sample.

[31]Data can be obtained at: http://www.bundesbank.de.

return is more difficult to estimate. Using the full sample of all 777 funds, we first calculate for each month the rolling difference between the log capital distributions over the last year and the log capital distributions over the previous year. We then approximate ρ_{PW} by the correlation between the rolling changes in log capital distributions and the corresponding continuously compounded rolling yearly MSCI World returns. This gives a correlation of 0.4411 for venture funds and of 0.1739 for buyout funds. However, we find that the magnitude of these correlations is mainly driven by a highly positive correlation between the variables from the year 2000 onwards.[32] We exclude this period from the analysis, as the largest portion of the capital distributions of our sample funds occurs before that period. This gives the substantially lower correlation coefficients stated in Table 4 which are used in the following.

5.4 Empirical Results

Having thus calibrated the model to historical fund cash flow data, we now turn to the empirical results. First, a simple consistency test is presented to assess the model's goodness-of-fit with empirical cash flow data. Second, the implications of our model with respect to valuation, illiquidity costs and expected returns of private equity funds are analyzed.

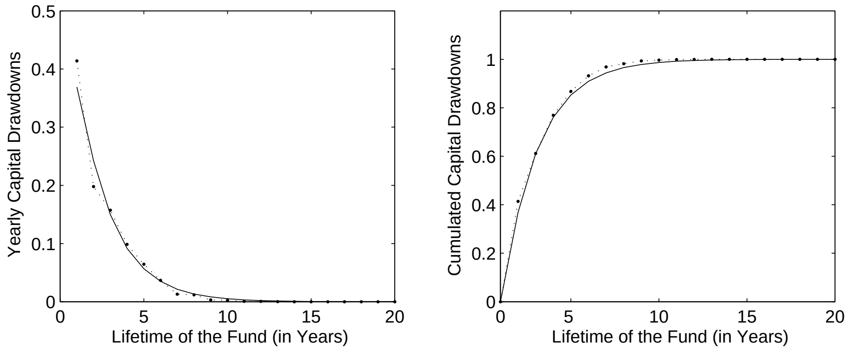
A. Goodness-of-Fit of the Calibrated Model

We begin our examination of the calibrated model by assessing the model's goodness-of-fit with empirical data. A simple way to gauge the specification of our model is to examine whether the model's implied cash flow patterns are consistent with those implicit in the time series of our defined data sample. That is, are the model expectations of the cash flows similar in magnitude and timing to the average values derived from their data samples counterparts?

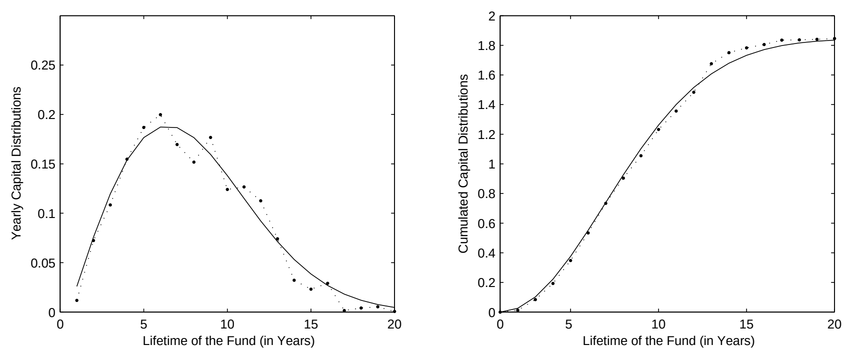
In this sense, Figure 5 compares the historical average capital drawdowns, capital distributions and net fund cash flows of all $N = 203$ sample funds to the corresponding expectations that can be constructed from our model by using the parameters reported in Table 3. Overall, the results from Figure 5 indicate an excellent fit of the model with the historical fund data. In particular, as measured by the coefficient of determination, R^2, our model can explain a very high degree of 97.73 percent of the variation in average yearly net fund cash flows. In addition, the mean absolute error (MAE) of the approximation is only 1.56 percent p.a. (measured in percent of the committed capital). Splitting the overall sample for venture and buyout funds, we find that the quality of the approximation differs somewhat between these two sub-samples. In general, the quality of approximation is slightly lower for the sub-sample of venture funds, where R^2 with 94.69 percent (MAE: 2.74 percent) takes a smaller value than the for buyout funds with 97.02 percent (MAE: 1.58 percent).

This form of consistency test of our model can certainly only provide an incomplete picture of the goodness of fit of our model with empirical cash flow, as it is only based on a comparison of the first distributional moment of the fund's cash flows. However, its economic relevance stems from the following line of argument. Correctly forecasting expected cash flows of private equity funds over time is a

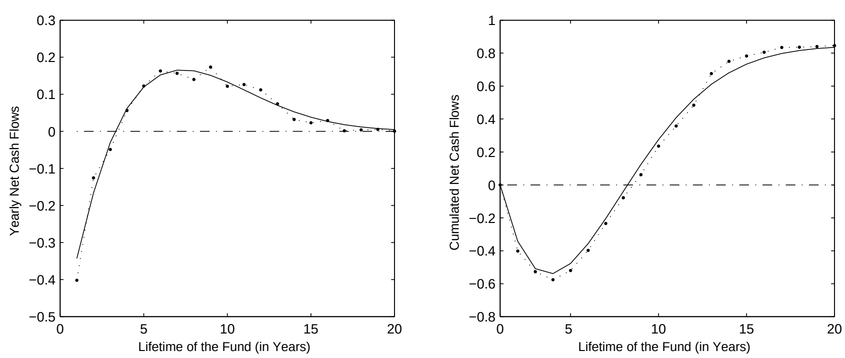
[32] This finding is consistent with very active IPO market in the years 2000 and 2001 and declining equity markets and vanishing exit possibilities for private equity funds thereafter. This, in particular, affects the magnitude of the correlation of venture funds.



(a) Yearly Capital Drawdowns (Left) and Cumulated Capital Drawdowns (Right)



(b) Yearly Capital Distributions (Left) and Cumulated Capital Distributions (Right)



(c) Yearly Net Fund Cash Flows (Left) and Cumulated Net Fund Cash Flows (Right)

Figure 5: Model Expectations Compared to Historical Data of All $N = 203$ Sample Funds (Solid Lines represent Model Expectations; Dotted Lines represent Historical Data)

main ingredient in a model's ability to correctly assess fund values. This follows as the value is just the difference between the discounted sum of all expected capital distributions and drawdowns, where expectations have to be evaluated under the risk-neutral probability measures. Although our analysis focuses on expectations under the objective probability measures here, the results suggest that our model is of economic relevance for the valuation of private equity funds. The following is concerned with its valuation implications.

B. Valuation Results

Having assessed the goodness-of-fit of our model, we can now turn to its empirical implications. We start with the implications of our model concerning the value and illiquidity costs of private equity funds. First, we employ the model to calculate risk-adjusted excess values and implicit illiquidity costs of the sample private equity funds. Second, we show how fund values and illiquidity discounts evolve over the lifetime of the funds.

B.1 Excess Value and Illiquidity Costs

Table 5 gives our valuation results for the sample funds overall and broken down by venture capital versus buyout funds. All reported values are derived under the null that our model holds with the calibrated parameter values shown in Tables 3 and 4. The first panel of Table 5 gives the present values V_0^P and V_0^D of the capital distributions and capital drawdowns, respectively, and the resulting fund values V_0^F at the starting date $t = 0$. The fund values V_0^F are equivalent to the ex-ante risk-adjusted net present values of investing in a fund. Intuitively, this can be thought of the present-valued return on committed capital, that is, the excess value created for each dollar committed. Thus, $1 committed to private equity funds is worth one plus V_0^F in present value terms. In addition, the second panel of Table 5 gives the instantaneous illiquidity costs c^{ill} that are derived implicitly by assuming that the risk-adjusted net present values V_0^F are compensation required by the investors for the illiquidity of the funds. Instantaneous illiquidity costs are annualized and are given as a percentage of the committed capital.

Table 5: Fund Value and Illiquidity Costs

The first panel of Table 5 gives the present values V_0^P and V_0^D and the resulting fund values V_0^F for the sample funds overall and broken down by venture capital versus buyout funds. The second panel gives the instantaneous costs c^{ill} that are derived implicitly by assuming that the ex-ante excess values V_0^F are compensation for holding an illiquid fund. Instantaneous illiquidity costs are annualized and are given as a percentage of the committed capital of the funds.

Model	Value			Illiquidity Costs
	V_0^P	V_0^D	V_0^F	c^{ill}
All	1.1143	0.8663	0.2480	2.49%
VC	1.0918	0.8770	0.2147	2.15%
BO	1.1113	0.8541	0.2572	2.58%

Several observations are in order. First, we find that private equity funds create excess value on a risk-adjusted basis. The results in Table 5 show that the risk-adjusted excess value (net-of-fees) of an average private equity fund in our sample is on the order of 24.80 percent relative to $1 committed. That is, $1 committed to a private equity funds is worth 1.2480 in present value terms. Second, these excess values hold for both venture and buyout funds, though, in our sample buyout funds create slightly more value. Third, by interpreting these excess values as a compensation required by investors for illiquidity of the funds, we can implicitly derive annualized illiquidity costs c^{ill}. Overall, the illiquidity costs are in the order of 2.49 percent of committed capital p.a.[33] The results also show that illiquidity costs depend on the type of fund. Interestingly, the results imply that buyout funds offer a higher compensation for illiquidity than venture funds. One possible explanation for this is that investors of buyout funds require higher compensation for illiquidity because of the larger size of the investments of these funds.[34]

A word of caution should be added in interpreting the above results. The illiquidity costs are derived under the implicit assumption that excess values are compensation for illiquidity of the funds only. The perils of doing so can be manifold. In addition to illiquidity, private equity investors often are not sufficiently diversified and some of the compensation may represent a premium for this non-diversification. Furthermore, private equity investors may require additional compensation for agency conflicts and asymmetric distribution of information. Overall, these additional sources could reduce the estimated illiquidity costs stated above. Therefore, the values given here should better be interpreted as an upper boundary for the illiquidity costs that investors require.

B.1 Value Dynamics and Illiquidity Discounts

We can also employ our model to illustrate the dynamics of the fund values over time. In this sense, Table 6 gives the dynamics of fund values under liquidity, V_t^F, the corresponding fund values under illiquidity, $V_t^{F,ill}$, and the illiquidity discounts, C_t^{ill}, for all three (sub-)samples of funds.[35]

It should first be noted that the dynamics of the fund values shown in Table 6 conform to reasonable expectations of private equity fund behavior. In particular, the fund values increase over time as the investment portfolios are build up and decrease towards the end as fewer and fewer investments are left to be exited. In addition, two interesting differences between venture and buyout funds become apparent. First, it can be inferred that buyout funds reach their maximum value before the venture funds and that their values decrease faster towards the end. This follows as venture funds have a slightly slower drawdown schedule and also distribute capital slower than the buyout funds. Second, venture fund reach higher maximum values. This holds as they, on average, also distribute substantially more capital to the investors than the buyout funds. Table 6 does also give the illiquidity discounts

[33] Illiquidity costs are calculated using equation (3.11) by assuming an average lifetime of the funds of 15 years.

[34] This hypothesis is supported by the results of Franzoni et al. (2009). Franzoni et al. (2009) show that investment size is a positive and significant determinant of compensation for illiquidity. They conjecture that larger investments are more sensitive to exit conditions than smaller investments and thus, via this channel, are more exposed to liquidity risk.

[35] Note that for all $t > 0$, we suppress the stochastic behavior of the fund values and illiquidity discounts over time by showing the unconditional expectations of these variables. This is done here for illustrative purposes.

Table 6: Value Dynamics over Time for Liquid and Illiquid Funds

Table 6 illustrates the dynamics of fund values under liquidity (V_t^F), the corresponding fund values under illiquidity ($V_t^{F,ill}$) and the illiquidity discounts (C_t^{ill}) for all three (sub-)samples of funds. Note that unconditional expectations of the fund values and illiquidity discounts are shown for all $t > 0$ for illustrative purposes.

	All Funds			VC Funds			BO Funds		
Year	V_t^F	$V_t^{F,ill}$	C_t^{ill}	V_t^F	$V_t^{F,ill}$	C_t^{ill}	V_t^F	$V_t^{F,ill}$	C_t^{ill}
0	0.2480	0.0000	0.2480	0.2147	0.0000	0.2147	0.2572	0.0000	0.2572
1	0.5756	0.3343	0.2413	0.5919	0.3829	0.2089	0.5340	0.2837	0.2502
2	0.7875	0.5533	0.2342	0.8314	0.6286	0.2028	0.7072	0.4644	0.2429
3	0.8803	0.6536	0.2267	0.9496	0.7534	0.1962	0.7634	0.5284	0.2350
4	0.8860	0.6673	0.2187	0.9797	0.7904	0.1893	0.7360	0.5092	0.2267
5	0.8313	0.6211	0.2102	0.9469	0.7650	0.1820	0.6550	0.4370	0.2179
6	0.7388	0.5377	0.2012	0.8717	0.6976	0.1742	0.5464	0.3378	0.2086
7	0.6271	0.4355	0.1916	0.7708	0.6049	0.1659	0.4305	0.2318	0.1987
8	0.5107	0.3291	0.1815	0.6576	0.5004	0.1572	0.3219	0.1337	0.1882
9	0.4000	0.2292	0.1708	0.5427	0.3948	0.1479	0.2289	0.0518	0.1771
10	0.3018	0.1424	0.1594	0.4338	0.2958	0.1380	0.1549	0.0000	0.1549
11	0.2195	0.0722	0.1474	0.3360	0.2084	0.1276	0.0998	0.0000	0.0998
12	0.1538	0.0193	0.1346	0.2519	0.1354	0.1165	0.0612	0.0000	0.0612
13	0.1037	0.0000	0.1037	0.1825	0.0777	0.1047	0.0356	0.0000	0.0356
14	0.0670	0.0000	0.0670	0.1271	0.0348	0.0923	0.0196	0.0000	0.0196
15	0.0411	0.0000	0.0411	0.0844	0.0053	0.0791	0.0101	0.0000	0.0101
16	0.0236	0.0000	0.0236	0.0523	0.0000	0.0523	0.0048	0.0000	0.0048
17	0.0120	0.0000	0.0120	0.0288	0.0000	0.0288	0.0020	0.0000	0.0020
18	0.0046	0.0000	0.0046	0.0120	0.0000	0.0120	0.0006	0.0000	0.0006
19	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

we have constructed by assuming that ex-ante excess fund values are compensation for illiquidity alone. In line with economic intuition, the results show that illiquidity discounts are an increasing function in the remaining lifetime of the private equity funds. That is, the higher the remaining lifetime of the fund, the higher are the illiquidity discounts of the fund.

C. Expected Return and Systematic Risk

Finally, we turn to the implications of our model regarding the systematic risk and expected return of private equity funds. Table 7 illustrates the dynamics of the systematic risk, as measured by the beta coefficients, and of the expected fund returns under liquidity and illiquidity that can be derived from our model for the overall sample of funds and for the sub-samples of venture and buyout funds.[36] These results reveal a number of interesting observations, some of which have not yet been documented in the private equity literature.

First, and possibly most important, the results show that the beta coefficients, β_t^F, and expected fund returns under liquidity, $E_t[R_t^F]$, follow distinct time-patterns. The highest values of these variables can be observed at the start of the funds. Over time, both variables decrease towards some constant level. Mathematically, this time-pattern can easily be explained by equation (4.6) for the beta coefficient of a private equity fund. Over time, as the fund builds up its investment portfolio and draws down capital from the investors, the present value of the capital drawdowns V_t^D decreases. On average, drawdowns occur earlier than capital distributions under our model specification. Therefore, the value V_t^D decreases faster over the fund's lifetime than the value V_t^P. As time passes by, V_t^D eventually decreases to zero. Therefore, the beta coefficient of the fund converges to the beta coefficient of the capital distributions $\beta_P = \sigma_{PW}/\sigma_W^2$ over time, and expected returns converge to $r_f + \beta_P(\mu_W - r_f)$. From an economic standpoint, this result follows as the structure of stepwise capital drawdowns of private equity funds acts like a financial leverage that increases the beta coefficients of the funds as long as the committed capital has not been completely drawn. Second, the results show that venture funds have higher beta coefficients and therefore generate higher ex-ante expected returns than buyout funds. Specifically, the results suggest that the beta risk of venture funds is higher than the market and that the beta risk of buyout funds is substantially lower than the market for all times during the lifetime considered. Third, we also document how illiquidity of private equity funds affects expected returns over time. All calculations here are again performed under the simplifying assumptions that the excess values of the funds reported in Table 5 are compensation for illiquidity alone and that illiquidity costs are constant over time. As expected, the results show that illiquidity increases expected funds returns. This holds, in particular, at the start and at the end of a fund's lifetime, where increases in expected returns are highest. The rationale behind this result is as follows. As discussed in Section 3.3, illiquidity costs represent expected liquidation costs of a fund. That is the product of the probability that an (average) investor has to liquidate his fund investment per unit time by the costs of selling the fund. If the liquidation probability and costs of selling a fund are assumed to be constant, regardless of the fund's stage in life, then relative illiquidity costs c^{ill}/V_t^F are highest when the fund values are low. That is at the start and end of a fund's lifetime. From this effect, one can also see

[36] Note that the beta coefficients and expected returns in Table 7 are calculated by using the unconditional expectations of the fund values shown in Table 6.

Table 7: Expected Return and Systematic Risk over Time

Table 7 illustrates the dynamics of the beta coefficients ($\beta_{F,t}$), the expected returns under liquidity ($E_t[R_t^F]$) and the expected returns under illiquidity ($E_t^{ill}[R_t^F]$) for all three (sub-)samples of funds.

	All Funds			VC Funds			BO Funds		
Year	$\beta_{F,t}$	$E_t[R_t^F]$	$E_t^{ill}[R_t^F]$	$\beta_{F,t}$	$E_t[R_t^F]$	$E_t^{ill}[R_t^F]$	$\beta_{F,t}$	$E_t[R_t^F]$	$E_t^{ill}[R_t^F]$
0	2.7838	19.37%	27.86%	5.8753	34.35%	42.85%	0.4396	8.00%	16.50%
1	1.2559	11.96%	15.60%	2.2638	16.85%	19.91%	0.2181	6.93%	11.00%
2	0.9239	10.35%	13.01%	1.6536	13.89%	16.07%	0.1614	6.65%	9.73%
3	0.7979	9.74%	12.12%	1.4311	12.81%	14.72%	0.1393	6.55%	9.40%
4	0.7356	9.44%	11.81%	1.3245	12.29%	14.15%	0.1282	6.49%	9.45%
5	0.7006	9.27%	11.79%	1.2662	12.01%	13.93%	0.1219	6.46%	9.78%
6	0.6791	9.16%	12.01%	1.2317	11.84%	13.93%	0.1180	6.44%	10.42%
7	0.6653	9.10%	12.45%	1.2100	11.74%	14.10%	0.1156	6.43%	11.47%
8	0.6559	9.05%	13.17%	1.1958	11.67%	14.44%	0.1140	6.42%	13.15%
9	0.6491	9.02%	14.27%	1.1860	11.62%	14.98%	0.1130	6.42%	15.85%
10	0.6440	8.99%	15.94%	1.1790	11.59%	15.79%	0.1122	6.41%	20.25%
11	0.6396	8.97%	18.50%	1.1735	11.56%	16.98%	0.1114	6.41%	27.61%
12	0.6352	8.95%	22.49%	1.1688	11.54%	18.75%	0.1102	6.40%	40.24%
13	0.6298	8.92%	28.88%	1.1641	11.51%	21.46%	0.1077	6.39%	62.23%
14	0.6218	8.88%	39.41%	1.1584	11.49%	25.72%	0.1023	6.37%	100.34%
15	0.6218	8.88%	58.59%	1.1584	11.49%	32.95%	0.1023	6.37%	182.19%
16	0.6218	8.88%	95.58%	1.1584	11.49%	46.14%	0.1023	6.37%	359.17%
17	0.6218	8.88%	178.70%	1.1584	11.49%	74.41%	0.1023	6.37%	799.04%
18	0.6218	8.88%	448.74%	1.1584	11.49%	163.08%	0.1023	6.37%	2341.95%
19	-	-	-	-	-	-	-	-	-
20	-	-	-	-	-	-	-	-	-

that illiquidity particularly increases expected returns at the end of the lifetime as fund value V_t^F decreases towards zero here.[37]

6 Conclusion

In this paper, we present a new stochastic model for the dynamics of a private equity fund. Our work differentiates from previous research in the area of venture and private equity fund modeling in the following respects. Our model of a fund's capital drawdowns and distributions is based on observable economic variables only. In this sense, we do not specify a process for the dynamics of the unobservable value of a fund's assets over time, as done in the existing models of Takahashi and Alexander (2002) and Malherbe (2004, 2005). Rather, we endogenously derive the market value of a fund by using equilibrium intertemporal asset pricing considerations. The combination of equilibrium asset pricing principles and appropriate economic modeling of the underlying stochastic processes allows us to derive a simple closed-form solution for the market value of a fund over time. This theoretical value enables us to show how the value of a fund is related to its cash flow dynamics and to general economic variables, such as the riskless rate of return and the correlation between the fund's cash flows and the returns on a market portfolio. With a simple model extension, we also show how illiquidity affects fund values and how illiquidity discounts of the funds change over time. Furthermore, we use our model to explore the dynamics of the expected return and systematic risk of private equity fund over time. This analysis complements the literature on the risk and return characteristics of private equity funds, as it reveals that the systematic risk of a funds will, in general, be non-stationary. To our knowledge, we are the first in the finance literature that acknowledge the existence and importance of this effect for private equity funds. The empirical part of our paper shows how our model can be calibrated to cash flow data of a sample of European private equity funds and analyzes its empirical implications. This provides a number of interesting insights, some of which have not been noted in the private equity literature before.

[37] However, note that this is not necessarily the effect we want to stress here. Rather, the point is to show how illiquidity affects returns during time periods in which fund values are not close to zero.

A Appendix: Derivation of the Value Under Liquidity

In this appendix, we derive the arbitrage free value of a private equity fund stated in Theorem 3.1. We start with the present value of the capital drawdowns V_t^D that can be stated as

$$V_t^D = E_t^{\mathbb{Q}}\left[\int_t^{T_l} e^{-r_f(\tau-t)} d_\tau d\tau\right] 1_{\{t\leq T_c\}}. \tag{A.1}$$

Note that we can first reverse the order of the expectation and the time integral in (A.1) because of Fubini's Theorem. That is

$$E_t^{\mathbb{Q}}\left[\int_t^{T_l} e^{-r_f(\tau-t)} d_\tau d\tau\right] = \int_t^{T_l} e^{-r_f(\tau-t)} E_t^{\mathbb{Q}}[d_\tau] d\tau \tag{A.2}$$

holds, as the riskless rate r_f is assumed to be constant. In addition, we have assumed the drawdown rate to carry zero systematic risk. Therefore, the expectation on the right hand side of (A.2) is similar under the risk-neutral measure $\mathbb{Q}$ and the objective probability measure $\mathbb{P}$, i.e. $E_t^{\mathbb{Q}}[d_\tau] = E_t^{\mathbb{P}}[d_\tau]$. Thus, inserting (2.8) directly yields

$$V_t^D = -U_t \int_t^{T_l} e^{-r_f(\tau-t)} C(t,\tau) d\tau 1_{\{t\leq T_c\}}, \tag{A.3}$$

where

$$C(t,\tau) = (A'(t,\tau) - B'(t,\tau)\delta_t) \exp[A(t,\tau) - B(t,\tau)\delta_t],$$

and $A(t,\tau)$, $B(t,\tau)$ are as given in (2.7)

We now turn to the present value of the capital distributions V_t^P. Similarly to above, applying Fubini's Theorem gives

$$V_t^P = \left[\int_t^{T_l} e^{-r_f(\tau-t)} E_t^{\mathbb{Q}}[p_\tau] d\tau\right]. \tag{A.4}$$

This reduces the problem to finding $E_t^{\mathbb{Q}}[p_\tau]$. Solving the risk-neutralized process (3.4) with $\lambda_P = \sigma_{PW}/\sigma_P$ yields

$$\begin{aligned} p_\tau = \alpha t(mC - P_t) \exp\{ &- \frac{1}{2}[\alpha(\tau^2 - t^2) + \sigma_P^2(\tau - t)] \\ &+ \sigma_P \epsilon_t \sqrt{\tau - t} - \sigma_{PW}(\tau - t)\}, \end{aligned} \tag{A.5}$$

for $\tau \geq t$.[38] Taking the conditional expectations of (A.5) gives

$$E_t^{\mathbb{Q}}[p_\tau] = \alpha t(mC - P_t) \exp\left\{-\frac{1}{2}[\alpha(\tau^2 - t^2) - \sigma_{PW}(t - s)\right\}, \tag{A.6}$$

Inserting this into (A.4), the present value of the capital distributions can be represented as

$$V_t^P = \alpha\ (m\ C - P_t) \int_t^{T_l} e^{-r_f(\tau-t)} D(\tau,t) d\tau, \tag{A.7}$$

[38] Note that the derivation of the risk-neutralized process (A.5) is similar to that shown in Section 2.3.

where

$$D(t,\tau) = \exp\left[\ln\tau - \frac{1}{2}\alpha(\tau^2 - t^2) - \sigma_{PW}(\tau - t)\right].$$

Finally, substituting (A.3) and (A.7) into the valuation identity, $V_t^F = V_t^P - V_t^D$, gives the result stated in Theorem 3.1.

B Appendix: Derivation of the Expected Return

The purpose of this appendix is to derive the expected return of a private equity fund stated in equation (4.4). The instantaneous return R_t^F of a private equity fund at time t is defined by

$$R_t^F dt = \frac{dV_t^F + dP_t - dD_t}{V_t^F}. \tag{B.1}$$

From an economic perspective, R_t^F gives the return that can be earned by investing in the fund over an infinitesimally short time interval $[t, t + dt]$. Dividing by the time increment dt on both sides of equation (B.1) yields

$$R_t^F = \frac{\frac{dV_t^F}{dt} + \frac{dP_t}{dt} - \frac{dD_t}{dt}}{V_t^F}. \tag{B.2}$$

Substituting the conditions, $V_t^F = V_t^P - V_t^D$ and $dV_t^F/dt = dV_t^P/dt - dV_t^D/dt$, equation (B.2) can be rewritten as

$$R_t^F = \frac{\frac{dV_t^P}{dt} - \frac{dV_t^D}{dt} + \frac{dP_t}{dt} - \frac{dD_t}{dt}}{V_t^P - V_t^D}. \tag{B.3}$$

Taking the conditional expectation $E_t^{\mathbb{P}}[\cdot]$ of (B.3), the expected instantaneous fund return is given by

$$E_t^{\mathbb{P}}\left[R_t^F\right] = \frac{E_t^{\mathbb{P}}\left[\frac{dV_t^P}{dt}\right] - E_t^{\mathbb{P}}\left[\frac{dV_t^D}{dt}\right] + E_t^{\mathbb{P}}\left[\frac{dP_t}{dt}\right] - E_t^{\mathbb{P}}\left[\frac{dD_t}{dt}\right]}{E_t^{\mathbb{P}}\left[V_t^P\right] - E_t^{\mathbb{P}}[V_t^D]}, \tag{B.4}$$

where $E^{\mathbb{P}}[dP_t/dt]$ and $E^{\mathbb{P}}[dD_t/dt]$ denote expected instantaneous capital distributions and capital drawdowns, respectively. Under the specifications of our model, the expected instantaneous change of the present value of the capital drawdowns $E_t^{\mathbb{P}}[dV_t^D/dt]$ can be represented as

$$E_t^{\mathbb{P}}\left[\frac{dV_t^D}{dt}\right] = -E_t^{\mathbb{P}}\left[\frac{dD_t}{dt}\right] + r_f\ V_t^D. \tag{B.5}$$

This result can essentially be derived by using to different ways: (i) The first way is to directly differentiate the value V_t^D given by equation (3.6) with respect to t and then take the conditional expectation of the result. After some tedious algebraic transformations, it follows that, in fact, (B.5) holds. (ii) The second and much faster way is to directly derive (B.5) by using the general equilibrium model given by equation (3.1). From this, it must hold that

$$E_t^{\mathbb{P}}\left[\frac{dV_t^D + dD_t}{V_t^D}\right] = r_f dt, \tag{B.6}$$

where equality with the riskless rate of return r_f follows from the fact that we have assumed capital drawdown to carry zero systematic risk. Multiplying by V_t^D and rearranging directly leads to (B.5).

Following a similar line of argument, it can be inferred that the expected instantaneous change of the present value of the capital distributions $E_t^{\mathbb{P}}[dV_t^P/dt]$ can be represented as

$$E_t^{\mathbb{P}}\left[\frac{dV_t^P}{dt}\right] = -E_t^{\mathbb{P}}\left[\frac{dP_t}{dt}\right] + (r_f + \sigma_{PW})V_t^P, \tag{B.7}$$

where now, compared to equation (B.5), the additional term $\sigma_{PW}V_t^P$ enters into the equation on the right hand side, as we have assumed (log) capital distributions and the return on the market portfolio to be correlated with a constant coefficient of covariation σ_{PW}.

Finally, substituting (B.5) and (B.7) into (B.4), the expected instantaneous fund return turns out to be

$$\mu_t^F = r_f + \sigma_{PM}\frac{V_t^P}{V_t^P - V_t^D}. \tag{B.8}$$

C Appendix: Estimation Methodology

In this appendix, we present our estimation methodologies for the processes of the capital drawdowns and capital distributions.

A. Capital Drawdowns

The modeling of the drawdown dynamics requires the estimation of the following parameters: the long-run mean of the fund's drawdown rate θ, the mean reversion speed κ, its volatility σ_δ and its initial value δ_0.

The objective is to estimate the model parameters θ, κ, σ_δ and δ_0 from the observable capital drawdowns of the sample funds at equidistant time points $t_k = k\Delta t$, where $k = 1, \ldots, M$ and $M = T/\Delta t$ holds. To make the funds of different size comparable, the capital drawdowns of all $j = 1, \ldots, N$ sample funds are first standardized on the basis of each fund's total invested capital. Let $D_{k,j}^{\Delta t}$ denote the standardized capital drawdowns of fund j in the time interval $[t_{k-1}, t_k]$. Using this definition, cumulated capital drawdowns $D_{k,j}$ of fund j up some time t_k are given by $D_{k,j} = \sum_{i=1}^{k} D_{i,j}^{\Delta t}$ and undrawn committed amounts $U_{k,j}$ at time t_k are given by $U_{k,j} = 1 - D_{k,j}$.

Using these definition, the (annualized) arithmetic drawdown rate $\delta_{k,j}^{\Delta t}$ of fund j in the interval $[t_{k-1}, t_k]$ can be defined as

$$\delta_{k,j}^{\Delta t} = \frac{D_{k,j}^{\Delta t}}{U_{k-1,j} \cdot \Delta t}. \tag{C.1}$$

To estimate the model parameters we use the concept of *conditional least squares* (CLS). The concept of conditional least squares, which is a general approach for estimation of the parameters involved in the conditional mean function $E^{\mathbb{P}}[X_k|X_{k-1}]$ of a stochastic process, was given a thorough treatment by Klimko and Nelson (1978).[39] The idea behind the CLS method is to estimate the parameters from

[39] An application of the CLS method to the CIR process considered here is given in Overbeck and Rydén (1997).

discrete-time observations $\{X_k\}$ of a stochastic process, such that the sum of squares

$$\sum_{k=1}^{M}(X_k - E^{\mathbb{P}}[X_k|\mathcal{F}_{k-1}])^2 \tag{C.2}$$

is minimized, where $\mathcal{F}_{k-1}$ is the σ-field generated by $X_1, \ldots, X_{k-1}$. This basic idea can be slightly adapted to our particular estimation problem. As we have time-series as well as cross-sectional data of the capital drawdowns of our sample funds, a natural idea is to replace the X_k in relation (C.2) by the sample average $\bar{X}_k$.

Let $\bar{U}_k$ denote the sample average of the remaining committed capital at time t_k of the sample funds, i.e., $\bar{U}_k = \frac{1}{N}\sum_{j=1}^{N} U_{k,j}$. An appropriate goal function to estimate the parameters θ and κ is then given by

$$\sum_{k=1}^{M}(\bar{U}_k - E^{\mathbb{P}}[U_k|\mathcal{F}_{k-1}])^2, \tag{C.3}$$

where $\mathcal{F}_{k-1}$ is the σ-field generated by $\bar{U}_1, \ldots, \bar{U}_{k-1}$. The conditional expectation can in discrete-time be written as:[40]

$$E^{\mathbb{P}}[U_k|\mathcal{F}_{k-1}] = \bar{U}_{k-1}(1 - E^{\mathbb{P}}[\delta_k^{\Delta t}|\mathcal{F}_{k-1}]\Delta t). \tag{C.5}$$

Under the mean reverting square root process defined by (2.5), the conditional expectation of the drawdown rate $E^{\mathbb{P}}[\delta_k^{\Delta t}|\mathcal{F}_{k-1}]$ is given by:[41]

$$E^{\mathbb{P}}[\delta_k^{\Delta t}|\mathcal{F}_{k-1}] = \theta(1 - e^{-\kappa\Delta t}) + e^{-\kappa\Delta t}\bar{\delta}_{k-1}^{\Delta t}, \tag{C.6}$$

where $\bar{\delta}_k^{\Delta t}$ denotes the average (annualized) drawdown rate of the sample funds that is defined by

$$\bar{\delta}_k^{\Delta t} = \frac{\frac{1}{N}\sum\limits_{j=1}^{N} D_{k,j}^{\Delta t}}{\frac{1}{N}\sum\limits_{j=1}^{N} U_{k-1,j}\Delta t}. \tag{C.7}$$

Substituting equation (C.6) and (C.5) into (C.3), the goal function to be minimized turns out to be

$$\sum_{k=1}^{M}\{\bar{U}_k - \bar{U}_{k-1}[1 - (\theta(1 - e^{-\kappa\Delta t}) + e^{-\kappa\Delta t}\bar{\delta}_{k-1}^{\Delta t})\Delta t]\}^2. \tag{C.8}$$

Appropriate estimates of $\hat{\theta}$ and $\hat{\kappa}$ can then be derived by a numerical minimization of relation (C.8). This also requires the knowledge of the initial values of the drawdown rate. For simplicity, this value is set to zero. That is, we assume $\hat{\delta}_0 = \bar{\delta}_0^{\Delta t} = 0$ in the following.

---

[40] This follows directly from the continuous-time specification $dU_t = -\delta_t U_t dt$. In discrete-time, this relation can be written as

$$U_{t_k} = U_{t_{k-1}}(1 - \delta_{t_k}^{\Delta t}\Delta t). \tag{C.4}$$

Taking the conditional expectation $E^{\mathbb{P}}[\cdot|\mathcal{F}_{k-1}]$ and replacing $U_{t_{k-1}}$ by the sample mean $\bar{U}_{t_{k-1}}$, it follows that equation (C.5) holds.

[41] See Cox et al. (1985), p.392.

The conditional variance of the capital drawdowns of fund j in the interval $[t_{k-1}, t_k]$ can be formulated as

$$
\begin{aligned}
E^{\mathbb{P}}[D_{k,j}^{\Delta t} - E^{\mathbb{P}}[D_{k,j}^{\Delta t}|\mathcal{F}_{k-1}]|\mathcal{F}_{k-1}]^2 &= Var^{\mathbb{P}}[\delta_{k,j}^{\Delta t} U_{k-1,j}\Delta t|\mathcal{F}_{k-1}] \\
&= (U_{k-1,j}\Delta t)^2 Var^{\mathbb{P}}[\delta_{k,j}^{\Delta t}|\mathcal{F}_{k-1}]. \qquad \text{(C.9)}
\end{aligned}
$$

Under the specification of the mean reverting square root process defined by (2.5), the conditional variance $Var^{\mathbb{P}}[\delta_{k,j}^{\Delta t}|\mathcal{F}_{k-1}]$ of the drawdown rate is given by:[42]

$$
Var^{\mathbb{P}}[\delta_{k,j}^{\Delta t}|\mathcal{F}_{k-1}] = \sigma_{k,j}^2(\eta_0 + \eta_1 \delta_{k-1,j}^{\Delta t}), \qquad \text{(C.10)}
$$

where

$$
\begin{aligned}
\eta_0 &= \frac{\theta}{2\kappa}\left(1 - e^{-\kappa\Delta t}\right)^2, \\
\eta_1 &= \frac{1}{\kappa}\left(e^{-\kappa\Delta t} - e^{-2\kappa\Delta t}\right).
\end{aligned}
$$

The conditional expectation $E^{\mathbb{P}}[D_{k,j}^{\Delta t}|\mathcal{F}_{k-1}]$ can in discrete-time be written as

$$
\begin{aligned}
E^{\mathbb{P}}[D_{k,j}^{\Delta t}|\mathcal{F}_{k-1}] &= E^{\mathbb{P}}[\delta_{k,j}^{\Delta t}|\mathcal{F}_{k-1}]U_{k-1,j}\Delta t \\
&= (\gamma_0 + \gamma_1 \delta_{k,j}^{\Delta t})U_{k-1,j}\Delta t, \qquad \text{(C.11)}
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma_0 &= \theta\left(1 - e^{-\kappa\Delta t}\right), \\
\gamma_1 &= e^{-\kappa\Delta t}.
\end{aligned}
$$

Substituting equation (C.10) and (C.11) into (C.9), an appropriate estimator of the variance σ_j^2 of the drawdown rate of fund j turns out to be

$$
\hat{\sigma}_j^2 = \sum_{k=1}^{M} \frac{[D_{k,j}^{\Delta t} - (\hat{\gamma}_0 + \hat{\gamma}_1 \delta_{k-1,j}^{\Delta t})U_{k-1,j}\Delta t]^2}{(U_{k-1,j}\Delta t)^2(\hat{\eta}_0 + \hat{\eta}_1 \delta_{k-1,j})}, \qquad \text{(C.12)}
$$

where $\hat{\gamma}_0$ and $\hat{\eta}_0$ are evaluated at $(\hat{\theta}, \hat{\kappa})$ and so on. In the following, the sample variance is then defined to be the simple average of the individual fund variances, i.e., $\hat{\sigma}_\delta^2 = \frac{1}{N}\sum_{j=1}^{N}\hat{\sigma}_j^2$.

B. Capital Distributions

The modeling of the distribution dynamics requires the estimation of the following parameters: the long-run mean of the fund's multiple m, the constant share of the mean reversion speed α and the volatility σ_P.

The objective is to estimate the model parameters m, α and σ_P from the observable capital distributions of the sample funds at equidistant time points $t_k = k\Delta t$, where $k = 1, \ldots, M$ and $M = T/\Delta t$ holds. To make the funds of different size comparable, the capital distributions of all $j = 1, \ldots, N$ sample funds are first standardized on the basis of each fund's total invested capital. Let $P_{k,j}^{\Delta t}$ denote the standardized capital distributions of fund j in the time interval $[t_{k-1}, t_k]$. Using this

[42] See Cox et al. (1985), p.392.

definition, cumulated capital distributions $P_{k,j}$ of fund j up some time t_k are given by $P_{k,j} = \sum_{i=1}^{k} P_{i,j}^{\Delta t}$.

From the definitions given above, the multiple M_j of fund j at the end of the lifespan T is given by

$$M_j = \sum_{i=1}^{M} P_{i,j}^{\Delta t}. \tag{C.13}$$

An unbiased and consistent estimator for the long-run mean m is given by the sample average, i.e.,

$$\hat{m} = \frac{1}{N} \sum_{j=1}^{N} M_j. \tag{C.14}$$

The second model parameter α cannot be observed directly from the capital distributions of the sample funds. However, it can be estimated by using the *conditional least squares* (CLS) method introduced above. In this case the conditional least squares estimator $\hat{\alpha}$ minimizes the sum of squares

$$\sum_{k=1}^{M} (\bar{P}_k - E^{\mathbb{P}}[P_k|\mathcal{F}_{k-1}])^2, \tag{C.15}$$

where $\bar{P}_k = \frac{1}{N} \sum_{j=1}^{N} P_{k,j}$ is the sample average of the cumulated distributions at time t_k and $\mathcal{F}_{k-1}$ is the σ-field generated by $\bar{P}_1, \ldots, \bar{P}_{k-1}$. By definition, the conditional expectation $E^{\mathbb{P}}[P_k|\mathcal{F}_{k-1}]$ of the cumulated capital distributions is given by:[43]

$$E^{\mathbb{P}}[P_k|\mathcal{F}_{k-1}] = mC - (mC - \bar{P}_{k-1}) \exp[-0.5\alpha(t_k^2 - t_{k-1}^2)]. \tag{C.16}$$

Substituting this condition into equation (C.15), the corresponding sum of squares to be minimized is given by

$$\sum_{k=1}^{M} \left\{ \bar{P}_k - \hat{m}C + (\hat{m}C - \bar{P}_{k-1}) \exp[-0.5\alpha(t_k^2 - t_{k-1}^2)] \right\}^2, \tag{C.17}$$

where the conditional expectation $E^{\mathbb{P}}[P_k|\mathcal{F}_{k-1}]$ is evaluated with $\hat{m}$ and $t_k = k\Delta t$. An simple estimate for the parameter $\hat{\alpha}$ can then be derived by a numerical minimization of statement (C.17).

In order to estimate the volatility of the capital distributions σ_P, we first calculate the variances of the log capital distributions in each time interval $[t_{k-1}, t_k]$ by

$$\hat{\sigma}_k^2 = \ln \left[\frac{1}{N} \sum_{j=1}^{N} (P_{k,j}^{\Delta t})^2 \right] - 2 \ln \left[\frac{1}{N} \sum_{j=1}^{N} P_{k,j}^{\Delta t} \right]. \tag{C.18}$$

The variance of the capital distributions σ_P^2 is then defined as the average of the individual variances $\widehat{\sigma}_k^2$ $(k = 1, \ldots, M)$, where weighting is done with the average distributions that occur in the given time period, i.e.,

$$\hat{\sigma}_P^2 = \sum_{k=1}^{M} \left(\frac{\frac{1}{N} \sum\limits_{j=1}^{N} P_{k,j}^{\Delta t}}{\hat{m}} \hat{\sigma}_k^2 \right). \tag{C.19}$$

[43] This can directly be inferred from equation (2.13).

The idea behind this is to weight the individual variances according to the magnitude of the average capital distributions that occur in this time period.

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