# Theory of the Conditional Affine Term Risk of Jumps: an Analytical Single Event Q-Jump Distribution is Derived and Tested

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## Abstract

A new transition probability function, the Q-jump, is derived and developed for the pricing of term risk due to jumps. The dynamic of the risk depends on the single jump with dispersion in the autocorrelation over conditional intertemporal variations in the time direction. The new Q-distribution is found by Fourier analysis, definition of marginals in frequency, and a new Nikon Radodym scaling in log time. The Q-jump model is first tested by an evaluation of its properties, then application to the term risk properties of yield curves and credit spreads. Q-jump models are shown capable of providing the required economic insights required of term risk if it is to justify the market risk (systematic risk over time) in jumps. As a single jump model, the Q-jump model solves the incomplete markets problem in Lévy processes, and avoids the need for the use, generally in current models, of arbitrary modelling assumptions (mean reversion and square root diffusion, etc). Unlike current models, the amounts and patterns of risk premiums are commensurate with large market premiums in term risk. Finally, it is noted that the Q-jump model measures risk orthogonally to Gaussian risk, in the time direction. The model is well suited to pricing conditional inter-temporal variations, essentially in underlyings that are rates, as a class. In combination with the Gaussian Wiener models, therefore, it should cover the full range of market risks.

**Key words:** Q-jump, term structure, Fourier analysis, inter-temporal pricing, credit risk

**JEL classifications:** G13, C46, E43

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## 1. *Introduction*

This paper examines the role of a new form of pure jump risk model, the Q-single jump model. The model proposed differs fundamentally from current models in the nature of its transition probability distribution function, and risk neutral transform. It depends for its main mechanism on the auto-correlative dispersion in time over a single event Poisson intensity process. Although the main empirical focus of the study is on the term structure of the yield curve and credit spreads, the model may also be applied to the problem of time varying conditional term risk in the general case. The pure jump model in the analytical form proposed in this paper has, to our knowledge, not been previously presented. This paper a) develops an original analytical model for the conditional jump risk in a single intensity Poisson process, and b) discusses methods for the empirical evaluation of the model on yields and credit spreads.

*Term risk due to jumps*

Empirical evidence suggests that financial markets are affected across the board by some form of term risk, where term risk is defined as the inter-temporal dependence in the conditional variability of returns over time. The extent of such risk is evident in the wide ranging occurrence for example, of implied volatility skews in options Cont and Fonseca (2002) or the autoregression patterns in time series Bollerslev (2001). Practitioners are careful to try to anticipate the amounts of such risk when pricing contingent claim models or managing their hedges. Although a number of modelling solutions are followed to quantify these effects, it is however fair to say that the treatment of such risk remains more qualitative than quantitive. Models popularly used may assume some form of stochastic volatility, jump risk, or a combination of inter-temporal mechanisms. Theoretical obstacles in these methods exist at the fundamental level: for example, the achievement of satisfactory risk neutral pricing or, where this fails, a satisfactory alternative; and also the availability of a risk neutral pricing of jump risk or where this fails a satisfactory approximation. The problem to be addressed is that since term risk is large enough to warrant being priced, then by the first law of finance (arbitrage free pricing) it must be due to systematic risk, since systematic risk has a unique Q pricing density function. Therefore although yet to be found, a market model for this risk must exist.

To assess the general nature of the modelling problem, we draw on the evidence already existing in markets, where a term structure dependence might be most evident. The term structure of interest rates might provide this strong focus for stochastic term structure modelling, as discussed by Cochrane and Piazzesi (2005), and Rebonato (1998). The models of Campbell and Cochrane (1999), or Bansal and Yaron (2004) seek to explain the term risk premium explicitly by risk preference in inter-temporal consumption-habit behaviour. The term structure of credit spread puzzle also suggests some form of inter-temporal term risk, not yet identified, Collin-Dufresne, Goldstein, Martin (2001). The fact that the current structural or reduced form models of credit default, and associated credit spread risk based respectively on the Merton (1974) theory and the Duffie, Pan, Singleton (2000) affine jump diffusion (AJD) models as examples, consistently underestimate the scale of the spread risk premiums, indicate a mechanism still to find. The skews in implied volatility surfaces are also attributed to some form of intertemporal effect. The current models based on stochastic volatility or jump risk, however, do not replicate these patterns well.

As further evidence on term risk, the possibility of a commonality in the problem of conditional time dependent variations, particularly linking equity and debt markets (credit spreads), has resulted in studies which look for the connection between risk premiums across asset classes. The question of a common process for term risk in markets is increasingly relevant in innovative credit derivative and structured finance markets. It is also questioned whether the long standing anomaly in the equity risk premium in CAPM might be captured by credit spread premium. It is for example usual practice to estimate equity premiums from elements of credit spreads as discussed by Chen, Roll and Ross (1986). The search for this connection continues, see the results of Campbell (1987), or Chen, Collin-Dufresne, Goldstein (2008). The question is more than intriguing since such connections might support the notion of a general risk premium mechanism for time risk. As another example, Carr and Wu (2007) look for a connection between the variation in sovereign spreads and their respective currency option volatility skews (which act as a proxy for a term or jump risk).

Despite a range of evidence, of an underlying term risk process across asset classes, current theory does not shed much light on what the common process for it might be. This, and the other problem that models to not match the scale of risk premiums found empirically, suggest a gap in theory still to be solved.

## *2. Current Models and Limitations*

A number of models exist for the conditional time dependence of risk behaviour. Each model may however rely on quite different primary mechanisms or mathematical descriptions. Models that are used extensively in theory and practice may be of the following form: stochastic volatility models Hull and White (1987) or Heston (1993), the exponential-Lévy process, see Sato (1999), and affine jump-diffusion models Duffie et al (2000). Econometric studies on the statistical analysis of return variations, Cont (2001), provide useful evidence on the general existence of term structure in financial time series, but fail to meet risk neutral pricing criteria. The practical role of the option theoretic models of Merton (1974), for debt valuation and credit risk pricing, also provides a conceptual interpretation of credit risk, which may have some links with the pricing kernel we seek. In the following, we examine the merits and demerits of current models in relation to our pricing objective.

### *Stochastic volatility models*

It is well understood that the Black Scholes Merton (1973) model contains no information on conditional transition probabilities for term risk, since it is a time homogeneous model. By combining stochastic volatility as a Wiener process with a Weiner diffusion for the asset price, a bivariate diffusion model is obtained, as proposed in the Hull and White (1987), or Heston (1993) models. Descriptions of volatility surface or term risk are obtained in the Bates (1996) or Barndorff-Nielson and Shephard (2001) (BNS) type models, via the correlation coefficient between these two Wiener processes. Also, it is possible to adapt stochastic volatility models for fixed income pricing, by adding functions to account for the stylized effects in interest rates, of a tendency to a mean, and a requirement for a positive interest rate. A typical model in the former case is an Ornstein-Uhlenbeck process, and in the latter, a square root diffusion. The stochastic volatility models in these forms can go some way to replicating leptokurtic effects and heavy tails in pricing densities, but in general provide much too little risk premium to account accurately for the behaviours seen. The general criticism is that the solutions they provide for the interpolation of implied volatility smiles or, estimation of interest rates, are phenomenological not fundamental in character.

*Exponential-Lévy models*

The theory of Lévy processes helps rationalise the building of jump risk models. Jump diffusion models typically describe stochastic price increments as a Gaussian diffusion process with jumps occurring at relatively infrequent random intervals. The Poisson jump component typically will be a compound (double stochastic) Poisson process, with a finite number of jumps but with randomness in the size of jumps. Alternatively the jumps can be modelled through the Lévy measure as a model with the capability of infinite number of jumps in a given time interval. These present the possibility of an infinite divisibility of jumps (in the Lévy measure). All the basic jump and stochastic volatility models just described are generic Lévy processes, i.e. processes with stationary independent increments.

Convergent integrands over probabilities, or measures, are required in pricing models, if they are to satisfy pricing boundary conditions, e.g. the pay-off condition of a contingent claim. An attraction of Lévy processes is that the smoothness and integrability of densities is generally tractable using Fourier analysis (i.e. the characteristic functions), over a rich range of Lévy process dynamics. Such models can be closely fitted to the option price conditional data from markets. To obtain parameterised Lévy models, some mechanism has to be added to ensure integrability over probabilities. New Lévy processes can be built on basic principles, for example the tilting of the Lévy process by an exponential damping. One reason for using an exponential Lévy process is the ease with which well behaved density functions are obtained in equivalent martingale form by Esscher transform. More details are provided in Cont and Tankov (2004), p310, on the range of techniques applied to ensure the smoothness and integrability of Lévy densities (linear transforms, Lévy subordination or exponential tilting).

Despite their tractability, two problems beset Lévy processes, if they are to price jump risk fundamentally. The first is that, since Lévy processes are, by definition, stationary independent increment processes, they cannot accommodate the time-inhomogeneity of term risk. The pricing of inter-temporal conditional randomness is required. The solutions offered for this generally include a deterministic time dependence rule. Alternatively, a stochastic volatility process in combination with a Lévy process may be added. The former process is not attractive, since it is not a conditional process of time dependence. Stochastic volatility is difficult to justify, since it presents no supporting *a priori* reason for the term risk.

The second major problem of Lévy processes derives from their infinite divisibility property. With many discontinuous jumps permissible, it is not possible to define a complete markets condition. As shown by Bellamy and Jeanblanc (1999), infinitely divisible Lévy processes are, by definition, semi-martingales. A unique pricing martingale cannot then be found.

*Jump-diffusion models*

Jump-diffusion models are currently the workhorse for the continuous time modelling of term risk behaviour, in interest rates and in credit spreads. The attraction of jump-diffusion is that it incorporates both mechanisms for the full understanding of financial randomness – both Poisson jumps and Brownian motion diffusion. As AJD models, they offer tractable and comprehensive solutions for contemporaneous risks in markets. Examples are the Merton (1976) jump-diffusion model with Gaussian jumps and the Kou (2002) model with double exponential jumps. A comparison can also be made with the Cox double stochastic pure jump risk model, see Lando (1998). Jump-diffusion of course plays a major part in the interest rate models of Das (1998) and Zhou (2001). These models can be constructed without explicit reference to the theory of Lévy processes, the theorems for the manipulation of Lévy distributions (the Lévy-Khintchin), or the structure of sample paths by Lévy-Ito decomposition. The Merton 1976 model is significant, as the starting point for the study of such models, defined as a Gaussian diffusion with the intermittent occurrence of jumps by Poisson process. Also, randomness is generally assumed in jumps size, as lognormal Gaussian processes.

A main problem arising with the Merton 1976 jump-risk model, is in its assumption that jumps are diversifiable, therefore not priceable, as systematic risk. The evidence from markets, that they are strongly affected by systematic jump risk, sits uncomfortably with the current theory for jumps in this respect. One consequence of this non-systematic nature view of jump risk, is that Poisson distributed events are then priced as arrival times of integer jumps, as objective (statistical) probabilities. To counter this effect, risk premiums may have been modelled with a component risk arbitrarily added, and calibrated to market prices, see Das (1998). It has been broadly assumed that the market premium for the pathway risk in jumps is small enough to ignore, a result that has been generally carried forward in subsequent models, see Zhou (2001). Then the martingale over pathways is obtained by a compensation of the Poisson distributed integer jump arrivals. The main jump risk is then modelled, risk neutrally, over jump size distributions of jumps.

A problem with infinitely divisible jump models is the impossibility of setting-up a risk neutral portfolio required in a complete markets setting. The only choice is then to apply a preference optimisation (over marginal utility of consumption) or a measure optimisation (quadratic or entropic minimisation).

*Combinations of models*

No one model dominates the field. The current route to selecting models is to seek the appropriate risk neutral modelling route by testing how well given models replicate the prices of traded options. This process of model calibration however may be insufficiently rigorous to find a general theory, since it provides ill-posed inverse solutions implying either no solution or, an infinite number of solutions. Confirming this, it is found that such models are unreliable over time. Also it is seen that better fitting is obtained by use of a combination of modelling features. If different concepts are equally possible, it is difficult to separate the causal effects of risk, and justify methods. The ill-posed nature of the mark to market process for validating models is endemic in current models.

*Unconditional distribution of returns*

Econometric studies concentrate on the characterisation of log financial returns (for example) in time series models. Although very many different parametric models have been tried, including $\alpha$ stable distributions, the student-*t* distribution, hyperbolic distributions, normal inverse Gaussian distribution, and more, these do not lead to pricing models. They need the additional step of a risk neutral or preference risk modelisation. Without this, the best that any unconditional distribution of returns can readily specify is the presence of term risk. The volatility clustering behaviour of Mandelbrot (2001), or the leptokurtosis in interest rate changes discussed by Das (1998) are symptomatic of such effects.

*Option theoretic structural models, the Merton (1974) model*

These models are very familiar in credit spread evaluation and commercial models. Structural models of this type relate spreads to leverage, asset volatility, risk free rate, loss given default, or distance to default variables. Although these models, when compared to Poisson intensity models, are more tangible in conceptual approach, they can be questioned on their economic validity. Amongst other things, they fail to provide reason for the much higher levels of risk premium across markets found in practice. They focus on firm level variables, when the main pricing may be at the market level. Further evolution of structural models has concentrated on developing the pricing kernel, in order to formulate a more realistic pricing kernel for distance to default. Calibrations have centred on market premiums derived from bond markets. Denzler et al (2005) in a KMV[1] white paper provide modified distributions of this type, which are more in line with the types of distribution that we might now find for the Q-jump.

***Stylized empirical facts on model limitations***

If a fundamental theory of term risk is to be found, a number of stylized limitations have to be solved across current models. We note that the following generic problems seem to affect all current models, to greater or lesser extent:

- Intrusiveness of Gaussian Wiener dynamic.

  The Gaussian Wiener dynamic is chosen for the randomness dynamic in many models even those supposedly concentrated on jump risk. Thus Gaussian risk is applied not only to bivariate stochastic diffusions, but also to otherwise pure jump risk models, e.g. in Cox double stochastic models, and in the jump amplitude component in jump-diffusion. More surprising, is the hold that Gaussian risk has over models that are primary dependent on yield curve dynamics, therefore term risk. The extent of the incursion of the Gaussian diffusion can be appreciated by its dominance in interest rate derivatives. Main spot

[1] KMV is a registered trade mark of Moody's KMV

interest rate models assume that the interest rate, bond price or priced by contingent related term structure variable are Gaussian lognormal. The advantages of using the Gaussian randomness infrastructure for analytical solutions are clear enough. It allows the linear correlation of risks, risk neutral measures through the Girsanov theorem, and techniques for manipulating the distribution through Ito decomposition.

If we need to legitimately replace this dynamic by another, for example to cater for jumps, the resistance to dropping the Gaussian solution with all its conveniences is obvious. Yet, this is the task that has to be set if a Q-jump model is to be found.

- Current models contain one or more untested *a priori* constraint.

Both current stochastic volatility based models and pure jump risk models need to add functionals if they are to replicate the time dependence effects. Whether these are the mean reversion or many functional features that improve the fitting of the yield curve or the deterministic time dependence adjustments to infinite divisible Lévy process, or the distribution assumptions in the size of jumps in the 'pure' jumps Cox double stochastic model, these are phenomenological features. To obtain solutions which respect the first law of finance requires a distribution for the conditional nature of risks not the arbitrary rules currently applied. Reliance assessing the suitability of models by estimation to markets data, fails to account for model error.

- Arbitrage free pricing appears to be ruled-out by the infinite divisibility of current Lévy, or jump enhanced diffusion, models. Without this, risk preference modules, e.g. maximising marginal utility, are needed.

The most important aspect of stochastic modelling should be the achievement of a complete market pricing distribution, as in the preference free logic of the Black Scholes model. Yet, in the current jump models a multiplicity of semi martingales occur~~s~~, since each of the infinitely divisible Lévy processes satisfies equivalent martingale measure (EMM) conditions. This type of jump risk model is unable to produce arbitrage free market pricing, since the condition for a pricing martingale of a complete market, required for an EMM, is not met.

An alternative is to constrain the model to one jump. The probability over one jump is 1, since over the whole distribution it must jump just once. This would ensure a complete market. The problem with this route, however, is the lack of a well defined probability density for the Poisson single jump. The solution found for this process, in its characteristic function, is a Cauchy distribution. The Cauchy distribution is however non-convergent and undefined in its cumulants. Since a Cauchy variance is not then available, it seems not possible to derive an equivalent martingale for the single Poisson jump. A variance would be required to obtain the drift change, using the Girsanov theorem. The alternative, which we try in this paper, is to find a totally new equivalent martingale, not requiring an integrating variance.

All current models fail aspects of a fundamental pricing rule, for one reason or another. This leaves little choice but to look for a new model, one which breaks with the tradition of the Gaussian Wiener solution, or current pure jump models. This is the route chosen in this paper. We propose a Q-jump model which exploits the mathematics available from the sciences, to obtain the pricing distribution required. We consider this in three steps: 1, Fourier analysis of the single jump (single jump ensures a complete market condition) to find the P transition probability distribution; 2, establish the marginal distribution in the Fourier transform to obtain the P density function; and 3, identify the equivalence martingale transform for the Q measure which ensures the market pricing conditions required for term risk. The model is tested by interpretation of its premiums, and by applying the model to term risk in various markets. We first try the model on the interest rate yield curve and then their credit spreads from those yields.

The structure of the rest of this paper is first to derive a pure jump model, on the assumption that a Q distribution does exist over the single jump pathway. Secondly, the properties of this model are described and evaluated. Thirdly, the application of the Q-jump risk model is interpreted on the term structure of yield curves and spreads, before concluding on the features of the model, and its potential for future applications.

## *3. Derivation of Affine Jump Risk Model*

In this section, we derive, then interpret, the pure jump risk theory of term risk. The route followed is the classical continuous time theory method, except in requiring the solution to be in the jump, not Gaussian Wiener diffusion component, of the stochastic integrals. It entails definition of the stochastic differential equations (SDE). A partial integro-differential equation (PIDE) is then obtained from this SDE to solve for the pricing probability as pricing kernel, over an equivalent martingale measure. The pricing distribution derived in this paper is non-Gaussian. The new mathematics and analysis required for this purpose are now described.

*Stochastic differential equation*

To illustrate the pricing process, consider a standard form of jump diffusion model. **Equation 1** below shows this in the mean-reversion jump-diffusion stochastic equation for interest rates with added jumps, as developed by Zhou (2001) for interest rates, with increments given by:

$$dr = \kappa(\theta - r)dt + \sigma dW + dZ(\lambda) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots\ldots (1)$$

In practice, this SDE may be further elaborated with further stochastic features depending on application, e.g. with deterministic terms, on the grounds of improved fitting to observed market prices. Some of the basic features are: the mean reversion term for the central tendency of interest rates to a mean rate, $\theta$, with rate of reversion $\kappa$. Randomness is modelled as a geometric Brownian diffusion $dW$ (Wiener increment) with instantaneous volatility, $\sigma$. Jump risk in **equation 1** is would typically has two sources of variability: Poisson process increments, $dN(\lambda)$, with intensity $\lambda$ to the Poisson process and stochastic size of jumps, $J$.

*Partial integro-differential equation*

Under equivalent martingale measures, the Feynman-Kac form of solution (in its probabilistic representation) for the above jump diffusion is:

$$\frac{\partial P}{\partial r}k(\theta - r) + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma^2 - \frac{\partial P}{\partial t} + JdN(t) = 0$$

With no known integrable solution for the P and Q distributions for the jump risk component, the risk neutral jump component is left in the integro-differential form: $\tilde{h}E^Q[P(Z+J) - P(Z)]$. $E^Q(.)$ is the jump risk over its equivalent martingale measure Q. Note that risk neutrality might be tried via a term of type: $\tilde{h} = h(1-\eta)$ where $\eta$ proxies for premium for market risk.

$$\frac{\partial P}{\partial r}k(\theta - r) + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma^2 - \frac{\partial P}{\partial t} + \tilde{h}E^Q[P(Z+J) - P(Z)] = 0 \ldots\ldots\ldots\ldots\ldots\ldots..\ldots\ldots\ldots\ldots\ldots (2)$$

The task now remains to solve the jump part of the PIDE.

*The PIDE for pure jumps*

To eliminate all reference to Gaussian Wiener risk, drop terms relating to the diffusion process. This produces a PIDE for the jump risk alone without encumbrance of Gaussian diffusion. No information is lost provided jump and Gaussian risks are mutually independent mechanisms. Gaussian risk can be added if it is needed, then:

$$\frac{\partial P_J}{\partial t} - E^Q[P(Z(\lambda))] = 0 \ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots\ldots\ldots (3)$$

where the jump process jump $P(Z(\lambda)$ is a Poisson process with intensity $\lambda$ and with a size of jump risk component $J$.

*Systematic risk in jumps (pathways and jump size)*

The key to solving the PIDE **equation 3** is to find the correct specification of its systematic risk(s). Both the treatment of pathways risk as objective probability processes in current models, and jump amplitudes in various randomness types of model are considered. In current models, the expected value of the jump increment, $dN(\lambda)$ assumes a Poisson process integer jumps with intensity $\lambda$ on integer jumps, such that the arrival of a jump is independent of the arrival of previous jumps, and that the probability of two simultaneous jumps is zero. Over the next small time interval, $dt$, one jump arrives with probability $\lambda dt$, and no jumps with probability $1 - \lambda dt$. The expectation over Poisson increments then becomes: $E[dN(t)] = 1.\lambda dt + 0.(1 - \lambda dt) = \lambda dt$. A martingale may be defined for the physical jump increments by a term of the form: $dM(t) = dN(t) - \lambda dt$. The compensation martingale thus formed when dM is zero. In some models for $dN(\lambda)$, further elaboration as a Gaussian randomness structure in $\lambda$ might be tried. The penalty of reduced tractability however normally negates this approach. The second source of randomness is then applied in most current models as randomness over the amplitude component, $J$. Sometimes a pure exponential jump distribution might be used, as in the Kou (2002) model.

Just as in the original Merton 1976 jump diffusion model, which considers the jump risk to be under objective probabilities (therefore not priceable), the current pure jump model also assumes the same principle at the individual jump level. In effect, the martingale is achieved over drift change by adjusting the jump distribution, weighted equally across pathways. The pattern of individual pathway probabilities is left unchanged. We note that the compensating martingale applies to all current popular jump risk models, including the Cox double stochastic process or the infinitely divisible Lévy measure models generally.

**Step 1 – The P probability density spectrum in frequency**

*The P probability density spectrum in frequency*

To account for the fact that markets jump systematically across whole markets, a systematic risk in the single jump is first sought. For this, we try a different approach in which dispersion is assumed to be in the single jump pathway, and that this is accessible by de-convoluting the Poisson exponential. To do this, the auto-correlative time distribution is transformed into its frequency distribution. Learning from the sciences, we know that, often, a frequency distribution exists when the time base~~d~~ process looks singular. This type of relationship is recognised in spectral analysis.

As example, consider the autocorrelation occurs over a single jump process of amount $Y_t$ in a time series. Assume that these correlations are represented by the form $\langle Y_t, Y_{t+s} \rangle$ where the bracket indicates the convolution over all the possible correlations in time. Then for a mean reversion with Wiener randomness with mean reversion rate, $\lambda$, then the autocorrelation over increments = $\langle Y_t, Y_{t+s} \rangle = e^{-\lambda s}$, as discussed by Cont and Tankov (2004), p472. The latter process is the Poisson single intensity AR(1) process. This is the auto-correlative single Poisson intensity process as a P dispersion.

To obtain a continuous dispersion of risk, when the single jump probability P is a step function then a multiple Poisson trajectory in time, seems counterfactual. However, such a distribution could exist if the P single jump is the integrated result of a dispersion in frequency. All that is required is the de-convolution of the exponential integral. All that is required then is to Fourier transform to analysis the single jump into its equivalent distribution in frequency measure. For the single Poisson jump pathway, the cumulative distribution function (CDF) is:

$$\int_0^\infty dN(\lambda)dt = CDF(t) = 1 - e^{-\lambda t}$$

or in its probability density (PDF) form:

$$dN(\lambda) = PDF(t) = d(1 - CDF(t))/dt = -\,\lambda.e^{-\lambda t}.1_{t\geq 0} \dots\dots\dots\dots\dots\dots\dots. \quad (4)$$

where the operator 1 denotes jump occurred 1, or not 0.

The fact that we can have a continuous P distribution under frequency measure when we only have a discrete distribution under time measure, stems for the properties of the Fourier transform. To confirm that a smooth function should technically exist, a mathematical check is given in **proposition 1**, using the Wiener Khintchin theorem.

**Proposition 1**
*In Stieltjes form, the characteristic function can have a continuous spectral distribution (under frequency measure), even when the probability distribution under time measure is a discrete step function.*

To find it the continuous distribution try the Fourier transform which gives the characteristic function solution, $\phi(\omega)$ in frequency:

$$\phi(\omega) = E^{*}\left[e^{-i\omega X(t)}\right] = \int_{-\infty}^{\infty} e^{-i\omega t}.PDF(t).dt \quad \dots\dots.\dots\dots\dots\dots\dots\dots..\dots\dots\dots (5)$$

In the limit $dt \to 0$, we cannot integrate over **equation 5** since the differential $d(CDF(t))/dt$ over a step jump is undefined. But **equation 5** can be rewritten in Fourier-Stieltjes form:

$$\phi(\omega) = E^{*}\left[e^{-i\omega X(t)}\right] = \int_{-\infty}^{\infty} e^{-i\omega t}.CDF(dt) \dots\dots\dots\dots\dots\dots\dots\dots.\dots\dots\dots\dots\dots.(6)$$

**Equation 6** has a solution by the Wiener-Khintchin theorem, as discussed by [27] Priestley (1981) for spectral analysis. If CDF is any positive semi-definite function, i.e. a positive continuous, it has a well-defined Fourier-Stieltjes transform as **equation 6**. Therefore a solution is obtained for the Poisson $CDF(t)$ since it is semi-positive over all times of interest in financial pricing. ■

**Required distributional components of $\boldsymbol{\phi(\omega)}$**

Substituting in the Fourier transform obtains the characteristic function $\phi(\omega)$:

$$\phi(\omega) = F_t[P(t)](\omega) = \frac{\sqrt{2}}{\pi}.\int_{0}^{\infty} e^{-i\omega t}.(-\lambda).\mathrm{e}^{-\lambda t}.\mathrm{dt} \dots\dots\dots\dots\dots\dots\dots\dots\dots.(7)$$

With the integration range selected to give a starting time of $t = 0$ , as required for a financial process, this solves by standard forms:

$$\phi(\omega) = \frac{\sqrt{2}}{\pi} . \frac{\lambda}{\lambda + i\omega}$$

multiplying through by $\lambda - i\omega$ obtains:

$$\phi(\omega) = \frac{\sqrt{2}}{\pi} . \frac{\lambda^2}{\lambda^2 + \omega^2} - \frac{i.\sqrt{2}}{\pi} . \frac{\lambda\omega}{\lambda^2 + \omega^2}$$

Define $\phi(\omega) = \phi^{'}(\omega) - i.\phi^{''}(\omega)$, this provides:

$$\phi^{'}(\omega) = Re.\phi(\omega) = \sqrt{2}.\frac{1}{1 + (\omega/\lambda)^2} \; ;$$

$$\phi^{''}(\omega) = Im.\phi(\omega) = \sqrt{2}.\frac{\omega/\lambda}{1 + (\omega/\lambda)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

**Equation 7** obtains the Fourier complex function components for the characteristic function. We note that in option pricing it is customary to use the complex component for option pricing, but either will do providing the equations yields a positive finite pricing integral. The components of the Fourier transform, however, can be put to even wider use as described in **step 2**.

**Step 2 – Specification of the 'marginals' (pricing density) in Fourier component $\boldsymbol{\phi^{''}(\omega)}$**

The use of the Fourier transform (characteristic function) is a popular technique in options pricing, since it is generally found easier to obtain tractable solutions in the characteristic function than the direct probabilities. The solution is first found by characteristic function then Fourier inverted to option price. Instead of using Fourier analysis as a mechanical means to option pricing, the use we put to the Fourier analysis is slightly different. It is used for its interpretive analytical properties of the transition probability function that we now describe.

Fourier analysis provides further properties in the specification of the distribution, which to our knowledge has not been used before in finance applications. This stems from the ability of the Fourier analysis to handle spectral analysis, a topic more at home in the sciences than in finance. The relationship between the pricing density and the imaginary component of the characteristic function $\phi(\omega)$ now used in this paper, was first proposed in the sciences. It has then been routinely used for spectral analysis in sciences for a number of decades. This theory was first developed by Kramers and Kronig (1927), based only on the fluctuation dissipation theorem (FDT). This theorem states that the response of a set to a disturbance is governed only by fluctuations in the equilibrium of ensemble of elemental states. Broadly, this is merely that the system of variations under change is linear. The end result is a relationship between the imaginary component and the distribution of marginals.

In essence, the Fourier analysis characteristic function $\phi(\omega)$ provides two components. These are important since they represent the dependence and non-dependence parts of the pricing function, uniquely[2]. The imaginary component of $\phi(\omega)$ then provides the pricing density in P, of the Poisson single jump. From **equation 8** we therefore obtain:

$$\phi''(\omega) = Im.\phi(\omega) = \frac{\sqrt{2}}{\lambda}.\frac{\omega/\lambda}{1 + (\omega/\lambda)^2}$$

The shapes of distributional components for the single jump can then be readily plotted from **equation 8**. These are shown in **figure 1a**. These distributions seem promising for pricing in that they show some structure in the pricing density. It is however a very asymmetric distribution with long tails and, to all appearances, indefinite in its right hand asymptote in frequency, i.e. at longer times. At first glance, this distribution seems to fail as a pricing distribution, due to its nonconvergent and undefined moments properties i.e., as a Cauchy distribution.

**Step 3 - The probability transition probability under Q measure (in frequency)**

Leaving the transition probability function as the P distribution, **equation 8**, does little to advance our study of the pricing of market jump risk. Whereas econometric studies operate on the basis of P probabilities, these are not acceptable for contingent claim option pricing. We require the risk neutral probability, Q, which defines the anticipation of a jump not the probability of the jump itself. Markets measure the risk premium of prospects of a change much more strongly than historical patterns of probability show, with the Peso effect as one example frequently quoted. We therefore should convert the P distribution **equation 8** into its equivalent Q. But the conventional route for finding its Q distribution directly from the P distribution by the Girsanov theorem, does not work for the Cauchy distribution. The latter theorem requires the variance measure common to the Gaussian world, but which is not available in the Cauchy form, since the Cauchy distribution has undefined moments.

Instead, we consider possibilities for a totally different change of measure. We were originally struck by the use of the log frequency (or log time) transform for several unconnected observations: firstly, the use of logarithmic frequency was commonly applied in spectral analysis in the sciences, when dealing with jump like terms. The laws of statistical mechanics confirmed the link between activation energy and the frequency (or rate) of reactions. These demonstrated the importance of using log time (or frequency), whenever modelling reaction rates. We might deduce that a log rescaling is the natural law relevant to jumps. Secondly, when we looked at the marginal expectation integral over its marginal distribution in **equation 8**, we noticed that a change in measure from frequency to log frequency makes the equation integrable. Thirdly, we noticed that, when we make the simple change in PDF or CDF from time to log time axis, the PDF collapsed to a symmetric curve, with the interesting new feature of a 'normalised' looking PDF.

[2] This arises because the main function of a Fourier analysis is to split the pricing function linearly into its sine and cosine (harmonic components). A common property of each component is its phase angle which describes the relationship between the dependent and non-dependent components. As the sine and cosine are at 90 degrees, they are non-interacting (correlatively independent), with the cosine components said to be in-phase, and the sine components out-of-phase. When out-of-phase, this implies that variations do not contribute cumulatively; they represent only noise. The sine component (i.e. imaginary factor) uniquely defines the non-dependent (marginal) components of the distribution.

In this paper, we document two ways to demonstrate the theory of this martingale solution. The first is the numerical transform from frequency to log frequency, or time to log time, e.g. by change to semi-log axis. The other is to obtain an analytical solution for a suitable equivalent Q transformation, by Radon Nikodym transform. In this paper, both methods are shown, starting with the analytical solution.

**Proposition 2**

*The equivalent martingale for the Q-jump single jump equivalent martingale obtains by a geometric law in frequency (or time). The analytical equation for the Q-pricing density is a log Cauchy distribution, square integrable in log frequency (or log time). Unlike the BSM, the Q-jump has a non-convergent P distribution (the Cauchy distribution) with undefined moments.*

To find an analytical version for the single jump P to Q transform, it is necessary to alter the integrands in the P expectation integral in some way, respecting the continuity of the distributions P and Q. There should be a definable Nikon Radodym differential $dQ/dP$ for:

$$E^Q[1_A] = E^P\left[1_A.\frac{dQ}{dP}\right]$$

1A is an indicator equal 1 if values in set A of probability values occur.

If we suspect a logarithmic rescaling is required as market pricing change in measure, the Radon Nikodym differential corresponding is:

$$\frac{dQ}{dP} = \frac{1}{\omega} = t \quad \text{(9)}$$

To find equivalent martingale expectation, $E^Q[1_A] = E^P\left[1_A.\frac{1}{\omega}\right]$, consider the transformation of the cumulative pricing distribution in P, i.e. $CDF^P(\omega) = E^P[1_A]$.

From **equation 8**, the cumulative P distribution $CDF^P(\omega)$ is given by:

$$CDF^P(\omega) = E^P[1_A] = \int_0^\infty \frac{\sqrt{2}}{\pi}.\frac{\phi''(\omega)}{\omega}.d\omega$$

$$= \int_0^\infty \frac{\sqrt{2}}{\pi}.\frac{\omega/\lambda}{1+(\omega/\lambda)^2}.\frac{d\omega}{\omega} \quad \text{(10)}$$

Apply the proposed Radon Nikodym differential, $\frac{dQ}{dP} = \frac{1}{\omega} = t$ **equation 9** to **equation 10**, obtains the cumulative pricing distribution in Q, $E^Q[1_A]$:

$$CDF^Q(\omega) = E^P\left[1_A\frac{1}{\omega}\right]$$

Gives:

$$= \int_0^\infty \frac{\sqrt{2}}{\pi} . \phi''(\omega) . \frac{1}{\omega} . d\omega = \int_0^\infty \frac{\sqrt{2}}{\pi} . \frac{\omega/\lambda}{1+(\omega/\lambda)^2} . dln\omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

In Radon Nikodym annotation, we see that the effect of the $1/\omega$ multiplier is, in effect, a logarithmic rescaling to replace $CDF^P(\omega) = E^P[1_A]$, which has integrand $\frac{1/\lambda}{1+(\omega/\lambda)^2}$ , by $CDF^Q(\omega) = E^Q[1_A] = E^P\left[1_A . \frac{dQ}{dP}\right]$ with the new integrand $\frac{\omega/\lambda}{1+(\omega/\lambda)^2}$.

**Equation 11** can be shown to satisfy the equivalent martingale measure (EMM) conditions which we recall are:

*Condition 1* – the Q-PDF should be equivalent to the P-PDF, i.e. can be continuously mapped from the real to the martingale probability PDFs
*Condition 2* - future prices are previsible. This condition specifies that future prices are only conditioned by past prices.
*Condition 3* - the PDF should be square integrable to a normalised value of 1.

These conditions are satisfied in either the numerical method by redrawing the graphs of PDF or CDF in log time, or by the analytical solution to confirm **proposition 2** above. ■

**Closed form equations for Q PDF and Q CDF**

By plotting the distribution in **equation 11** (both its PDF the integrand and CDF the integral under log frequency measure), it is quite easy to see that the EMM conditions 1, 2 and 3 above are satisfied. This is shown in **figure 1**, as a comparison between the PDF and CDF in frequency given in **figure 1a**, with the same distributions then plotted against log frequency in **figure 1b**. Although we could leave the analysis in this graphical form, it would be more useful to develop the Q PDF and Q CDF into analytical forms. By using closed form equations, pricing would be much more tractable. Their formulae might help yield valuable insights and add intuition.

To find the Q PDF and Q CDF equations, we first note that the EMM pricing is driven by the pricing kernel of **equation 11**. We also observe that, although the integration measure is in $ln\omega$, the integrand is expressed in $\omega$. These observations suggest that, by re-expressing the integrand PDF/CDF in $ln\omega$, we should arrive at the required EMM pricing distribution.

Consider the integrand in **equation 11**:

$$\phi''(\omega) = \frac{\omega/\lambda}{1+(\omega/\lambda)^2}$$

Rewrite showing the variables in log form:

$$\phi''(\ln \omega) = \frac{e^{\ln(\omega/\lambda)}}{1 + e^{2\ln(\omega/\lambda)}}$$

Substituting $x$ for $\ln(\omega/\lambda)$ then

$$\phi''(\ln \omega) = \frac{e^{x}}{1 + e^{2x}}$$

Multiply through by $e^{-x}$:

$$\phi''(\ln \omega) = \frac{e^{x}.e^{-x}}{(1 + e^{2x}).e^{-x}} = \frac{1}{e^{-x} + e^{x}}$$

Expanding $e^{-x}$ and $e^{x}$:

$$\phi''(\ln \omega) = \frac{1}{1 - (-x) + \frac{(-x)^2}{2!} - \frac{(-x)^3}{3!} + O.\frac{(-x)^4}{4!} - \ldots\ldots\ldots \quad +1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + O.\frac{x^4}{4!} + \ldots\ldots}$$

Since odd powers cancel leaving even functions, the distribution is square integrable and symmetric in $\ln(\omega/\lambda)$. It is found that $\phi''(\ln \omega)$ equates to a Cauchy distribution in log frequency (or log time, see below), for all values $x$ to leading order:

$$PDF^{Q}(jumps) = \phi''(\ln(\omega)) = \frac{\sqrt{2}/\pi}{2 + (\ln(\omega) - \ln(\lambda))^2}$$
$$= \frac{\sqrt{2}/\pi}{2 + (\ln(t) - \ln(\tau))^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

The PDF equation has an analytical cumulative Cauchy distribution of the form:

$$CDF^{Q}(jumps) = 0.5 + (1/\pi).arctan((\ln(\omega) - \ln(\lambda))/\sqrt{2})$$
$$= 0.5 + (1/\pi).arctan((\ln(t) - \ln(\tau))/\sqrt{2}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

**Equations 12 and 13** as equivalent martingale probability measures (i.e. Q-jump PDF/CDFs) define the risk-neutral probabilities of default (or variation in spread premia) at time *t*, for all times $\forall t$. These analytical forms for PDF and CDF are very practical for pricing applications. For example, by substituting the Q-jump equations into existing BSM formulae by replacing the BSM cumulative lognormal, $N(d)$, it should be possible to change the dynamic from one of diffusion to one of jumps, according to the nature of risks.

**Log time - log frequency equivalence**

The pricing **equations 12 and 13** are square integrable in the log of the variable. Since frequency and time are inversely related with $\omega = 1/t$, then $\log t = -\log \omega$. The consequence of this relationship, and the square integrability in the driving variable, is that the pricing equations are the same, whether expressed in either $\log \omega$ or $\log t$. This provides simplification and also some intuition on the meaning of the probability distributions for jump risk. In most pricing applications, the log time version is mostly used.

## 4. *Properties of Model*

In the theory of Lévy processes, jumps and Gaussian diffusion should cover the full range of financial variations since the jumps and diffusions elements can be arbitrarily chosen. The capability of the BSM to fulfil market pricing is well understood. In this section we test and interpret the capabilities of the Q-jump model in comparison to the BSM. The differences in dynamic between the BSM and the Q-jump model are detailed, in order to gain understanding of how the Q-jump model contributes to the overall financial pricing of markets.

### *1. Arbitrage free pricing kernels*

The risk neutral pricing capability is provided in both BSM and Q-jump models. Based on new evidence provided by the Q-jump distribution, the dynamics of jumps must however be considered very different from those of diffusion. Thus diffusion models can in no circumstances replicate the risks due to jumps and pricing models based on this premise will be correspondingly misleading. The randomness in the BSM is a steady slowly varying risk due to noise by instantaneous volatility. The source randomness in the Q-jump on the other hand represents uncertainty due to a conditionality based over time, i.e. an anticipation of an uncertain-in-time rare event. The attraction of both models is their availability of analytical pricing kernels for the PDF and CDF, in the case of the BSM through *N(d)* the cumulative of the lognormal distribution, and for the Q-jump its log Cauchy PDF and CDF:

$$PDF^{Q}(jumps) = \frac{\sqrt{2}/\pi}{2 + (\ln(t) - \ln(\tau))^2}$$
$$CDF^{Q}(jumps) = \ 0.5 + (1/\pi).arctan((\ln(t/\tau))/\sqrt{2}\,)$$

### *Jumps and diffusion dynamics differ absolutely*

The absolute difference in jumps and diffusion randomnesses is evident in their distinctive generic pricing kernels. The differences in pricing kernel reflect, as expected, a difference in the equivalent martingale measure (EMM) transforms for each process. In the BSM, the P to Q is obtained by a change a drift through the Girsanov theorem acting on variance. In the Q-jump, the P to Q is found by a logarithmic re-scaling in time or frequency. This can be summarised in the Radom Nikodym differentials for the BSM and Q-jumps respectively:

$\frac{dQ}{dP} = \exp\left(-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2\, dt\right)$ where $\gamma_t$ is a previsible process for drift and $dW_t$ a Gaussian Wiener process,

$\frac{dQ}{dP} = \frac{1}{\omega} = t$ this results in a log time or log frequency rescaling

*Orthogonality differences*

The other important aspect of jumps versus diffusion in their pricing kernels is in the nature of variables driving each process. The variables canonical to the Q-jump model are time (as log time), a risk aversion time constant $\tau$, the lower boundary condition of interest rate at short times (presumably a monetary policy driven choice), and the amount of single jump from 0 to 1. In comparison, the variables canonical to the BSM are the underlying asset price, its instantaneous volatility, risk free interest rate, and an upper bound condition, typically a strike price. The dynamics of jump and diffusion are then separable with behaviours that are truly orthogonal – a feature that justifies the treatment of diffusion and jump risk as mutually independent processes.

*2. Amounts and pattern of risk premiums*

Both BSM and Q-jump share the property of closed form solutions for pricing kernels and the invaluable property of parsimony in key variables. The Q-jump is particularly lean in this respect, with two term risk driving variables: one the risk aversion time constant $\tau$ and the second weighting indicator of upward-downward yield curve $\beta$, in the range 0-1. As is well known, the BSM is driven by value of underlying, volatility, strike price and risk free rate.

However, this is where similarities end. The PDFs and CDFs in the BSM and Q-jump are very different in amounts and pattern over time. This is most evident in graphs of PDFs and CDFs for each process as functions of time, given in **figures 1a and 1b.** One advantage of having PDFs and CDFs in analytic form, for both diffusion and jumps, is the ready method this provides to compare the premiums delivered by Q-jump model and BSM. This helps also in assessing the capability of Q-jumps to describe spread and term risk premium where the BSM fails.

**Figure 1a**

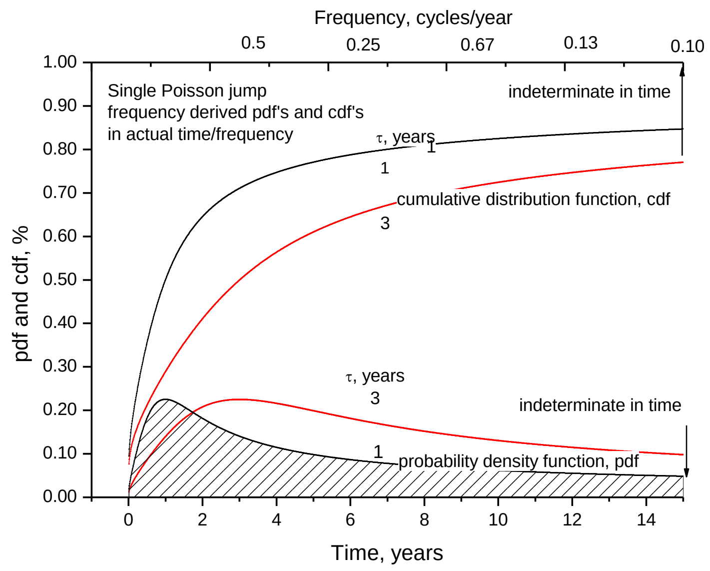
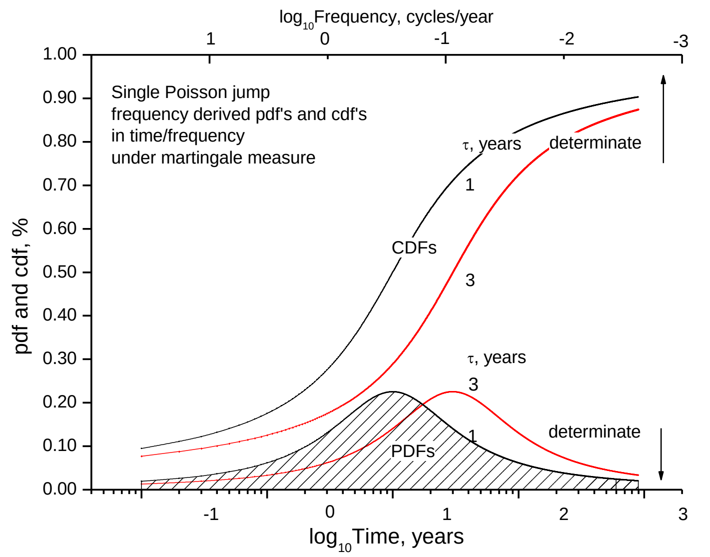



**Figure 1b**




The results for the PDF and CDF of the Q-single jump, when replotted in log time (frequency), reveal key features on the risk premium and martingale character of the model:

- The P distribution, although highly asymmetric and nonlinear in time, demonstrates the martingale condition, which is specified in **equations 12 and 13** above.

- The amounts of risk premium for the single jump in Q PDFs are consistently large, even at short times, and much greater than can be found in current model P distributions. As an example, Zhou (2001) evaluates the cumulate probabilities for

Poisson default jumps using the Poisson jump pathway model ($\lambda t$ as proxy), in comparison with the diffusion approach using the lognormal cumulative $N(-d_2)$. Based on realistic market leverage and volatility assumptions, the risk premiums over one year for P jumps although greater than diffusion are in the range 0.4% (cf. diffusion risk of even less 0.01%). For Q-jumps, however, risk premiums in the range 20 to 40% are readily achieved over one year, see **figure 1a**.

- The low levels of risk premium in P jumps is also revealed in the empirical study of Das (1998), on the term risk in interest rates due to P jump models. Das finds that typically a very large number of jumps need to be calibrated into models (40+ jumps in the year for the Poisson-Gaussian model and 30+ jumps for the ARCH-Poisson-Gaussian model) to obtain observed risk premiums. This suggests a rather low amount of risk premium occurs in P jumps.

The comparison with existing pure jump risk models above confirms that the most obvious difference of the Q-jump model to current jump models is its use of risk neutral (EMM) measure compared to the latter's dependence on objective P probabilities.

## *3. Structural linear properties of Q-jumps and BSM diffusion*

The ability to interpret the risk in the time dimension is key to the Q-jump solution. This has consequences in providing fundamental meaning on the nature of jump risk, leading to new intuition. The randomness effects in the BSM and in the Q-jump are orthogonal. That means that no correlations exist in the dynamics, between the two mechanisms. This is also understood in the timescales relevant to the variations being modelled. In the BSM, the risk is due to instantaneous constantly occurring variations. In the Q-jump, the time-scale of effects vary from near term, weeks to long term, years. The build-up of risk premium in jumps therefore anticipates the risk over the medium to longer term, and needs bear no relation to the instantaneous diffusion type of volatility. The jumps behaviour is structural in relation to market behaviour, the diffusion may relate more to variations linked to Gaussian correlation behaviour.

Since both Q-jump and BSM are martingale models, they both must be linear models in their outputs. This greatly simplifies the use of these models, since most applications entail the pricing of portfolios not individual instruments. We can briefly review the difference that the pricing kernels of jumps and diffusion make to the dependence properties, which drive the pricing of portfolios, in relation to the nature of the linearity and its superposition properties between the two processes. In the Gaussian structure, the dependence and complete independence between entities should be found in the distribution. These are separable in the Gaussian case by specifying only the correlation coefficients between entities. The risks are reinforced or diversified across entities leading to principles on how risk should be priced, as systematic (only the dependent) part of risks.

The dependence in Q-jumps is, however, governed by a totally different mechanism. In the Q-jump, the correlation driving the systematic risk occurs across time. The correlations are therefore relevant and possibly diversifiable, but only in the time dimension. No correlations (therefore dependencies) are expected for Q-jumps across entities.

### *5. Yield and Credit Spread Results*

A good test of the jumps model is to examine its application to term structure, in deep and liquid markets which display jumps. The market pricing capabilities of the Q-jump should then be captured. In this paper, we therefore choose yield and credit spreads data as the first example of Q-jump behaviour; although we note that the Q-jump risk should be significant to any market where the underlying is in the form of a rate process, with an associated term structure to model.

### *Yield curve model under Q-jumps*

To test the Q-jump model for term risk in the yield curve, we examine a simple model based on the shapes and levels of the interest rate yield. The first demonstration is given on the government spot rates from published UK data. Essentially we assume the yield curve follows as an exponential affine process in which the process is stochasticity determined by the Q pure jump pricing kernel. The shape of the yield curve frequently shows an upward yield curve over much of the economic cycle. It is also observed that the yield curve can take inverted and humped shapes in the yield curve. It will be important to model this behaviour, which is treated in the empirical paper on the Q-jump model accompanying this paper forthcoming. In this paper we concentrate on the basic upward curve. The assumptions are designed to fit the overall notion that the yield curve occurs over its full range in time one week to 15+ years.

The equation for the yield curve then follows an exponential affine equation of the type:

$$r = e^{-Z(\lambda,t)}$$

i.e. $$dlnr = dZ(\lambda, t)$$

To set the lower bound of very short time rates, assume the entire all the term risk is Q-jump risk driven. Then all but the lowest bound is determined by the model. Substituting for the Q-jump dynamic **equation 13**, obtains:

$$lnr = lnr_0 + \beta.\left[(0.5 + \frac{1}{\pi}.\arctan\left(\frac{\ln(t/\tau)}{\sqrt{2}}\right)\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots\ldots (14)$$

Keeping the initial rate $lnr_0$ separate as a given, there are two explanatory variables in **equation 3** for the behaviour of the yield curve. We qualify this by noting that inverse and hump-backed curves are excluded in this scheme. This can be overcome as we show in an accompanying paper shortcoming that two mechanisms of the type **equation 3** are needed for upwards and downwards yield curves. The two variables are: the term risk aversion time constant $\tau$, and the amount of upward/downward component $\beta$ (not to be confused with CAPM $\beta$). The time constant $\tau$ represents the markets view of risk for waiting in the market. This is likely to be a composite of time affected risks: market credit risk or liquidity risk apply. $\tau$ shortens with greater time effect risk in the market, and lengthens with less of it. It is tempting to relate this to the liquidity risk in markets. This could be a valuable insight, since, despite many efforts, it has proved difficult to provide measures of liquidity risk in past theory. The $\beta$ factor varies according to the steepness of the $ln(r_0)$ yield curve in the range 0 to 1. If all term risk is due to one jump, then the exponential affine model equation above, $\beta$ would tend to 1.

Returning to the nature of the initial interest rate to the yield, this represents the lowest bound of the yield curve. In the model proposed this falls outside the realm of the term risk effect. The initial interest rate will vary according to the setting of the short rate by governments/momentary policy committee. To that extent, it is a given and outside the market dynamic of term risk.

*Parameterisation of the yield curve model*

To test the model we parameterise the model to the yield curves found for the government benchmark data of the UK over the years 2001 to 2003 with the results shown in **figures 2**. **Figure 2a** provides the data plotted against time, and **figure 2b** the same data against log time. The data in **figure 2b** enables the parameters of the Q-jump model to be determined for each curve. The method is to fit the curves by Levenberg-Marquardt nonlinear fitting routine to establish the state variables: $\tau$, $\beta$, and $r_0$, of the Q jump model.

Also shown in **figures 2** are the yield curves for the corporate bond data of UK AAA bonds. We show these to illustrate the relative position of the risky curves to the riskless benchmark curves. We later analyse these with the Q jump model for their credit spread term structure in the next section. The corporate data in this paper are for corporate bonds from the Merrill Lynch database. The latter are collated over S&P ratings and maturity buckets, enabling the analysis we describe.

**Figure 2a**

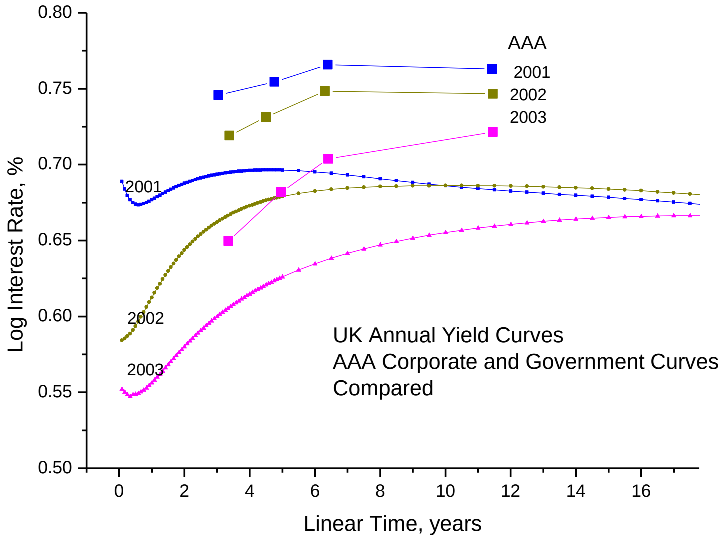
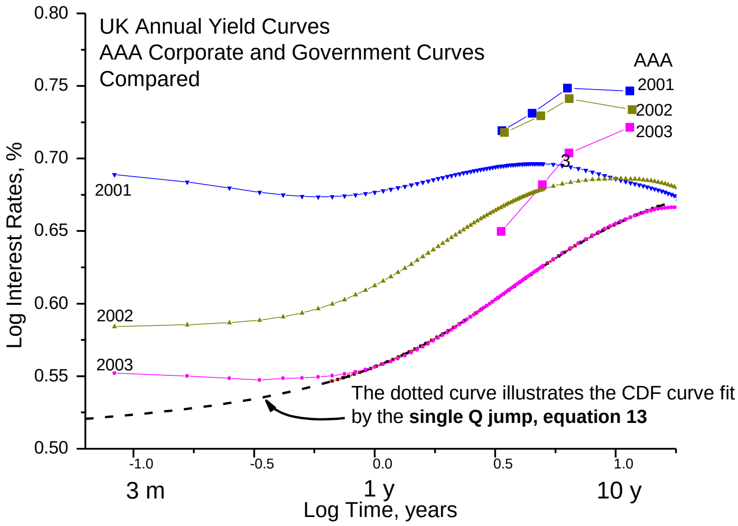


**Figure 2b**




The results in **figures 2a and 2b** show the variation of the yield curves over a period of market decline then recovery over the economic cycle 2001 to 2003. The analytical points:

- The annual benchmark curves capture the initial downward adjustment in markets as an inversion and shortening of the time constant $\boldsymbol{\tau}$. The yield curve deepens and the time constant $\boldsymbol{\tau}$ progressively lengthens, as recovery is anticipated.

- The fitting of the log Cauchy distribution of the single Q jump to yield term structure is clear cut when yield curves are plotted in log time.

- Corporate bond yield curves follow the trend of benchmark curves. The data points are much more limited for the corporate bond yields. On the other hand, armed with the model and realistic assumptions about beginning and end points, accuracy should be improved by taking advantage of the benchmark curve's greater accuracy, due to sample size. In analysis two co-integrations were considered – that the risky bonds are sensibly parallel to the benchmark or that risky and riskless bonds share the same lower bound, short rate. So far we have found the lower bound approach the more useful.

***Parameterisation of the credit spreads model***

To model the credit spreads, we take the relatively simple model as for the interest rate yield curve above, and compare the two sets of curve: risky and riskless, with the equation:

$$dln\lambda =$$

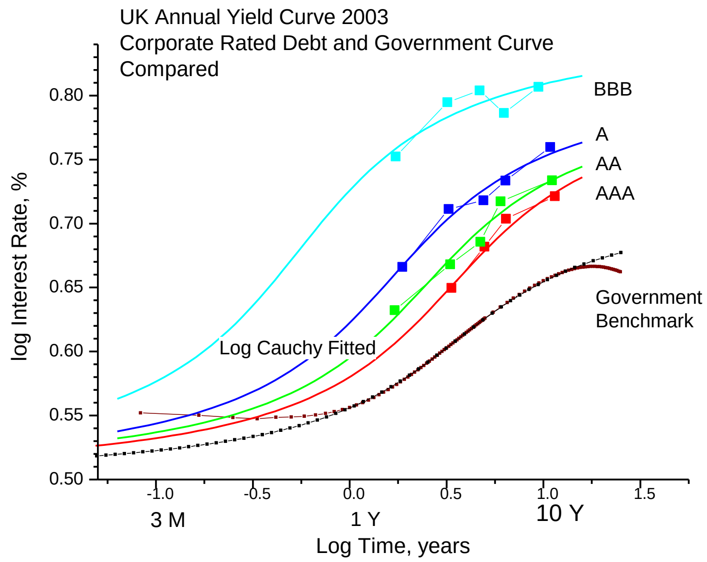
$$\beta_B\left(.5 + (1/\pi).\,arctan((\ln \mathrm{t} - \ln\tau_B)/\sqrt{2}\right)$$
$$-\beta_G\left(.5 + \quad (1/\pi).\,arctan((\ln \mathrm{t} - \ln\tau_G)/\sqrt{2}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots.(15)$$

The spreads are calculated by the difference between the risky and riskless bond yield curves in **figure 3.** This entails two equations of type **equation 14**, which summarise to **equation 15** above**.** If the yield curves are sensibly parallel, then $\beta_1 = \beta_2$ , and **equation 15** simplifies to dependence on term risk aversion time constants $\tau_B$ and $\tau_G$ only.

*Parameterisation of the credit spreads model*

Using the annual yield curves for the UK over the year ending 31 December 2003 for the UK we obtain the spread results now shown in **figure 3**.

**Figure 3**




*Interpretation of Q-jump yield and spreads results*

The credit spread results for the UK for 2003, are calculated from the risky and riskless yield curves shown in **figure3**. These show the upward sloping yield curve with the UK benchmark curve of government bonds, and composite yield curves for the rating categories AAA, AA, A, and BBB (S&P ratings).

The results shown in **table 1** provide the calibration coefficients, $\tau$ and $\beta$ for the spreads in **figure 3** based on the Q jump spread structure **equation 15**, and the co-integration assumptions, already set.

**Table 1 - UK 2003 annual average Q-jump spreads and calibration coefficients by rating, by Exponential Log Cauchy Model**

| Government bond or S&P Rating | Single Jump Intensity, $\beta_G$ or $\beta_B$ | Time constants $\tau_G$, $\tau_B$ years, y months, m | Q-jump spread, bps |
|---|---|---|---|
| Government benchmark | 0.52 | 3y 6m$\pm$0.3m | |
| AAA | 0.66 | 2y 9m$\pm$3.2m | 41.7$\pm$17.1 |
| AA | 0.68 | 2y 4m$\pm$2.3m | 51.4$\pm$10.2 |
| A | 0.72 | 1y 8m$\pm$1.4m | 86.6$\pm$7.2 |
| BBB | 0.81 | 4.8m$\pm$1.3m | 190.2$\pm$29.3 |

**The lowest bound interest rate, $r_0$ = 3.1%**
**Errors to 95% confidence limits**

*Meaning of the Q-jump model and extensions*

The above yield and spread results are modelled on Q-jumps exclusively. The results shown, represent only a limited sample, however of a wider analysis that we have been carried out over several corporate bond populations and economic cycles. They are representative of the upward yield curve only, i.e. an economy anticipating growth mode. Analysis of the other forms of curve: inversions and downward humped yields, are left to an accompanying paper forthcoming. Essentially, it is found that inversion can be modelled by assuming competing single Q-jumps. The Q-jump can occur as one or other of an upward downward term risk, or a combination of these two in competition over the full range of the yield curve.

The results in **table 1** on term spreads are broadly consistent with the model of the yield curve, in **equation 14**. The credit spread varies with rating quality, correspondingly in the risk aversion constant tau. But also as the economic situation varies $\tau$ shortens as the primary component of the spread risk. The spread is fairly flat in this model over a wide range of time. The log scale tends to generate this condition since the flatness of the spread is maintained over a wide range of times at the short end. The beta factor which denotes how much of the single jump from 0 to 1 is required varies from the benchmark curve at .52 to .81 for the riskiest category. For consistency of interpretation, a upward one jump dominated curve should produce a theoretical value of $\beta = 1$, if the randomness is due to the anticipation of a single jump.

The Q-jump results also help distinguish between diffusion risk and jump risk behaviour. In these results, the Q-jump model performs well in producing the large amounts of term risk premium in yield curves and spreads. In comparison, current models whether on stochastic volatility or pure jumps, have difficulty in generating such levels.

An advantage of the Q-jump model is that it should be possible to measure the relative occurrence of jumps and diffusion risk, in a more controlled way. This has been a stumbling block in jump diffusion and Lévy modelling to date, when it has not been possible to readily separate the empirical effects in each process. From our results so far, it seems that fixed income pricing might be more strongly affected by the term risk effects from Q jumps, whereas the dominant risk dynamic in equity pricing might be dominated by diffusion risk, as exemplified in BSM Gaussian dynamics. An interesting feature of the empirical evidence we have so far collected is the coherence of this interpretation, even in limited sample. However, it would be naive to expect the financial markets to be so cleanly divided. A combination of risks is more likely, with sometimes one effect more prominent than the other. For an option written on a credit spread, for example, then jump risk might determine the down-and-out component, and the diffusion element account for the diffusion risk, due to the effect of general noise in the market. The advantage of the Q-jump model is that it should allow jumps and Gaussian diffusion to be modelled together, within current option pricing formulae. Both distributions are independently linear (as unique EMMs), therefore can be added as mutually exclusive events to obtain:

$$CDF^Q(option) = {CDF_J}^Q + {CDF_D}^Q - {CDF_J}^Q . {CDF_D}^Q$$

where $J$ denotes jumps and $D$ diffusion. The availability of an analytical Q-jump would then offer a route to the risk neutral option pricing of risk, across both jumps and diffusion.

## *6. Conclusions*

The Q-jump model was developed for the market pricing of term risk due to jumps exclusively. To derive the Q-jump model, advantage was taken of equivalent models in the sciences for Poisson jumps. In stages, the first step was Fourier analysis to provide a P distribution in frequency for the single process Poisson jump, the second step was to obtain the distribution of marginals from the Fourier components in frequency to provide the P marginals for jumps. Then an original equivalent martingale transform was deduced for the single jump process.

A number of conceptual problems were addressed before arriving at the route to solution. A key realisation was that jump risk must occur over single jumps within their pathway dynamic not their compensated Poisson distribution arrivals or their amplitude of jumps risk. The received wisdom that pathways are always integer and therefore controlled by objective probabilities had to be rejected and the concept that a stochasticity process must exist in the single jump pathway accepted. The Fourier analysis was used to deconvolute the integral to find the distribution. To find the latter also invoked the concept from the sciences that the out-of-phase Fourier component is the probability density of the process. Once this was appreciated, the equivalent martingale measure was forthcoming as a new EMM transform, as a logarithmic rescaling in frequency (or time). An analytical equation was found by either graphical or analytic means. The EMM PDF and CDF of the Q-jump model were then obtained as the log Cauchy (a Cauchy distribution rescaled into log time or log frequency).

The model was first tested first on the properties of its analytical equations, and then by applying the Q-jumps to the interpretation of yield curves and spreads in interest rates. Several economic insights were possible. Much larger and more instant risk premiums could be obtained through the Q-jump pricing distributions than the Gaussian Wiener diffusion risk premium. A large part of this was due to the much larger risk premiums in the Q-jump model compared to the P distribution models. We could count the Merton 1976 jump-diffusion model, or pure jump models such as the Cox double stochastic model and the infinitely divisible Lévy models as P distribution models. The structural models could also be eliminated for low risk premiums, amongst other reasons.

The most important assumption, verified experimentally, was that as jump risk occurred over a single jump must be a unique trajectory; the complete markets condition was therefore assured. The Q-jump's martingale equation was a complete markets pricing martingale. In contrast, Lévy pure jump models, including the Cox double stochastic model, provided only semi martingale solutions.

Fundamental insights were obtained on the structural linear properties of Q jumps compared to Gaussian diffusions. The BSM and Q-jumps dynamics were fundamentally different in their correlation processes, dependencies and systematic risks. The correlation driving variables for jumps (and term risk) were in the time direction. The correlations in BSM Gaussian diffusion are on the other hand across entities, and time-homogeneous. The Q-jump and BSM Gaussian models, therefore, offered a method to price risks across the two effects systematically, in combined yet distinctive processes.

Another insight from the nature of the Q-jump randomness was its possibility for use in any randomness affected term risk, its applicability to rates, and other risks of a conditional term risk nature. This then could be a new means to price derivatives for forward rates generally: e.g. in interest rates, spreads, FX or cost of carry rates, as an area where a new fundamental theory of conditional inter-temporal variations could be the most useful.

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