Staged Venture Capital Contracting with Ratchets and Liquidation Rights: An Analysis of Financing Constraints

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Abstract

While the theoretical literature strongly argues in favor of ratchets (anti-dilution protections) and liquidation rights, recent empirical studies documented limits in their use. This paper studies firms' financing constraints in a model of staged venture capital financing where new-money raised and post-money values evolve randomly over time. Our analysis reveals that ventures fail if new money raised is "too small" or "too large"; in the latter case this is despite *positive* pre-money value. We analyze the conditions under which a venture capitalist will accept modifications of her contract provisions and document quantitatively the relevance of our results.

Keywords

options, venture capital, preference rights

JEL classification

G24, G30, G13

1 Introduction

Ratchets (anti-dilution protections) and liquidation rights are common preference rights in US venture capital contracts, but neither are they used in *all* US financings, nor are they a popular feature internationally, see, e.g. Cumming and Johan (2006) and de Bettignies (2008). Despite this empirical evidence, a large body of the theoretical VC literature studies agency problems and concludes that preference rights are optimal contracting provisions[1]. Our paper studies the feasibility of contracting for an entrepreneur and compares the financing constraints she faces with ratchets and liquidation rights.

We introduce a model of staged venture capital, where two different venture capitalists (VCs) provide financing at two subsequent dates. To capture the uncertain prospects of the company across time, both new money raised and the post-money (venture) value evolve randomly over time. Liquidation rights give VCs a minimum return (called multiple) on their invested capital; ratchets protect a VC from dilution when the value of their share decreases from one round to the next. These preference rights are common to most venture contracts, see Kaplan and Strömberg (2003)[2]. Throughout we compare different multiples and study two extreme situations: one where the VC holds no ratchet and one where she holds a full ratchet. In line with the literature, see, e.g., Amit et al. (1989), and Admati and Pfleiderer (1994), we assume that VCs invest at zero net present value and use this to derive their stakes. For pricing we use the real-options approach, see, e.g. Dixit and Pindyck (1996), Duffie (2001), Trigeorgis (1996).

We prove that, at both dates, either with or without ratchet, a VC can contract only within some interval for new money raised in relation to current post-money value. When the liquidation multiple is larger than the bond return, we show that "small" financings are *not* be feasible *with* a multiple; the reason for the failure is that the multiple then pays "too much," i.e. the multiple is "too large". Furthermore we prove that, if the financing gets "too large," contracting also fails.

[1]References include, among others, Gompers (1995), Bergemann and Hege (1998), Cornelli and Yosha (2003), and Schmidt (2003).

[2]Sahlman (1990) describes in detail preference rights as control mechanism in Venture Capital: cash-flow rights, staging of capital and syndication. Our focus is on the allocation of risk and return; in addition to the cash-flow rights that we study, dividend preferences are common. However, dividend preferences play a role similar to liquidation preferences; therefore, for simplicity, we only study liquidation preferences.

Clearly, new money raised cannot be larger than the post-money value; however our analysis shows that financing *also* fails when it is strictly smaller than post-money value. This destruction of positive pre-money value seems counterintuitive, but in staged financings the multiple of either venture capitalist (VC) impacts the other VC in such a way that she *cannot* recover her invested money if the amount is "too large." Furthermore, we characterize the situations under which the ratchet applies and when it has to be waived. Finally we discuss the interplay between excessive multiple, dilution, ratchet waiver and venture liquidation despite positive pre-money value. We document, among others, when second round post-money value is small compared to the post-money value in the previous round, there is no way to evade liquidation. Also, while multiples are a financing constraint for small financings, for intermediate sizes of financings they help to evade share price dilution and the need to waive ratchets.

One may counter that this failure and the financing constraints they impose are qualitative observations that are quantitatively not relevant. To address this, we illustrate our results in numerical examples using parameters taken from the empirical literature. We then document that the intervals within which VCs contract are fairly small: The company will be liquidated despite positive pre-money value if the new money raised is at roughly 2/3 of post-money value, i.e. if new money is more than twice pre-money value. With a multiple of 3 the minimum financing amount is at 20% of post-money value. We also find up to 90% probability that the company is either liquidated or the multiple reduced, and about 40% conditional probability that the ratchet will be waived when the share price decreases. Finally, we see that the liquidation multiple affects the company stake considerably, but that the ratchet makes no big difference for the first VC.

Our paper is related to the literature that looks at the use of preference rights in VC. While a large body of the literature focused on US venture capital, recent empirical evidence added an international perspective: Lerner and Schoar (2005), Cumming (2005), Kaplan et al. (2007), and Cumming (2008) all found that outside the US preferred equity with ratchets and liquidation rights is not the predominant form of venture capital financing. Addition to this, Gilson and Schizer (2003) have forcefully argued that tax rationales explain the prevalent use of preference rights in US. Despite this, the theoretical literature is dominated by the view that ratchets and liquidation rights are optimal contracting provisions between the entrepreneur and VCs: they come to this conclusion in analysis of the principal agent relationship between entrepreneur and VC (Amit et al. (1989), Admati and Pfleiderer (1994), Gompers (1995), Bergemann and Hege (1998), and Schmidt (2003)), of the double moral hazard problem when VC and entrepreneur to provide effort (Hellmann (2006)), and in analysis of window-dressing by the entrepreneur (Cornelli and Yosha (2003)).

To our knowledge, financings constraints that stem from these rights have not been studied so far[3]. The advantages of cash-flow rights as incentivization devices have been studied and stressed extensively in the literature so far; however, our paper points out that these preference rights impose severe financing constraints on the venture. First of all, we show that ventures may fail despite positive pre-money value. This matches the concern often voiced that ventures fail despite promising business prospects. While Boyle and Guthrie (2003) attribute this failure to informational asymmetries, we relate it to financing constraints. Venture capital is an important source of financing for entrepreneurs; indirectly, our paper thereby also relates to the literature on the availability of financing for new businesses, see Kerr and Nanda (2009). Second, our paper shows that new financings need to involve considerable amounts of money relative to post-money value; this restricts the amount of money to be raised. While it is often said that small amounts of money are too costly to be raised from VCs we point out, that this is due to the liquidation preferences in venture capital contracts and not necessarily a feature of venture financings itself. Finally, we stress the importance of adjusting contract provisions, i.e. waiving ratchets or reducing multiples.

The remainder is organized as follows: the next section presents our three-date setup. Section 3 looks at date 2 events, including the decision problem of a new VC coming in and the effect of any ratchet on contracting; section 4 then looks at the date 1 decision problem of the initial VC. Section 5 concludes the paper. All proofs are postponed to the appendix.

[3]Our paper is related to the valuation of ratchets and liquidation rights, but this problem has been ignored in the literature with the exception of the continuous-time analysis of Cossin et al. (2002).

2 Model

Detailed overviews of the VC Market are available, e.g., in Fenn et al. (1995), Gompers and Lerner (1999), Sahlman et al. (1999) and Metrick (2007). Ratchets and liquidation preferences are discussed in detail, e.g., in chapter 9 of Metrick (2007) and in chapter 13 of Bagley and Dauchy (2002). This section introduces our model of VC financings over two rounds with ratchets and liquidation rights.

2.1 Events

A new venture typically requires many financing rounds, but for simplicity we assume that there are exactly two. We look at a new venture company that has been founded by an entrepreneur at date 0; the founder holds all its equity until date 1. At dates 1 and 2 the venture requires new financing from a VC; at the final date 3 the company is liquidated either through a sale, an IPO or a liquidation of assets. To keep notation simple, we assume that in each financing round exactly one VC enters and that the VC in the second round is different from the VC of the first round.

At date 1 the entrepreneur raises capital in the amount $N_1 > 0$ from the first VC by issuing new shares to the VC; we denote the stake[4] of the entrepreneur (first VC) after the financing by α_0 (α_1). At date 2 the company requires additional financing in the amount $N_2 > 0$. If the second VC provides financing, we denote α_2 the company stake she contracts; the company stakes of the entrepreneur (VC) will be adjusted to α_0^a (α_1^a).

We derive a notion of fair contracting from fair pricing: at date 2, conditional on P_2, N_2, we first determine the value $V_2(\alpha_2)$ as a function of the second VC's stake α_2; we then choose α_2 as the one for which the investment is priced fairly, i.e. $V_2(\alpha_2) = N_2$. Similarly, we determine the date 1 value $V_1(\alpha_1)$ of the first VC's ownership; fair contracting means that we look for α_1 with $N_1 = V_1(\alpha_1)$. Note that the date 1 value function will have to take account of the date 2 dilution in ownership, any ratchets and any liquidation preference of the second VC.

[4]Throughout this paper we do not study the numbers of shares that a party holds, because only relative share holdings matter. We focus instead on the *fraction* of a company a party holds, the so-called company *stake*.

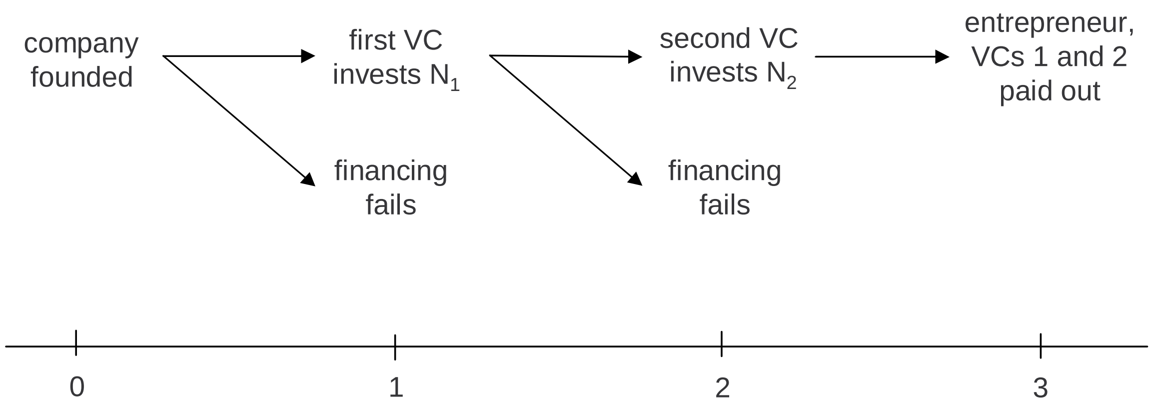

Figure 1: The timeline of events.

If a VC requires a stake larger than 100%, the financing fails and the company will be liquidated. If the ownership required is negative, we assume that contract specifications will be adjusted by the VC. If the company has not been liquidated before date 3, it will be liquidated at this date and all remaining values paid out to the parties according to their liquidation preferences. Figure 1 depicts the timeline of events.

2.2 Preference Rights

Liquidation preferences are expressed as a multiple of invested money with a typical multiple of 1, 2 or 3, see, e.g., the quarterly Fenwick and West (2010) market surveys. We denote $M_1 \geq 1$ and $M_2 \geq 1$ the multiples of the first and second VC, respectively. For simplicity of exposition we only look at participating liquidation preferences, i.e. after holders of preferred stock receive their full liquidation preference, they then also share with the other shareholders (in particular with the entrepreneur holding common stock) in the remaining amounts. Furthermore, we assume that the second VC gets senior liquidation preference over the first VC, i.e. she receives her liquidation preference before the other VC. Finally we assume, unless stated otherwise, that the first and the second VC contract the same multiple $M_1 = M_2 = M \geq 1$. (In some situations the second VC will reduce M_2; this will be discussed later.)

Ratchets (partially) protect a VC against value losses in financing rounds when the price per share decreases, a so-called down-round; the so-called full ratchet is an extreme case. The idea is to grant old VCs a sufficient number of new shares so that their cost per share equals that of the new VC coming in[5]. (When there is not a down-round the anti-dilution protection does not impact old VCs' share holdings.) In our three date model, there is no financing after the second VC gets in and so it makes only sense that the first VC holds a ratchet; it applies *only* when the price per share decreases between dates 1 and 2.

To analyze the ratchet we need to study the number of shares issued and, only for this purpose, we assume throughout the remainder of this subsection that the entrepreneur holds one share (before date 1 and thereafter); this assumption will neither affect the *relative* share price changes over time, nor any party's stakes. We denote S_1 and S_2 the number of shares the first and second VC contract at date 1, respectively. (Note that $\alpha_1 = S_1/(1+S_1)$ gives the stake the first VC contracts at date 1.) Share prices are based on new money raised and new shares issued in any financing round, i.e. they are $p_1 = N_1/S_1$ and $p_2 = N_2/S_2$.

Let us look at date 2 and assume that the post-money value is P_2 and new money raised is N_2. The fair contracting stake for the second VC from date 2 forward fulfills $\alpha_2 = S_2/(1+S_1+S_2)$ under the assumption that the ratchet will not take effect. This translates into

$$S_2 = \frac{(1+S_1)\alpha_2}{1-\alpha_2} \text{ and } p_2 = N_2/S_2. \tag{1}$$

When this share price is not lower than the date 1 share price ($p_2 \geq p_1$), the second VC will know that the ratchet does not take effect and accept S_2 shares in exchange for N_2. Because the second VC wants to contract α_2, the first VC's stake will be reduced by issuing new shares. The stake was α_1 between dates 1 and 2; after the financing, the first VC's stake is α_1 of the "remaining" company, i.e.

$$\alpha_1^a = \alpha_1(1-\alpha_2). \tag{2}$$

If the share price p_2 is lower than p_1, and the first VC holds a full ratchet, then the ratchet applies. She receives as many additional shares that she holds S_1^a in total afterwards[6], where

[5] There are other ways to adjust the number of shares, including the so-called weighted-average ratchet. In this paper we only study the two extremes, either that the second VC has no ratchet or that shes has a full ratchet.

[6] In practice a conversion ratio is introduced that determines the ratio by which preferred shares convert into common shares. This ratio is adjusted by the anti-dilution protection. However, to simplify our presentation throughout this paper we think of all shares on a so-called "fully converted basis,"such that the effect of the anti-dilution protection is to issue additional shares to old VCs. Note that the liquidation preference of a VC is based on the total capital invested and will not be affected by adjustments in the number of shares.

this new number of shares of the first VC is set in such a way that the (fictitious) date 1 price of her shares N_1/S_1^a equals p_2, i.e. $S_1^a = N_1/p_2 = p_1/p_2 S_1$. However, there is a "second round effect" to this: the second VC will anticipate that the first VC will hold more shares and that her company stake reduced below α_2 *if* she contracts S_2 shares. To counter this from the start, she will ask for as many shares S_2^a such that, taking account of the ratchet, she holds exactly the company share α_2 afterwards, i.e.

$$\alpha_2 = \frac{S_2^a}{1 + S_1^a + S_2^a}.$$

Then the first VC will hold the stake

$$\alpha_1^a = \frac{S_1^a}{1 + S_1^a + S_2^a} = \frac{S_1^a}{S_2^a}\alpha_2 = \frac{N_1}{N_2}\alpha_2. \tag{3}$$

2.3 Values

We denote the (physical) time between dates 1 and 3 by T; for simplicity of exposition we assume that the second financing date 2 is halfway between the first financing date 1 and the final date 3, i.e. at (physical) time $T/2$ from each. Over time, the so-called post-money value of the company and the new money raised are bivariate lognormal distributed[7], i.e. for dates $i = 1, 2$:

$$P_{i+1} = P_i \exp\left\{\left(\mu_P - \frac{\sigma_P^2}{2}\right)\frac{T}{2} + \sigma_P\sqrt{\frac{T}{2}}Z_i^P\right\}, \tag{4}$$

$$N_{i+1} = N_i \exp\left\{\left(\mu_N - \frac{\sigma_N^2}{2}\right)\frac{T}{2} + \sigma_N\sqrt{\frac{T}{2}}Z_i^N\right\}. \tag{5}$$

Each of the pairs (Z_1^P, Z_1^N) and (Z_2^P, Z_2^N) are bivariate standard normal random variables; we assume the correlation of Z_i^P and Z_i^N is constant over time and denote it by κ_{PN}. Note that the so-called post-money value in venture capital financings includes new money raised in a new financing round. To simplify exposition, for the remainder of this paper we think of all values as fractions of date 1 post-money value and set $P_1 = 1$.

[7] Cochrane (2005) reports that the distribution of post-money values is close to a lognormal. To our knowledge the distribution of post-money values and new money raised has not been analyzed, yet. Our choice is made for convenience.

At any date, when the company is liquidated, the liquidation value will be distributed to all shareholders according to their cash flow (liquidation preference) rights. (The entrepreneur receives the entire date 1 liquidation value; we will neither model nor discuss it, because we are interested in the impact of the VC's contracting terms on the feasibility of financing.) We assume that the date 2 liquidation value $L(P_2, N_2)$ is a (typically fairly small) fraction λ of the date 1 post-money value P_1, i.e. $L(P_2, N_2) = \lambda P_1$. The idea is that, in case of liquidation, investors will recover some fraction of the initial value. At date 3 we assume that the liquidation value is given by the date 3 post-money value P_3.

Both value functions V_1, V_2 require the valuation of future contingent payoffs. For this we use the risk-neutral (real options) approach, i.e. we denote the constant (continuously-compounded) interest rate by r and redefine the instantaneous mean of P and N to be equal to the risk-free bond return for the time-horizon of interest[8], $\mu_P = \mu_N = r$. At date 2 we can simplify our calculations: Because the distribution of post-money values is (conditional) lognormal, the value of date 2 call options on (terminal, date 3) post-money values is given by the well-known Black-Scholes formula $BS(r, \sigma_P, P_2, K, T/2)$; here P_2 denotes the (conditional) date 2 post-money value and K the strike price of the option. The parameters $r, \sigma_P, T/2$ refer to the interest rate, the volatility of post-money values and the physical time between the valuation date and maturity of the option. They are fixed; therefore we write $BS(P_2, K)$ for simplicity throughout this paper.

3 The Investment Problem for the Second VC

This section studies how the second VC contracts at date 2. She observes the post-money value P_2 and is asked to provide financing in the amount N_2; throughout this section, our analysis is conditional on these values, but to keep notation simple we usually drop dependence.

If the second VC refuses financing at date 2, the company is liquidated and all assets distributed according to the liquidation preference of the first VC. The resulting payouts to the entrepreneur and the first VC will be studied later in subsection 4.1 when we look at the date 1 investment problem of the first VC. If the second VC provides financing, the company stakes of

[8]For a detailed discussion we refer the interested reader to Dixit and Pindyck (1996), and Trigeorgis (1996).

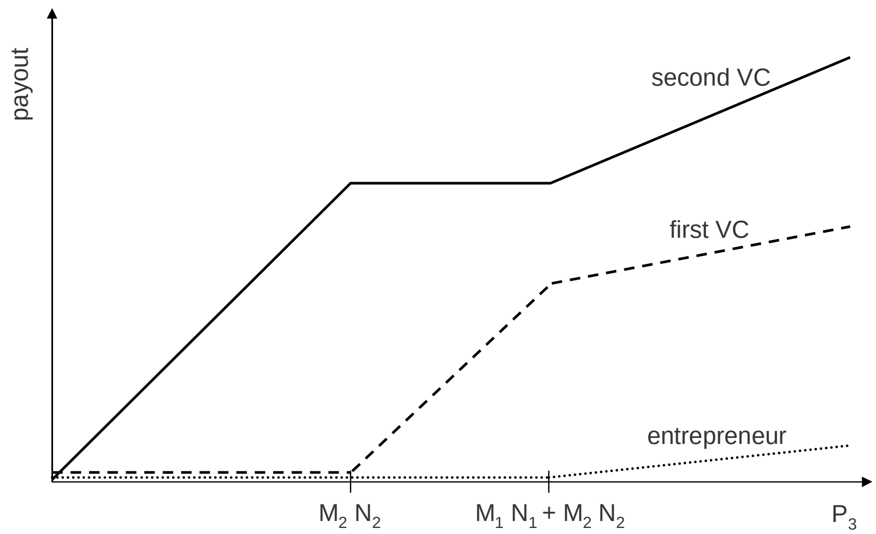

Figure 2: The date 3 payoff profiles.

the entrepreneur and the first VC will be adjusted to α_0^a, α_1^a.

3.1 The Second VC's Stake

If the second VC invests she has seniority over the first VC at date 3, i.e. she will be paid before any other party until she received a multiple $M_2 N_2$ of her date 2 investment N_2. (The value to be paid out is the date 3 liquidation value P_3.) After the second VC has received her multiple, the first VC will be next in line and be paid up to a multiple $M_1 N_1$ of her date 1 investment N_1. Finally, after the two VCs received their multiples, any remainder is distributed pro-rata among the two VCs and the entrepreneur. Figure 2 depicts the payout profiles as a function of date 3 post-money value.

The payout of the second VC can be interpreted as a portfolio of call options (with maturity at date 3) and the (so-called) "underlying security," her the date 3 venture value: one unit in the venture, -1 unit in the call option with strike $M_2 N_2$ and $\alpha_2(P_2, N_2)$ units in the call option with strike $M_1 N_1 + M_2 N_2$. Then the value V_2 of the second VC's claim can be calculated and expressed using the Black-Scholes formula:

$$V_2(\alpha_2) \quad = \quad P_2 - BS(P_2, MN_2) + \alpha_2 \cdot BS(P_2, M_1 N_1 + M_2 N_2).$$

Fair contracting requires that the value of the second VC's stake of the company equals N_2;

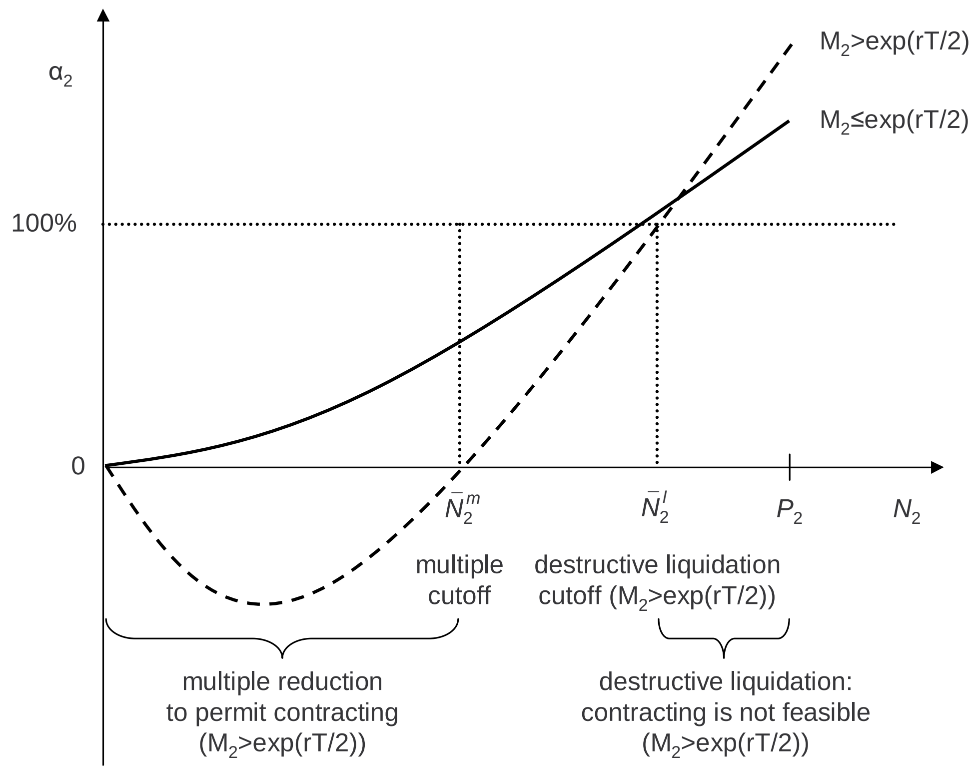

Figure 3: The shape of the second VC's stake as a function of N_2.

it gives us her stake:

$$\alpha_2(P_2, N_2) = \frac{N_2 - P_2 + BS(P_2, M_2 N_2)}{BS(P_2, M_1 N_1 + M_2 N_2)}. \tag{6}$$

Proposition 1 *For given $P_2 > 0$ we study the second VC's stake as a function of $0 \leq N_2 \leq P_2$. The function is 0 at $N_2 = 0$. If M_2 is smaller than or equal to exp$(rT/2)$, then the function increases monotonically on the interval 0 to P_2. If M_2 is larger than exp$(rT/2)$, then the interval 0 to P_2 splits into two intervals and the function decreases (increases) monotonically for smaller (larger) values of N_2. For all multiples $M_2 \geq 1$, the stake α_2 gets larger than 100% for some N_2 strictly smaller than P_2.*

Note that zero financings are not economically meaningful; here we study them only as limiting cases. Proposition 1 implies that for $M_2 \leq \exp(rT/2)$ the stake α_2 is always positive while for $M_2 > \exp(rT/2)$ it gets negative initially. For given post-money value P_2 figure 3 illustrates the shape of α_2 as a function of $0 \leq N_2 \leq P_2$ for both cases $M_2 > \exp(rT/2)$ (dashed line) and $M_2 \leq \exp(rT/2)$ (solid line). (The different cutoffs and intervals will be introduced in the next subsection.) For later reference we note that proposition 1 directly implies:

Proposition 2 *Assume $P_2 > 0$. For a multiple $M_2 \geq exp(rT/2)$ there exists a single value $\bar{N}_2^m > 0$, called the* multiple cutoff*, such that the stake α_2 is strictly for $0 < N_2 < \bar{N}_2^m$ and positive for $N_2 \geq \bar{N}_2^m$. For a multiple $M_2 \geq 1$ there exists a single value $\bar{N}_2^l < P_2$, called the* liquidation cutoff*, such that the function is larger than 100% for all $N_2 > \bar{N}_2^l$.*

Although we do not make this explicit, note that cutoffs depend on date 2 post-money value P_2 and multiple M_2. For $M_2 > exp(rT/2)$ and given date 2 post-money value P_2, figure 3 illustrates both cutoffs: If new money N_2 is below the multiple cutoff $\bar{N}_2^m$, stakes α_2 are negative; the liquidation cutoff $\bar{N}_2^l$ is smaller than the post-money value P_2 and for asset values in between the stake is larger than 100%. Note that when M_2 is smaller than or equal to $\exp(rT/2)$ the function is always positive and so there is only a liquidation but no multiple reduction cutoff. The next subsection analyzes separately stakes α_2 that are negative and those larger than 1.

3.2 Destructive Liquidation and Multiple Reduction

When new money N_2 is *larger* than the post-money value P_2 (which includes the new money), the pre-money value is negative; no investor will provide financing and the company be liquidated. Therefore, proposition 1 looks only at new financings that do not exceed P_2. It states that the company will be liquidated for all financings between the liquidation cutoff and the post-money value: for these the company stake α_2 is larger than 100% *even* when N_2 is *less* than P_2, i.e. even if pre-money value $P_2 - N_2$ is *strictly positive*. This destroys the positive pre-money value and therefore we refer to this as *destructive liquidation*.

At first, destructive liquidation may seem counterintuitive. To see it makes sense, let us look at figure 2 and assume that N_2 is close to P_2. We know that the date 3 venture liquidation payoff profile is linear with slope 1, intercept 0 and that this payoff profile has value P_2. However, the payoff profile for the second agent initially has slope 1 and intercept 0, but from M_2N_2 onwards it remains flat and only picks up with the participation. For fair contracting, the payout profile of the second agent needs to have a value equal to N_2. Therefore, the participation needs to have slope larger than 100% to make up for the flat payout profile between M_2N_2 and $M_2N_2 + M_1N_1$.

The underlying economic reason for destructive liquidation is therefore the preferential treatment that the previous VC receives through her multiple: before any other party, in particular before the entrepreneur, she receives (a multiple of) her invested money. Here, the payout profile for the second VC is flat between M_2N_2 and $M_1N_1 + M_2N_2$, and so she needs to recover the remainder from her participation, asking for more than 100% company stake. We want to stress that seniority of the second VC is not the source of destructive liquidation: if the first VC would have seniority over the second, the payoff profile would also be flat over some interval.

Figure 3 illustrates the destructive liquidation cutoff for multiples $M_2 > \exp(rT/2)$: for smaller (larger) financings the stake α_2 is smaller (larger) than 100%. (The cutoff depends on the multiple; we do not depict it for multiple $M_2 \leq \exp(rT/2)$.)

Proposition 3 *Assume $P_2 > 0$. We have $\frac{\partial \alpha_2}{\partial M_2} < 0$ for $0 < N_2 \leq \bar{N}_2^l$ and $\frac{\partial \bar{N}_2^l}{\partial M_2} > 0$.*

Proposition 3 states that outside destructive liquidation the stake α_2 decreases as we increase the multiple M_2. Figure 3 illustrates that below the horizontal 100% line the solid line for multiples $M_2 \leq \exp(rT/2)$ is always above the dashed line for multiples $M_2 > \exp(rT/2)$, i.e. that the liquidation cutoff increases with the multiple M_2. In consequence, proposition 3 states also that the first derivative of destructive liquidation cutoff w.r.t. the multiple is positive.

If a new financing is so large that destructive liquidation is inevitable, a larger multiple moves the liquidation cutoff to the right (proposition 3). However, the liquidation cutoff is *always* below the post-money value (proposition 2) and it remains unclear if this can always remedy liquidation. We do not pursue this avenue further here.

So far we studied in this subsection stakes larger than 100% and discussed destructive liquidation. Next we want to study negative stakes α_2. The economic reason for this is the following: When N_2 is "small" compared to P_2, it is likely that the date 3 value P_3 is large enough to pay the multiple M_2N_2 in full and *on top* the participation $\alpha_2 \cdot \max\{(P_3 - (M_1N_1 + M_2N_2)), 0\}$. Intuitively, the multiple ensures a return that is too large when new money raised is small. To price the value of the second VC's claim fairly at value N_2, the participation needs to *reduce* her payout, i.e. α_2 must be negative. Proposition 1 states that this situation comes up whenever $M_2 > \exp(rT/2)$.

When new money raised is then less than the cutoff value, the second VC requires a negative stake α_2 and the financing fails. Based on proposition 1 we know that negative stakes do not show up when $M_2 = 1$. To remedy the situation so that the second VC can invest and the company is not liquidated, we assume the second VC then reduces her multiple to $M_2 = 1$. Therefore we refer to this situation as *multiple reduction*.

Proposition 4 *If $M_2 < \exp(rT/2)$, there exists $\bar{\gamma} > 0$ such that for all $0 < \gamma < \bar{\gamma}$ exists $\bar{P}_2(\gamma)$ with the property that for all $0 < P_2 < \bar{P}_2(\gamma)$: $\alpha_2(P_2, \gamma P_2) > 1$.*

If $M_2 > \exp(rT/2)$, there exists $\bar{\gamma} > 0$ such that for all $0 < \gamma < \bar{\gamma}$ exists $\bar{P}_2(\gamma)$ with the property that for all $0 < P_2 < \bar{P}_2(\gamma)$: $\alpha_2(P_2, \gamma P_2) < 0$.

Note that $\bar{\gamma}$ and $\bar{P}_2$ depend on N_1, M_1, M_2, but we do not make this dependence explicit to simplify our presentation. Previously our analysis fixed post-money value P_2 and varied new money raised between 0 and P_2. Here we take a different look: we vary P_2 and study $N_2 = \gamma P_2$ for all γ and P_2 below suitably chosen values $\bar{\gamma}$ and $\bar{P}_2$. This proposition permits us to get further insights into destructive liquidation and multiple reduction for "small" date 2 post-money values P_2 in relation to the date 1 post-money value $P_1 = 1$.

For $M_2 < \exp(rT/2)$ and *all* sufficiently small γ, proposition 4 (together with propositions 1, 2) states that for $P_2 \to 0$ destructive liquidation occurs for $N_2/P_2 \geq \gamma$, i.e. destructive liquidation makes up an increasing fraction of the interval 0 to P_2. It also means that the cutoff $0 \leq \bar{N}_2^l \leq \gamma P_2$; therefore $\bar{N}_2^l/P_2 \to 0$. Recall that we set $P_1 = 1$; therefore $P_2 \to 0$ means that date 2 post-money value in *relation* to date 1 post-money value P_1 tends to 0. Our interpretation of the previous result is therefore that, it gets harder to evade destructive liquidation, if the venture's date 2 post-money value P_2 gets smaller in comparison to the previous date 1 post-money value $P_1 = 1$.

Similarly, for $M_2 > \exp(rT/2)$, proposition 4 implies that $\bar{N}_2^m/P_2 \to 0$ as $P_2 \to 0$. This means that as the date 2 post-money value P_2 (in relation to date 1 post-money value P_1) gets smaller it gets harder to evade negative stakes α_2. This proposition will play a role below when we look at negative stakes and in particular in subsection 3.4 when we study combinations of post-money value and new money.

Proposition 4 points to an interesting interplay of multiple reduction with destructive liquidation. When $M_2 > \exp(rT/2)$ the proposition states that for sufficiently small P_2 (in relation to date 1 post-money value $P_1 = 1$) multiple reduction leads to liquidation: if both γ and P_2 are sufficiently small then reducing the multiple M_2 to one, the stake α_2 gets larger than 100% and consequently the company will be liquidated. (Put loosely, this situation comes up when $N_2 < P_2$ and both values are close to 0.) In this situation there is no way to chose a multiple that permits financing and the company has to be liquidated. We will illustrate this in subsection 3.4.

3.3 Share Price Dilution

When the date 2 share price is lower than the date 1 share price we have a so-called down round, see our discussion in subsection 2.2. This subsection discusses what amounts of new money raised lead to dilution and ratchet waivers.

Proposition 5 *Assume a fixed $P_2 > 0$. If the share price is diluted for some N_2 with $0 < \alpha_2(P_2, N_2) < 1$, then there exists a cutoff $\bar{N}_2^d$ such that the share price is diluted for all $\bar{N}_2^d < N_2 < \bar{N}_2^l$ but not diluted for $0 < N_2 < \bar{N}_2^d$. The cutoff $\bar{N}_2^d$ increases with increasing M_2.*

This proposition tells us, when the share price is diluted for some N_2, it will also be for all larger financings. To characterize it in further analysis, we use the cutoff $\bar{N}_2^d$. A consequence of the proposition is also that there can be a down-round *only* for sufficiently large N_2. Because the dilution cutoff increases, there are intermediate sizes of financings, where a smaller multiple would lead to share price dilution but larger multiples evade dilution.

We described in subsection 2.2 the company stake α_1^a the first VC holds *after* the second VC provided money. With ratchet, when the required total company stake $\alpha_1^a + \alpha_2$ of both VCs together is larger than 100%, contracting is impossible. To remedy the situation we then assume that the first VC waives his ratchet; otherwise the second VC would not invest and the company would have to be liquidated. Without ratchet, the company will be financed unless destructive liquidation and/or multiple reduction make it impossible, see our earlier discussion. The following provides further insights when the ratchet will be waived:

Proposition 6 *Assume a fixed $P_2 > 0$. If the share price is diluted and the ratchet waived for some N_2 with $0 < \alpha_2(P_2, N_2) < 1$, then there exists a cutoff $\bar{N}_2^w$ such that the share price price is diluted and the ratchet waived for all $\bar{N}_2^w < N_2 < \bar{N}_2^l$, but not waived despite dilution for all $\bar{N}_2^d < N_2 \leq \bar{N}_2^w$. The cutoff $\bar{N}_2^w$ decreases with increasing M_2.*

This proposition tells us that the dilution interval $\bar{N}_2^d \leq N_2 < \bar{N}_2^l$ that we noted in proposition 5 is subdivided into two intervals, one where the ratchet is waived and one where it is not. Furthermore, it tells us that the ratchet is waived only for sufficiently large new financings. Finally note that proposition 5 tells us that the dilution cutoff increases with increasing multiple, while proposition 6 tells us that the ratchet waiver cutoff decreases. We conclude that increasing the multiple, the interval where the ratchet is waived takes up an ever larger portion of the dilution interval, because $\bar{N}_2^d < \bar{N}_2^w < \bar{N}_2^l$. We will illustrate and discuss this further in the next subsection.

3.4 Interplay of Destructive Liquidation, Multiple Reduction, Dilution and Ratchet Waived

Our analysis of the previous subsections showed that, at date 2, four special events can happen: the multiple of the second agent may be reduced, the company may be liquidated, the share price may decrease such that the ratchet would apply (price "dilution") and the first VC may waive her ratchet if applicable. This subsection discusses how these four events interplay; for this we study combinations of post-money value P_2 and new money raised N. Our analysis looks first at the situation where the second VC's multiple $M_2 < \exp(rT/2)$, i.e. when the multiple never has to be reduced; we then study in a second step how the insights carry over to the situation where the multiple may have to be reduced, i.e. when $M_2 > \exp(rT/2)$.

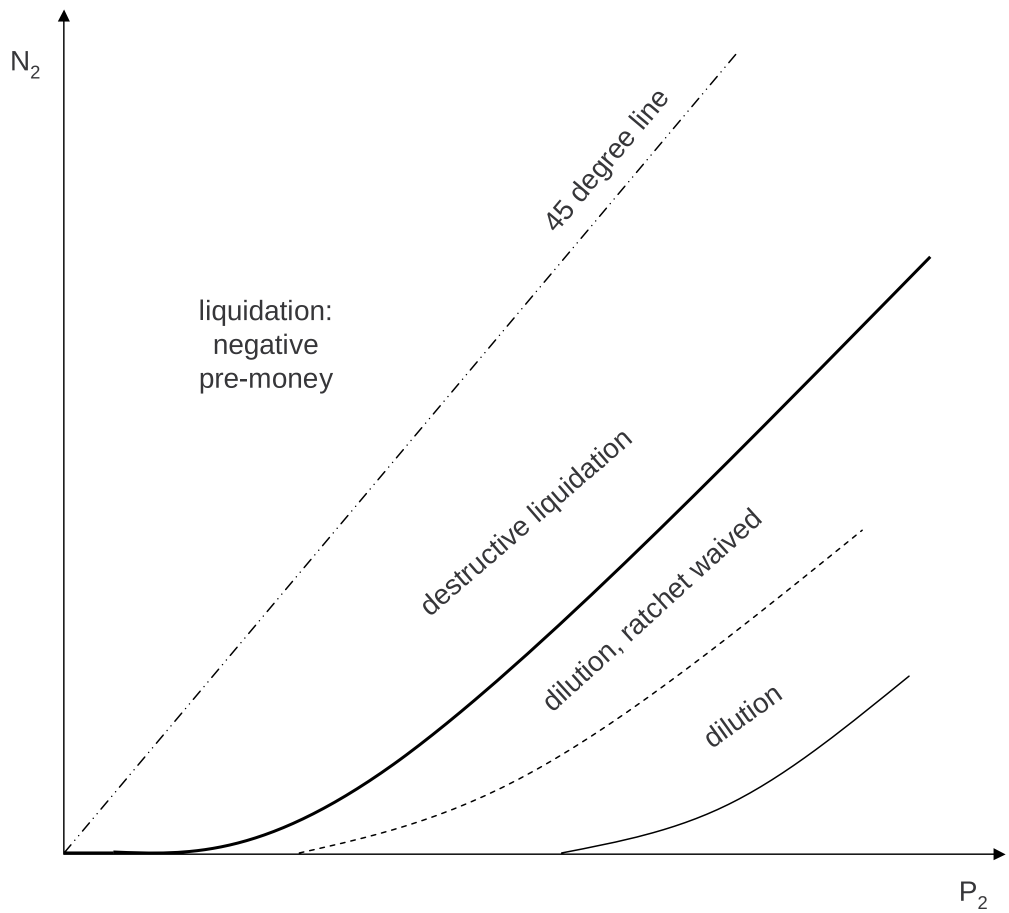

Figure 4: Date 2 combinations of post-money value and new money in which the company is liquidated, the multiple reduced, the share price decreased ("dilution") and the ratchet waived; multiple $M_2 < \exp(rT/2)$.

When the multiple $M_2 < \exp(rT/2)$, it never has to be reduced; so only three events can happen: the company may be liquidated, the share price may decrease such that the ratchet would apply (price "dilution") and the first VC may waive her ratchet if applicable. For combinations of post-money value P_2 and new money raised N_2 figure 4 shows the different areas where these events interplay. For completeness we show there the area above the 45 degree line where we have $N_2 > P_2$, i.e. negative pre-money and the company will be liquidated. However, this area has not been and will not be the object of our study: Below the 45 degree line pre-money is positive and this is the area we are interested in.

With the exception of proposition 4 we always looked at fixed post-money value P_2 and varied N_2; graphically within figure 4, this means that we previously looked at vertical lines for every possible date 2 post-money value P_2. All the cutoffs that we introduced before appear as points on these vertical lines. Varying P_2, the cutoffs define the areas that we see in figure 4: on any vertical line we see that directly below the 45 degree line is destructive liquidation, where the company is liquidated despite positive pre-money (proposition 2); propositions 5 and 6 state that area with dilution but ratchet waiver is directly below the destructive liquidation and above an area with dilution but without ratchet waiver; below all other areas is where we contract α_2 without share price dilution. Further analysis in proposition 4 looks at tilted lines through the lower left corner of figure 4, i.e. it looked at lines with intercept 0 but different slopes γ; this analysis shows that the line separating the destructive liquidation region approximates the P_2 axis smoothly, as depicted. (This will be relevant when we look at the interplay of our four different contracting events with a multiple $M_2 > \exp(rT/2)$.) Overall, figure 4 illustrates the location of the three different special events (destructive liquidation, dilution with/without waiver).

Finally, we study the situation when the second VC's multiple $M_2 > \exp(rT/2)$. We know from propositions 1 and 2 that for any given post-money value P_2, there is a multiple cutoff and we need to reduce the multiple for smaller financings. The dotted area in figure 5 refers to those combinations of post-money value and new money where the multiple must be reduced; here we reduce it to one and inside the dotted area we are back to figure 4. Inside, the events are ordered exactly as there: we see an area where the share price decreases ("dilution") which is further subdivided into one where the ratchet has to be waived by the first VC to permit contracting. There is also the region of figure 4 where the company needs to be liquidated, according to proposition 4 and our discussion thereafter.

Outside the dotted area but below the 45 degree line the mutiple is larger than inside the dotted area. There, the areas are ordered in the same way as in figure 4, in general. However, the lines that separate destructive liquidation, dilution and ratchet waiver do not have smooth continuations between the dotted and the non-dotted area. The origin of this is that for any P_2 value, we know from propositions 3, 5 and 6 that the liquidation and dilution cutoffs show

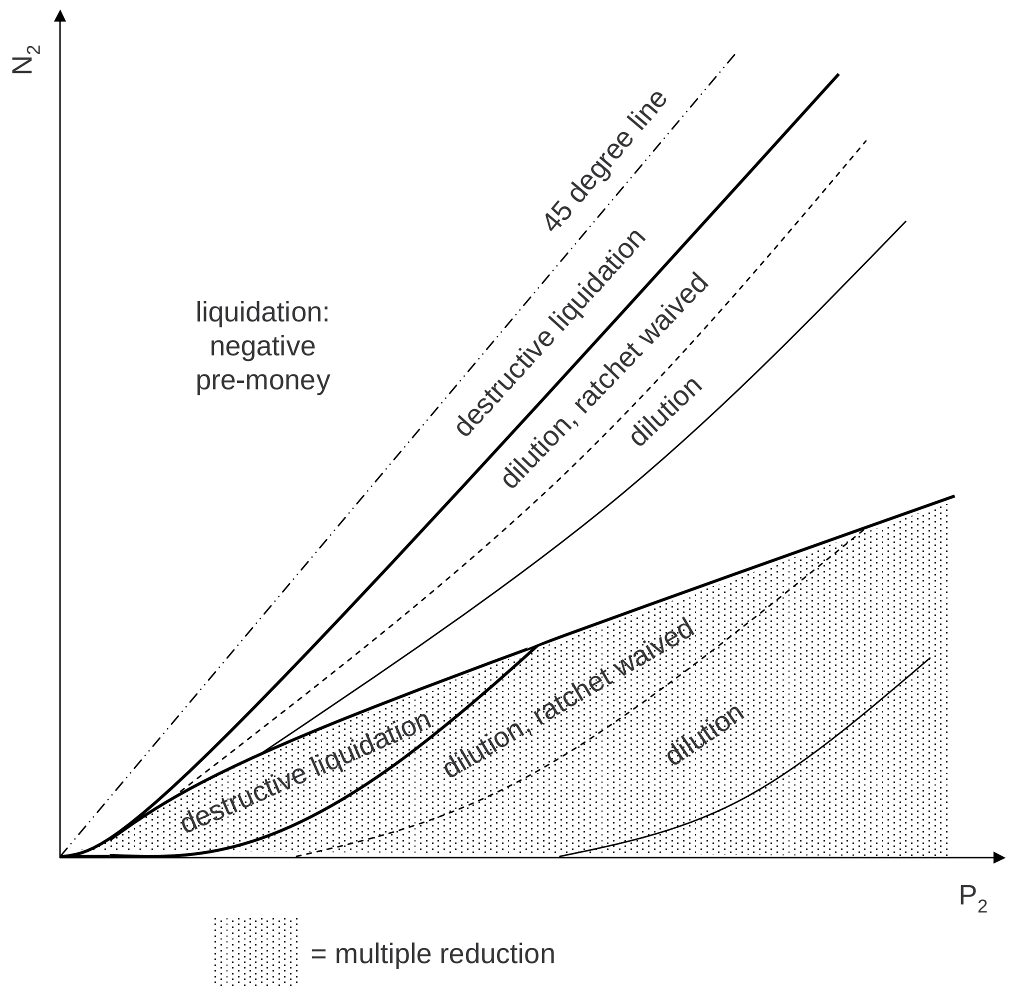

Figure 5: Date 2 combinations of post-money value and new money in which company is liquidated, the multiple reduced, the share price decreased ("dilution") and the ratchet waived; multiple $M_2 > \exp(rT/2)$.

up at higher N_2 values as the multiple increases, i.e. in comparison to figure 4 they all move upward as far as they are outside the dotted area. (Put differently: if new money raised N_2 is "too small" compared to post-money value to support a given multiple $M_2 > \exp(rT/2)$, we need to reduce it but then destructive liquidation, dilution and ratchet waived will all apply at much lower values N_2.)

In the lower left part we depict an odd situation which we pointed out after proposition 5: if the size of date 2 post-money value P_2 in relation to date 1 post-money value P_1 is "small," then there is no financing at all. If the multiple $M_2 > \exp(rT/2)$ then the company will be either liquidated because of destructive liquidation with the given multiple, or the multiple will have to reduced and then there is destructive liquidation. (This follows from proposition 4: the line separating the multiple reduction area approximates the destructive liquidation line smoothly, as depicted. We noted earlier with multiple $M_2 < \exp(rT/2)$ that the line separating the destructive liquidation region approximates the P_2 axis smoothly, also as depicted.)

If the size of post-money value P_2 is intermediate, the following interesting interplay between our four events arises as we vary the amount of new financing N_1 from zero to P_2. Near zero, the multiple needs to be reduced; as we increase it we encounter dilution without ratchet waiver, then dilution with ratchet waiver; once N_2 is sufficiently large that the multiple does not have to be reduced any more, the second VC contracts *as is*; however, as we further increase N_2 we encounter again dilution, first without ratchet waiver and then with waiver; ultimately we run into destructive liquidation.

4 The Investment Problem for the First VC

At date 1, the first VC will contract the company stake α_1 for her investment N_1, to be held between dates 1 and 2. She needs to take account of the optimal date 2 actions of the second VC, i.e. either the company will be liquidated or her fractional ownership will be adjusted to α_1^a at date 2. We look separately at the case where the first VC has no ratchet (anti-dilution protection) and the case where she holds the so-called full ratchet.

4.1 Date 2 Values of the First VC's Claims

If the second VC refuses financing at date 2, the company is liquidated and the liquidation value $L(P_2, N_2) = \lambda P_1$ distributed among the entrepreneur and the first VC according to the liquidation preference. The first VC will then receive a multiple M_1 of her initial investment N_1, i.e. before anything goes to the entrepreneur, the first VC receives all distributions until she has received $M_1 N_1$; any remaining values are distributed pro-rata according to their company stakes α_0, α_1 that have been contracted at date 0. Figure 6 plots the resulting payoff profile. In our later analysis, the first VC's payout plays an important role; for later reference we denote her

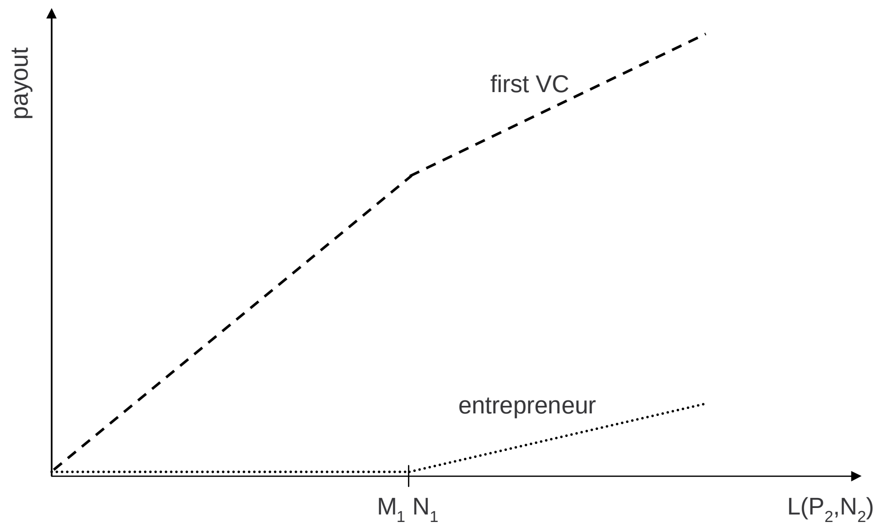

Figure 6: Date 2 liquidation payoff profiles.

date 2 liquidation payout by $L_{\mathrm{VC}}(P_2, N_2)$.

If the company is not liquidated at date 2, the second VC invests N_2 in exchange for a stake α_2 in the company according to equation (6). (The multiple will be $M_2 = M_1$, but may be reduced to $M_2 = 1$, see our earlier discussion in subsection 3.1.) The stake α_1^a that the first VC holds between dates 2 and 3 is then given by equations (2, 3) depending on whether the ratchet applies or not. The date 2 claim of the first VC is then a portfolio of calls, see figure 2, with value

$$BS(P_2, M_2N_2) + (\alpha_1^a - 1) \cdot BS(P_2, M_1N_1 + M_2N_2).$$

4.2 The First VC's Date 1 Stake Without Ratchet

This subsection assumes the first VC does not hold a ratchet (anti-dilution protection). According to equation (2), the first VC's stake α_1^a between dates 2 and 3 will be the fraction α_1 of the remainder of the second VC's stake, i.e. $\alpha_1^a = \alpha_1(1 - \alpha_2)$. She anticipates this reduction when setting her date 1 stake in the venture; we will now analyze how she contracts.

Further analysis in this subsection needs to distinguish the events when the second VC does and when she does not provide financing; for this we introduce two indicator variables on mutually exclusive events: I_{fin} (I_{liq}) takes the value 1 (0) when the company is financed, and 0 (1) otherwise, i.e. when it is liquidated. Based on our analysis in the previous subsection the date 1 value of the first VC's investment as a function of the stake α_1 is:

$$\begin{aligned} V_1(\alpha_1) &= e^{-r\frac{T}{2}} E\Big[L_{\mathrm{VC}}(P_2, N_2) I_{\mathrm{liq}}\Big] + e^{-r\frac{T}{2}} E\Big[BS(P_2, M_2 N_2) I_{\mathrm{fin}}\Big] \\ &\quad + e^{-r\frac{T}{2}} E\Big[\big(\alpha_1(1-\alpha_2) - 1\big) \cdot BS(P_2, M_1 N_1 + M_2 N_2) I_{\mathrm{fin}}\Big]. \end{aligned}$$

For fair contracting we require $N_1 = V_1(\alpha_1)$, which gives

$$\alpha_1 = \frac{N_1 - e^{-r\frac{T}{2}} E\Big[L_{\mathrm{VC}}(P_2, N_2) I_{\mathrm{liq}} + \big\{BS(P_2, M_2 N_2) - BS(P_2, M_1 N_1 + M_2 N_2)\big\} I_{\mathrm{fin}}\Big]}{e^{-r\frac{T}{2}} E\Big[(1-\alpha_2) BS(P_2, M_1 N_1 + M_2 N_2) I_{\mathrm{fin}}\Big]}. \tag{7}$$

Here the discounted expectation in the numerator is the date 1 value of the payouts to the first VC without the date 3 participation: the term $L_{\mathrm{VC}}(P_2, N_2)$ describes the payoff for the first VC *if* it is liquidated at date 2; when the company is financed (*not* liquidated), $\{BS(P_2, M_2 N_2) - BS(P_2, M_1 N_1 + M_2 N_2)\}$ describes the date 2 value of the second VC's multiple taking account of the first VC's seniority. This discounted expectation is then subtracted from the initial investment. Therefore, the numerator captures what value the first VC needs to recover from the participation. The denominator describes what value would be available from the participation *after* the two VCs received their multiples. Overall, the ratio of numerator to denominator determines her stake in the company.

Proposition 7 *For given $P_1 > 0$ we study the first VC's stake α_1 as a function of $0 \le N_1 \le P_1 = 1$. The function is 0 at $N_1 = 0$. If M_1 is smaller than $\frac{\exp(rT/2)}{Prob[financing]}$, then the function is always positive. If M_1 is larger than that, there exists a value $\bar{N}_1^m > 0$, called* multiple cutoff, *such that the stake α_1 is negative for $0 < N_1 < \bar{N}_1^m$ and positive for all $N_1 > \bar{N}_1^m$. For all multiples $M_1 \ge 1$ there exists a value $\bar{N}_1^l < P_1$, called* liquidation cutoff, *such that the first VC's stake is larger than 100% for all $N_1 > \bar{N}_1^l$.*

As before, we study zero financings only as limiting cases. Note that the multiple and liquidation cutoffs depend on M_1. Proposition 7 gives date 1 results for the first VC that are similar to those about the date 2 multiple and liquidation cutoffs in propositions 1 and 2 for the second VC; the difference is that before the shape depended on the distinction of multiples

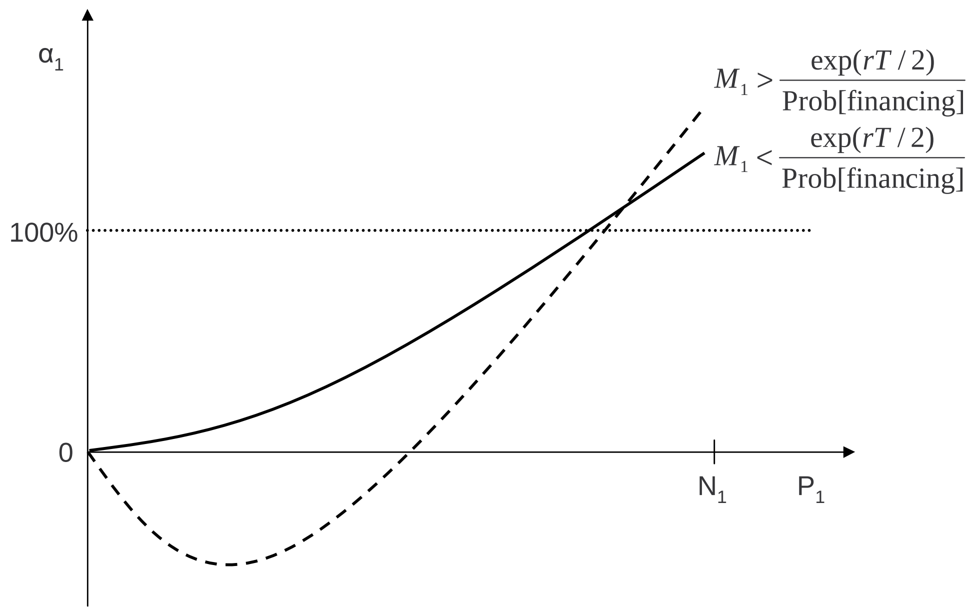

Figure 7: The first VC's company stake as a function of N_1.

being larger/smaller than $\exp(rT/2)$ while here it depends on the distinction of multiples being larger/smaller than the ratio of $\exp(rT/2)$ to the probability of date 2 financing, Prob[financing]. Figure 7 illustrates the shape of α_1 as stated in proposition 7; qualitatively it is similar to figure 3.

We recall the date 2 problem with the multiple: it gave a very good chance for a payment M_2N_2 in exchange for an investment of N_2; the negative company stake was necessary to ensure a fair contract. Here something similar happens. For illustration, we *focus* on the situation where $N_1 = N_2$ close to zero as well as $P_1 = P_2 = P_3$, and *ignore* any distribution. Then, if N_1 is small (close to 0), N_2 will also be small, the multiple will pay in full and contribute M_1N_1 to the value of the first VC's stake; to price this fairly at N_1 the value needs to be decreased using the participation; hence the stake should be negative if $M_1 > 1$. As before at date 2 the remedy to this is to reduce the multiple.

In addition, proposition 7 the company will be liquidated when new money raised gets too large despite positive pre-money value (destructive liquidation). To provide intuition for this, we *focus* on the situation where N_1, P_2 and P_3 are close to P_1. Then, at date 2, if the company is liquidated, the liquidation value λP_1 is small compared to N_1 and will pay only a small amount;

if the company is financed (not liquidated) at date 2, then the senior liquidation preference attributes any remaining value P_3 to the second VC's multiple $M_2 N_2$ and virtually nothing to the first VC. Taking account of the distribution of values, we note that, on top, the first VC still has a (small) chance of a payout from participation; a large payout from the participation is then needed so that the date 1 value equals the invested money N_1, i.e. α_1 needs to get large.

Our results imply that the VC does not have the freedom to invest any amount of new money N_1. If N_1 is too small and M too large, then α_1 is less than than 0%; but in this situation a smaller multiple ($M = 1$) will always work. If N_1 is large, but still smaller than P_1 we may not be able to contract at all; this is surprising as there is a positive pre-money value $P_1 - N_1$, i.e. after subtracting new money coming in there is a positive value to the company but it will still be liquidated, because the first VC has no chance of recovering her initial investment.

4.3 The First VC's Date 1 Stake With Full Ratchet

This subsection assumes the first VC holds a full ratchet (anti-dilution protection). When the ratchet does not apply or is waived, her stake is adjusted to $\alpha_1^a = \alpha_1(1 - \alpha_2)$ as in the case without ratchet; otherwise it is adjusted to $\alpha_1^a = N_1/N_2\alpha_2$, see equation (3) of subsection 2.2. The first VC anticipates this when setting her date 1 stake in the venture; we will now analyze how she contracts.

In the previous subsection we only had to distinguish between two date 2 events: liquidation versus financing. In addition we need to distinguish here if the ratchet applies or not when the company is financed. For this, we introduce the indicator variable $I_{(\text{fin, rat})}$ which takes the value 1 when the company is financed and the ratchet applies; in all other cases it takes the value 0. Similarly $I_{(\text{fin, no rat})}$ takes the value 1 when the company is financed and the ratchet does not apply; in all other cases it takes the value 0. (The ratchet applies when the share price decreases and it is not waived. The ratchet does not apply either when the share price does not decrease or when it does but the ratchet is waived.) The indicator variables I_{liq} and I_{fin} remain as in the previous subsection. Note that for a company that is financed the ratchet either applies or not, so that $I_{(\text{fin, rat})} + I_{(\text{fin, no rat})} = I_{\text{fin}}$.

Based on our analysis in subsection 4.1 we then determine the date 1 value of the first VC's investment as a function of the stake α_1:

$$\begin{aligned} V_1(\alpha_1) &= e^{-r\frac{T}{2}} E[L_{\mathrm{VC}}(P_2, N_2) I_{\mathrm{liq}}] + e^{-rT/2} E[BS(P_2, M_2 N_2) I_{\mathrm{fin}}] \\ &\quad + e^{-r\frac{T}{2}} E\left[\left(\frac{N_1}{N_2}\alpha_2 - 1\right) BS(P_2, M_1 N_1 + M_2 N_2) I_{(\text{fin, rat})}\right] \\ &\quad + e^{-r\frac{T}{2}} E\left[(\alpha_1(1-\alpha_2) - 1) BS(P_2, M_1 N_1 + M_2 N_2) I_{(\text{fin, no rat})}\right]. \end{aligned}$$

Fair contracting requires $N_1 = V_1(\alpha_1)$, which gives

$$\alpha_1 = \frac{N_1 - e^{-r\frac{T}{2}} E\Big[L_{\mathrm{VC}}(P_2, N_2) I_{\mathrm{liq}} + \{BS(P_2, M_2 N_2) - BS(P_2, M_1 N_1 + M_2 N_2)\} I_{\mathrm{fin}}\Big]}{e^{-r\frac{T}{2}} E\Big[(1-\alpha_2) BS(P_2, M_1 N_1 + M_2 N_2) I_{(\text{fin, no rat})}\Big]} \tag{8}$$

$$- \frac{E\Big[\frac{N_1}{N_2}\alpha_2 BS(P_2, M_1 N_1 + M_2 N_2) I_{(\text{fin, rat})}\Big]}{e^{-r\frac{T}{2}} E\Big[(1-\alpha_2) BS(P_2, M_1 N_1 + M_2 N_2) I_{(\text{fin, no rat})}\Big]}. \tag{9}$$

In the event that the ratchet applies at date 2, the stake α_1 that the first VC contracts at date 1 does not impact the payout from participation: it is then $\alpha_1^a = N_1/N_2\alpha_2$. Therefore it will not determine what the VC needs to recover from participation. This is exactly what the denominator in equations (8, 9) gives us: it is conditioned on the joint event that the company is financed and that the ratchet does not apply.

The terms in equations (8, 9) have interpretations similar to our earlier analysis without ratchet: together, both numerators capture what value the first VC needs to recover from the participation and the (identical) denominator describes what is available from participation *after* the two VCs received their multiples. (In comparison to the stake without ratchet, the term in equation (9) is added; it captures the additional money the first VC recovers from her ratchet.)

Proposition 8 *For given $P_1 > 0$ we study the first VC's stake α_1 as a function of $0 \leq N_1 \leq P_1 = 1$. The function is 0 at $N_1 = 0$. If M_1 is larger than $\frac{\exp(rT/2)}{Prob[financing]}$, there exists a value $\bar{N}_1^m > 0$, called* multiple cutoff*, such that the stake α_1 is negative for $0 < N_1 < \bar{N}_1^m$ and positive for all $N_1 > \bar{N}_1^m$. For all multiples $M_1 \geq 1$ there exists a value $\bar{N}_1^l < P_1$, called* liquidation cutoff*, such that the first VC's stake is larger than 100% for all $N_1 > \bar{N}_1^l$.*

The proof of this proposition, see the appendix, shows that for M_1 smaller (larger) than $\frac{\exp(rT/2)}{\text{Prob[financing]}}$ the term on the right-hand side in equation (8) is positive (negative) for small financings. The term in equation (9) captures the impact of the full ratchet and always makes a positive contribution to the stake so that, overall, for M_1 larger than $\frac{\exp(rT/2)}{\text{Prob[financing]}}$ the stake is negative. We expect that only for multiples M_1 up to a value smaller than $\frac{\exp(rT/2)}{\text{Prob[financing]}}$ the stake is positive for small financings; however there is no expression to this. Other than that, proposition 8 is similar to proposition 7 and has similar interpretations.

5 Numerical Examples

Table 1 provides a summary of the literature on VC characteristics of new ventures. Peng (2001) does not report risk-parameters from which the volatility σ_P could be inferred; Quigley and Woodward (2003) are the only ones that provide pre- and post-money values from which the fraction of new money N can be inferred. Given that venture capital is a high risk environment where many ventures fail, the standard deviation reported by Quigley and Woodward (2003) strikes us as surprisingly low in comparison to broad stock market indexes, e.g. to the S&P500; therefore we adopt the value reported by Cochrane (2005). To our knowledge the literature has not studied the volatility of new moneys raised and the correlation with post-money values; we take the volatility of new money raised equal to that of the post-money value; a successful venture increases in value and requires new financing, so that we expect post-money value and new money raised to be positively correlated and simply take the correlation equal one-half. Furthermore we adopt a value of $\lambda = 0.1$ for fractional recovery of initial post-money value. Overall, our baseline parameters throughout this section are

$$T = 5; N_1 = 0.27; \sigma_P = \sigma_N = 0.89; \kappa_{PN} = 0.5; r = 5\%; \lambda = 0.1.$$

Table 1: Literature on Venture Characteristics.

Author	Sample Description	Data Source	Findings
Peng (2001)	12,946 rounds of venture financing with 5,643 venture-backed firms from January 1987 to December 1999	Compiled from several sources, including VentureOne database, SDC plantinum service, MarketGuide, and other online resources.	He finds an average geometric return of 55%.
Cochrane (2005)	16,613 financing rounds with 7,765 companies between January 1, 1999 and June 30, 2000	VentureOne database, SDC platinum corporate new issues and Mergers and Acquisition (M&A) databases, MarketGuide, and other online resources	Measures the mean, standard deviation, alpha and beta of venture capital investments. Shows venture returns are approximately lognormal distributed. A lognormal distribution calibrated to this has 55% mean and 89% standard deviation in logs.
Quigley and Woodward (2003)	5,607 private firms with a total of 12,553 rounds of financing between January 1, 1987 and March 31, 2000	Sand Hill Econometrics database	Construct a venture capital index. The annual real return has mean 4.1% and standard deviation 15.6% (tabel 5). The venture typically needs 3-5 years to exit from the first financing and average new-money raised in relation to post-money value is about 27% (table 2).
Cumming and Walz (2010)	Cash flow information at the level of the individual investment for the 5,038 portfolio firms of 221 private equity funds in 39 countries between 1971 and 2003	Dataset collected by Center of Private Equity Research (CEPRES) in Frankfurt, Germany	Average internal rate of return of exited investments is 68.2%.

M_2	$P_2 = 0.25$	$P_2 = 0.5$	$P_2 = 0.75$	$P_2 = 1$	$P_2 = 1.25$	$P_2 = 1.5$	$P_2 = 1.75$	$P_2 = 2.0$
				Panel A: multiple cutoff				
1	NA	NA	NA	NA	NA	NA	NA	NA
2	0.100	0.200	0.299	0.399	0.499	0.599	0.699	0.798
3	0.147	0.295	0.442	0.589	0.737	0.884	1.031	1.179
				Panel B: destructive liquidation cutoff				
1	0.199	0.443	0.691	0.940	1.189	1.438	1.688	1.938
2	0.202	0.445	0.693	0.941	1.190	1.439	1.689	1.939
3	0.209	0.452	0.699	0.947	1.196	1.446	1.695	1.945
				Panel C: distance cutoff from post-money value				
1	0.051	0.057	0.059	0.060	0.061	0.062	0.062	0.062
2	0.048	0.055	0.058	0.059	0.060	0.061	0.061	0.061
3	0.041	0.048	0.051	0.053	0.054	0.055	0.055	0.055

Table 2: Multiple cutoff (Panel A), destructive liquidation cutoff for N_2 (Panel B) and its distance to date 2 post-money value (Panel C) for different combinations of date 2 post-money value and multiple M_2.

5.1 The Second VC's Investment Problem

Table 2 uses baseline parameters and presents for different choices of date 2 post-money value P_2 the multiple cutoff (Panel A), the destructive liquidation cutoff (Panel B) and how far that cutoff is from the post-money value, i.e. the difference $P_2 - \bar{N}_2^l$ (Panel C). Each panel looks at date 2 post-money values P_2 ranging from 1/4 to 2; rows one to 3 look separately at multiples 1 to 3.

For $M_2 = 1$ there is no multiple cutoff in Panel A because the stake is always non-negative. For $M_2 = 2, 3$, whatever the value P_2, the cutoff is always strictly positive and increases as we increase P_2; also, the cutoff increases as we increase M_2 from 2 to 3. Note that the cutoff is sizeable; e.g. for $P_2 = 1$ it is 0.399 (0.589) for multiples $M_2 = 2$ ($M_2 = 3$). We do not report the numbers here, but an easy calculation based on Panel A shows that, the quotient of multiple cutoff to post-money value P_2 is fairly constant, quantitatively, but that the multiple impacts results considerably.

	no ratchet		full ratchet	
M_1	multiple	liquidation	multiple	liquidation
1	NA	0.657	NA	0.657
2	0.064	0.657	0.103	0.657
3	0.141	0.655	0.194	0.655

Table 3: Date 1 multiple (reduction) and (destructive) liquidation cutoffs without ratchet and with full ratchet.

We see in Panel B that the liquidation cutoff is a sizeable number that will likely affect the second VC's contracting. Furthermore, as P_2 or the multiple increases, the cutoff increases; with respect to the multiple this result is in line with proposition 3. Surprisingly, the cutoff does not change much for fixed P_2 as we increase the multiple. Similarly, we see in Panel C that the difference between the post-money value and the cutoff value does not change much as we increase the post-money value or the multiple. (It seems to decrease (increase) slightly as we increase the multiple (the post-money value).) This means that relative to post-money value destructive liquidation gets more and more relevant, in line with proposition 4.

5.2 The First VC's Investment Problem

Table 3 presents the date 1 multiple (reduction) and (destructive) liquidation cutoffs; we distinguish "no ratchet" from "(full) ratchet." When the multiple $M_1 = 1$ we calculate that it never has to be reduced. As we increase the multiple, there is a cutoff for multiple reduction and it increases with the multiple. Comparing the cutoff for multiple reduction without ratchet and the one with (full) ratchet we see that the latter is always larger. Intuitively this makes sense: for the *same* stake the date 1 value of the first VC is always larger with than without ratchet; under fair contracting, the participation needs to compensate this with an even larger negative stake; consequently the multiple needs to be reduced with ratchet in more situations than without ratchet.

Table 3 suggests that there is (almost) no impact on the (destructive) liquidation cutoffs whether the first VC has or does not have a ratchet: for the same multiple the numbers without ratchet and with full ratchet are the same. Getting back to our discussion in subsection 3.4, in particular to figure 5, we recall that, for any date 2 post-money value P_2, varying the amount N_2 both dilution and ratchet waiver are directly below destructive liquidation. This means that when new money raised is so large that the company will be liquidated despite having positive pre-money value, then for slightly smaller financings there is dilution but the ratchet will be waived. So, it does not make a big difference if the VC has a ratchet or not.

As we increase the multiple, it appears in table 3 that the liquidation cutoff decreases. It is surprising, however, that the liquidation cutoff does not depend quantitatively much on the multiple.

Finally it is important to note the size of the cutoffs. The liquidation cutoffs are roughly at two-third, which means that even if pre-money value is roughly one-third of post-money value, the venture will not be financed. The multiple reduction cutoffs can also be fairly large; with a multiple of $M = 3$ and full ratchet its is approximately 20%. Overall, with some contracting constellations (large multiple, full ratchet) the interval in which VCs can contract can become fairly small.

In subsection 3.4 we noted that four date 2 events are relevant for the first VC: multiple reduction, destructive liquidation, dilution and ratchet waiver. (These events may occur jointly.) In subsection 3.4 we studied in figures 4 and 5 the date 2 combinations of post-money value with new money and presented the areas in which the mentioned events occur. Here we are interested in date 1 and want to assess the (joint) probability of the four events. The assessment depends on the company stake contracted at date 1; we use the date 1 fair contract of subsection 4.2 (without ratchet) and 4.3 (full ratchet) for the first VC for baseline paramters

Table 4 presents in Panel A the date 1 probabilities of the four date 2 events and in Panel B the probability of multiple reduction joint together with liquidation, share price decrease and ratchet waiver, respectively. We distinguish three different multiples $M_1 = 1, 2, 3$ (rows one to three in both tables). When the multiple is one, we calculate that the multiple does not have to be reduced.

Each of the probabilities in Panel A of table 4 increases as we increase the multiple, but the

M_1	multiple red.	liquidation	price decrease	ratchet waived
Panel A: Single events				
1	NA	0.24	0.36	0.18
2	0.61	0.24	0.39	0.17
3	0.71	0.27	0.47	0.19
Panel B: Events joint with multiple reduction				
M_1		liquidation	price decrease	ratchet waived
1		NA	NA	NA
2		0.01	0.29	0.12
3		0.06	0.41	0.16

Table 4: Date 1 probabilities of events joint with multiple reduction; based on date 1 fair contract for first VC.

sensitivity varies: the increase is very strong for the event "multiple reduction" and any joint events with this but basically does not depend on the event "liquidation."

Note that the probabilities are sizeable: the probability that the company is liquidated is about 1/4; the probability that the multiple is reduced goes to more than 70% with a multiple of 3. The probabilities that relate to the ratchet are also of considerable size: the probability that the share price decreases is in the range 35% to 45%, and that the ratchet is waived is about 18%. Furthermore, in Panel B, the events "multiple reduction" and "liquidation" intersect partially, but only on a small set of conditional date 2 events with small probability; with a multiple of $M = 3$ this is less than 6%; therefore the sum of the probabilities of both events indicates that, with a multiple of 3, the probability is about 90% that the multiple will have to be reduced or the company liquidated. Finally, we calculate based on Panels A and B that the conditional probability that the ratchet is waived if the share price decreases is 40% to 50%. Furthermore, note that even when the multiple is reduced the joint probability of a share price decrease is 40%.

Panel A: Varying T										
	no ratchet					full ratchet				
M	-15%	-5%	0%	+5%	+15%	-15%	-5%	0%	+5%	+15%
1	0.232	0.236	0.238	0.240	0.243	0.221	0.226	0.227	0.229	0.232
2	0.165	0.179	0.185	0.190	0.200	0.154	0.169	0.175	0.181	0.191
3	0.095	0.119	0.129	0.139	0.155	0.077	0.103	0.114	0.125	0.142

Panel B: Varying N_1										
	no ratchet					full ratchet				
M	-15%	-5%	0%	+5%	+15%	-15%	-5%	0%	+5%	+15%
1	0.186	0.220	0.238	0.257	0.296	0.175	0.209	0.227	0.246	0.285
2	0.133	0.167	0.185	0.204	0.244	0.123	0.157	0.175	0.194	0.234
3	0.079	0.112	0.129	0.148	0.188	0.062	0.096	0.114	0.134	0.174

Panel C: Varying σ_P										
	no ratchet					full ratchet				
M	-15%	-5%	0%	+5%	+15%	-15%	-5%	0%	+5%	+15%
1	0.224	0.235	0.238	0.241	0.245	0.215	0.224	0.227	0.230	0.233
2	0.141	0.173	0.185	0.195	0.210	0.129	0.162	0.175	0.186	0.202
3	0.058	0.109	0.129	0.146	0.174	0.030	0.092	0.114	0.133	0.163

Panel D: Varying σ_N										
	no ratchet					full ratchet				
M	-15%	-5%	0%	+5%	+15%	-15%	-5%	0%	+5%	+15%
1	0.242	0.240	0.238	0.237	0.234	0.230	0.228	0.227	0.226	0.224
2	0.192	0.187	0.185	0.182	0.177	0.183	0.178	0.175	0.173	0.167
3	0.138	0.132	0.129	0.126	0.119	0.123	0.117	0.114	0.111	0.104

Panel E: Varying κ_{PN}										
	no ratchet					full ratchet				
M	-15%	-5%	0%	+5%	+15%	-15%	-5%	0%	+5%	+15%
1	0.233	0.237	0.238	0.240	0.245	0.222	0.226	0.227	0.229	0.233
2	0.179	0.183	0.185	0.187	0.191	0.170	0.173	0.175	0.177	0.181
3	0.123	0.127	0.129	0.131	0.135	0.109	0.112	0.114	0.116	0.120

Table 5: Comparative statics for the first VC's company stake α_1: each panel varies exactly one parameter by $\pm 15\%, \pm 5\%$ to baseline parameters (0%).

5.3 Comparative Statics of the First VC's Stake

The previous subsections studied qualitatively and quantitatively when the multiple is reduced and when there is destructive liquidation. This subsection carries out comparative statics of the first VC's stake with respect to the input parameters.

Table 5 presents the size of the first VC's stake. It is divided into five Panels and each of them is divided further into two sub-panels, left-hand without ratchet and right-hand with full ratchet. Each sub-panel starts with the baseline parameters (0%) and varies exactly one of them by ±5% and by ±15%; Panel A varies only time T between dates 1 and 3, Panel B only date 1 new financing N_1 (which is in relation to date 1 post-money value P_1), Panel C only the log standard deviation of post-money value, Panel D only the log standard deviation of new money and finally Panel E only varies the correlation κ_{PN} between new money raised and post-money value. (We do not vary the interest rate here.)

Some observations can made for all choices. First, as we increase the multiple, the stake decreases. Second, the company stake with full ratchet is always lower than without ratchet. (These two observations have been noted before and come at no surprise.)

The sign of the sensitivity varies. It is positive for time remaining to liquidation (T), new money required (N_1), post-money risk (σ_P) and correlation between post-money risk and new money required (κ_{PN}), but negative for the risk of new money required over time (σ_N). This is as expected from the literature on call options and spread options: the option value of the multiple contributes most to the claim value and increases, e.g. with time T; only the remainder needs to be made up by the participation which comes from the company stake and, therefore, decreases as we increase time T.

In general, the sensitivity to changes in the parameters is fairly small when the multiple is one. (The exception here is new money N_1 which almost doubles as we go from -15% of the baseline value for N_1 to $+15\%$.) The sensitivity increases (remains almost unchanged) for T, N_1, σ_P (σ_N, κ_{PN}) as we increase the multiple.

Interestingly, overall, the difference between no ratchet and full ratchet is not very large; with a multiple of one it is about 10% and even with a multiple of three it is typically less than

20%. (The exception here is the situation where σ_P is reduced by 15%). Furthermore, some of the parameter choices do not appear very critical: varying σ_N and κ_{PN} does not impact much the company stake. This is good news as these parameters have not been studied directly in the literature and we could only infer rough estimates of them in subsection 2.3. The parameter choices time T, new money N_1 and log post-money standard deviation, however, are either known (N_1) or we have previously been studied in the literature. This implies that remaining time to IPO, new money raised and post-money risk are critical model parameters, quantitatively, while risk (volatility) of new money raised and correlation are less important.

6 Conclusion

This paper introduced a model of venture capital, in which the post-money value and new money raised are random and two VCs could provide staged financing at different dates. We determined their zero net present-value stakes, analyzed their optimal investment policies over time both qualitatively and quantitatively, and discussed the resulting financings constraints.

Appendix

Throughout this appendix, we study stakes α at both dates for small and large moneys raised in asymptotic expansions. As it is common in such expansions, we use the symbol $\mathcal{O}(\cdot)$ to denote higher-order expansion terms. For our analysis we need a first-order series expansion of the well-known Black-Scholes call option pricing formula and note that the first derivative w.r.t. to the strike K is over a time-interval of length $T/2$:

$$\frac{\partial BS}{\partial K}(K) = -\exp(-rT/2)\mathcal{N}(d_2) \text{ with } d_2 = \frac{\ln(P_2/K) + (r - \sigma^2/2)T/2}{\sigma\sqrt{T/2}}. \tag{10}$$

(To simplify notation we usually only write out the strike.)

Proof of proposition 1. For $N_2 = 0$ we have $BS(P_2, M_2N_2) = BS(P_2, 0) = P_2$, and $BS(P_2, M_1N_1 + M_2N_2) = BS(P_2, M_1N_1) > 0$. So, according to equation (6): $\alpha_2(P_2, 0) = 0$.

Next we study the behavior of α_2 when N_2 is close to zero. Note that at $N_2 = 0$ we have $\frac{\partial BS}{\partial K} = -\exp(-rT/2)$. We then expand $BS(P_2, K)$ in a series around $K = 0$, evaluate it at

$K = M_2 N_2$ and get from equation (6)

$$\alpha_2(P_2, N_2) = \frac{N_2 - \exp(-rT/2)M_2N_2 + \mathcal{O}(N_2^2)}{BS(P_2, M_1N_1 + M_2N_2)} = N_2\frac{1 - \exp(-rT/2)M_2}{BS(P_2, M_1N_1 + M_2N_2)} + \mathcal{O}(N_2^2).$$

Note that an option always has positive value, $BS(P_2, M_1N_1 + M_2N_2) > 0$. If $M_2 > \exp(rT/2)$ this implies that, as N_1 tends to zero, α_2 will take negative values; however, if $M_2 < \exp(rT/2)$ it will be positive.

If $M_2 = \exp(rT/2)$ we use the probabilistic representation of the Black-Scholes value, i.e. $BS(P_2, \exp(rT/2)N_2) = \exp(-rT/2)E[(P_3 - \exp(rT/2)N_2)^+] = E[(\exp(-rT/2)P_3 - N_2)^+]$, see, e.g. Duffie (2001). The positive part defines a convex function and so Jensen's inequality implies that this is larger than $E[\exp(-rT/2)P_3] - N_2)^+] = P_2 - N_2$. Together with equation (6) this implies $\alpha_2(P_2, N_2) \geq 0$ for all $0 \leq N_2 \leq P_2$.

Next we prove the stated monotonicity properties. The denominator of α_2 in equation (6) is strictly decreasing in N_2 and therefore its inverse is strictly increasing. For further analysis we define the numerator of α_2 in equation (6) as a function $f(N_2) = N_2 - P_2 + BS(P_2, M_2, N_2)$. The first and second derivative of this function are

$$\frac{df}{dN_2} = 1 + M_2\frac{dBS}{dK}(P_2, M_2N_2), \frac{d^2f}{dN_2^2} = M_2^2\frac{d^2BS}{dK^2}(P_2, M_2N_2).$$

The sign of the first derivative near 0 has been determined above in this proof: it is positive (negative) if M_2 is smaller (larger) than $\exp(rT/2)$. The second derivative is always positive. This implies that with increasing N_2, the function f is decreasing initially and then increasing. Because the inverse of the denominator is strictly increasing this implies the stated monotonicity of α_2.

It remains to study the stake α_2 when N_2 is close to P_2. We rewrite equation (6) as

$$\alpha_2(P_2, N_2) = 1 + \frac{N_2 - P_2 + BS(P_2, M_2N_2) - BS(P_2, M_1N_1 + M_2N_2)}{BS(P_2, M_1N_1 + M_2N_2)}. \tag{11}$$

The sign of the numerator then determines whether $\alpha_2(P_2, N_2)$ is larger or smaller than one. Because $BS(P_2, M_2N_2) > BS(P_2, M_1N_1 + M_2N_2)$, the numerator is strictly positive and we have $\alpha_2(P_2, N_2) \geq 1$ for $N_2 \geq P_2$. Note that the function $\alpha_2(P_2, N_2)$ is continuous in N_2; we showed above that for $N_2 = 0$, $\alpha_2(P_2, N_2) = 0$. Therefore, it will be larger than one on some interval before P_2 and there is a cutoff for N_2 beyond which the stake is larger than 100%. $\blacksquare$

Proof of proposition 3. Taking derivatives of the stake α_2 in equation (6) shows:

$$\begin{aligned}\frac{\partial \alpha_2}{\partial M_2} &= \frac{\frac{\partial BS}{\partial K}(P_2, M_2N_2)M_2BS(P_2, M_1N_1 + M_2N_2)}{BS(P_2, M_1N_1 + M_2N_2)^2} \\ &\quad - \frac{(N_2 - P_2 + BS(P_2, M_2N_2))\frac{\partial BS}{\partial K}(P_2, M_1N_1 + M_2N_2)M_2}{BS(P_2, M_1N_1 + M_2N_2)^2}.\end{aligned}$$

Using equation (10) we find:

$$\begin{aligned}\frac{\partial \alpha_2}{\partial M_2} &= -e^{-rT/2}\frac{\mathcal{N}(d_2(M_2N_2))M_2}{BS(P_2, M_1N_1 + M_2N_2)} \\ &\quad + e^{-rT/2}\mathcal{N}(d_2(M_1N_1 + M_2N_2))M_2\frac{(N_2 - P_2 + BS(P_2, M_2N_2))}{BS(P_2, M_1N_1 + M_2N_2)^2}.\end{aligned}$$

The second term can be expressed using the stake α_2, see equation (6). Under our assumption $\alpha_2 \leq 1$ it is smaller than $1/BS(P_2, M_1N_1 + M_2N_2)$. Therefore we have

$$\frac{\partial \alpha_2}{\partial M_2} \leq e^{-rT/2}M_2\frac{\mathcal{N}(d_2(M_1N_1 + M_2N_2)) - \mathcal{N}(d_2(M_2N_2))}{BS(P_2, M_1N_1 + M_2N_2)}$$

Because $M_1N_1 + M_2N_2$ is greater than M_2N_2 and d_2 is a decreasing function, the numerator is strictly negative. This proves that $\frac{\partial \alpha_2}{\partial M_2} < 0$, which in turn implies $\frac{\partial \bar{N}_2^l}{\partial M_2} > 0$. $\blacksquare$

Proof of proposition 4. A first-order approximation of the Black-Scholes formula around strike 0 gives

$$\gamma - 1 + BS\left(1, M_2\gamma\right) = \gamma - 1 + 1 - e^{-rT/2}M_2\gamma + \mathcal{O}(M_2^2\gamma^2) = \gamma\left(1 - e^{-rT/2}M_2\right) + \mathcal{O}(M_2^2\gamma^2).$$

If $M_2 < \exp(rT/2)$, this implies existence of $\bar{\gamma}$ such that $\gamma - 1 + BS\left(1, M_2\gamma\right) > 0$ for all $0 < \gamma < \bar{\gamma}$. In the following we assume γ with this property. It is well known that numeraire invariance of the Black-Scholes formula implies $BS(P_2, M_2N_2) = P_2BS\left(1, M_2\gamma\right)$ and $BS(P_2, M_1N_1 + M_2N_2) = P_2BS\left(1, M_1N_1/P_2 + M_2\gamma\right)$. Therefore,

$$\alpha_2(P_2, \gamma P_2) = \frac{\gamma - 1 + BS\left(1, M_2\gamma\right)}{BS\left(1, M_1\frac{N_1}{P_2} + M_2\gamma\right)}. \tag{12}$$

As P_2 tends to zero, M_1N_1/P_2 tends to infinity and so the denominator tends to zero. Because the numerator is strictly positive, $\alpha_2(P_2, \gamma P_2)$ tends to plus infinity. This implies the existence of $\bar{P}_2$ with the stated property.

If $M_2 < \exp(rT/2)$, we proceed similarly: then exists $\bar{\gamma}$ such that $\gamma - 1 + BS\,(1, M_2\gamma) < 0$ for all $0 < \gamma < \bar{\gamma}$; so, the numerator in equation (12) is negative, $\alpha_2(P_2, \gamma P_2)$ tends to minus infinity and there exists $\bar{P}_2$ with the stated property. ■

Proof of proposition 5. First, we study the statement about dilution and existence of the cutoff. We recall that the share price is diluted, if $p_2 < p_1$, where the date 1 share price is $p_1 = N_1/S_1$ and the date 2 share price is given by equation (1). Because $\alpha_1 = S_1/(1+S_1)$, this condition translates into

$$\frac{N_1}{\alpha_1} > N_2\frac{1-\alpha_2}{\alpha_2} = \frac{N_2}{\alpha_2} - N_2. \tag{13}$$

For further analysis we define $f(N_2) = (N_2 - P_2 + BS(P_2, M_2N_2))/N_2$ and calculate

$$\begin{aligned} \frac{\partial f}{\partial N_2} &= \frac{\left(1 + \frac{\partial BS}{\partial K}(M_2N_2)M_2\right)N_2 - (N_2 - P_2 + BS(P_2, M_2N_2))}{N_2^2} \\ &= \frac{P_2 - BS(P_2, M_2N_2) + \frac{\partial BS}{\partial K}(M_2N_2)M_2N_2}{N_2^2}. \end{aligned}$$

We have that the numerator

$$\begin{aligned} & P_2 - BS(P_2, M_2N_2) + \frac{\partial BS}{\partial K}(M_2N_2)M_2N_2 \\ = \; & P_2 - \left(P_2\mathcal{N}(d_1(M_2N_2)) - M_2N_2e^{-rT/2}\mathcal{N}(d_2(M_2N_2))\right) - e^{-rT/2}\mathcal{N}(d_2(M_2N_2))M_2N_2 \\ = \; & P_2\,(1 - \mathcal{N}(d_1(M_2N_2))) > 0. \end{aligned}$$

This implies that f is strictly increasing in N_2. Because $BS(P_2, M_1N_1 + M_2N_2)$ is strictly decreasing in N_2, we find that

$$\frac{N_2}{\alpha_2} = \frac{BS(P_2, M_1N_1 + M_2N_2)}{f(N_2)}. \tag{14}$$

is strictly decreasing in N_2. Obviously, $-N_2$ is strictly decreasing as we increase N_2. We conclude that the right-hand side in equation (13) is strictly decreasing as N_2 increases. This implies the stated property about dilution. The cutoff $\bar{N}_2^d$ is then defined as the value of N_2 for which the right-hand side in equation (13) is equal to its left-hand side.

It remains to prove that the cutoff $\bar{N}_2^d$ increases as we increase the multiple M_2. According to equation (13) it is enough to prove that $\partial\left(\frac{N_2}{\alpha_2}\right)/\partial M_2 \geq 0$. Using equation (14) we find

$$\begin{aligned}\frac{\partial\left(\frac{N_2}{\alpha_2}\right)}{\partial M_2} &= \frac{\frac{\partial BS}{\partial K}(M_1N_1+M_2N_2)N_2 f - BS(P_2, M_1N_1+M_2N_2)\frac{\partial f}{\partial M_2}}{f^2} \\ &= \frac{\frac{\partial BS}{\partial K}(M_1N_1+M_2N_2)\alpha_2 BS(P_2, M_1N_1+M_2N_2) - BS(P_2, M_1N_1+M_2N_2)\frac{\partial f}{\partial M_2}}{f^2}.\end{aligned}$$

Because $\alpha_2 \leq 1$ and $\frac{\partial BS}{\partial K}$ is negative this is larger than

$$\frac{\frac{\partial BS}{\partial K}(M_1N_1+M_2N_2)BS(P_2, M_1N_1+M_2N_2) - BS(P_2, M_1N_1+M_2N_2)\frac{\partial f}{\partial M_2}}{f^2}.$$

Equation (10) describes $\frac{\partial BS}{\partial K}$; using this representation and $\frac{\partial f}{\partial M_2} = \frac{\partial BS}{\partial K}(M_2N_2)$ we know that

$$\frac{\partial\left(\frac{N_2}{\alpha_2}\right)}{\partial M_2} \geq \frac{BS(P_2, M_1N_1+M_2N_2)}{f^2}e^{-rT/2}(\mathcal{N}(d_2(M_2N_2)) - \mathcal{N}(d_2(M_1N_1+M_2N_2)).$$

The difference in brackets on the right-hand side is strictly positive because d_2 is decreasing in the strike. This implies the statement. ■

Proof of proposition 6. If the share price decreases, the ratchet is waived when $1 < \alpha_2 + \alpha_2^a$ where α_2^a is given in equation (3); this happens if

$$\alpha_2 \frac{N_1+N_2}{N_2} > 1, \tag{15}$$

i.e. if N_2 gets sufficiently large. Proposition 3 states that $\frac{\partial \alpha_2}{\partial M_2} < 0$; this means that if we increase M_2 we need to decrease N_2 to get to the same level on the left-hand side in equation (15) and so $\bar{N}_2^w$ must decrease. ■

Proof of proposition 7. Using equation (6) together with equation (7) we rewrite the first VC's stake as

$$\alpha_1 = \frac{N_1 - e^{-r\frac{T}{2}}E\Big[L_{\text{VC}}(P_2, N_2)I_{\text{liq}} + \{BS(P_2, M_2N_2) - BS(P_2, M_1N_1+M_2N_2)\}I_{\text{fin}}\Big]}{e^{-r\frac{T}{2}}E\Big[\{BS(P_2, M_1N_1+M_2N_2) - BS(P_2, M_2N_2) + P_2 - N_2\}I_{\text{fin}}\Big]}. \tag{16}$$

This will be used throughout this proof.

First we study the statement about the multiple cutoff and new financings N_1 close to zero. A series expansion gives

$$BS(P_2, M_1N_1+M_2N_2) - BS(P_2, M_2N_2) = \frac{\partial BS}{\partial K}(M_2N_2)M_1N_1 + \mathcal{O}(N_1^2),$$

where the $\mathcal{O}(N_1^2)$ term is uniformly bounded across N_2. Therefore,

$$E[\{BS(P_2, M_1N_1 + M_2N_2) - BS(P_2, M_2N_2)\}I_{\text{fin}}] = E\left[\frac{\partial BS}{\partial K}(M_2N_2)I_{\text{fin}}\right] M_1N_1 + \mathcal{O}(N_1^2).$$

We have Prob[financing] $= 1 - \mathcal{O}(N_1^2)$ from the properties of the lognormal distribution, because the company will be financed at date 2 as long as its future new money N_2 is below the cutoff $\bar{N}_2^l$. Therefore, we have $e^{-r\frac{T}{2}}E[\{P_2 - N_2\}I_{\text{fin}}] = P_1 - N_1 + \mathcal{O}(N_1^2)$ and $E[L_{\text{VC}}(P_2, N_2)I_{\text{liq}}] = \mathcal{O}(N_1^2)$. So, we can write (16) as:

$$\alpha_1 \;=\; \frac{N_1 + E\left[\frac{\partial BS}{\partial K}(M_2N_2)I_{\text{fin}}\right] M_1N_1 + \mathcal{O}(N_1^2)}{E\left[\frac{\partial BS}{\partial K}(M_2N_2)I_{\text{fin}}\right] M_1N_1 + P_1 - N_1 + \mathcal{O}(N_1^2)}. \tag{17}$$

Finally, we note that $\frac{\partial BS}{\partial K}(M_2N_2) = -\exp(-rT/2)\mathcal{N}(d_2(M_2N_2))$, that this is bounded, and that $\mathcal{N}(d_2(M_2N_2))$ tends to 1 as N_2 tends to 0. Therefore,

$$E\left[\frac{\partial BS}{\partial K}(M_2N_2)I_{\text{fin}}\right] = -\exp(-rT/2)\text{Prob[financing]} + \mathcal{O}(N_1^2).$$

As N_1 tends to 0, the denominator tends to $P_1 = 1$. We can therefore write this using the numerator as

$$\alpha_1 \;=\; N_1\Big(1 - \exp(-rT/2)\text{Prob[financing]}M_1\Big) + \mathcal{O}(N_1^2). \tag{18}$$

This tends to zero as N_1 tends to zero; asymptotically we see negative stakes α_1 if

$$M_1 > \frac{\exp(rT/2)}{\text{Prob[financing]}}.$$

However, if this M_1 is smaller than the right-hand side, then stakes α_1 are positive near 0.

We now study the first VC's stake for new financings N_1 close to P_1. Using equation (16) we get

$$\alpha_1 \;=\; 1 + \frac{N_1 - e^{-r\frac{T}{2}}E\Big[L_{\text{VC}}(P_2, N_2)I_{\text{liq}} + \{P_2 - N_2\}I_{\text{fin}}\Big]}{e^{-r\frac{T}{2}}E\Big[\{BS(P_2, M_1N_1 + M_2N_2) - BS(P_2, M_2N_2) + P_2 - N_2\}I_{\text{fin}}\Big]}.$$

The denominator in this equation is equal to the denominator in equation (7), i.e. equal to $e^{-r\frac{T}{2}}E\Big[(1-\alpha_2)BS(P_2, M_1N_1 + M_2N_2)I_{\text{fin}}\Big]$; this term is strictly positive, because the date 1 probability of date 2 liquidation is *not* 100%. Therefore, α_1 will be larger than one, when the numerator $N_1 - e^{-r\frac{T}{2}} E[L_{\text{VC}}(P_2, N_2) I_{\text{liq}} + \{P_2 - N_2\} I_{\text{fin}}] > 0$.

Recall that the company will be liquidated at date 2 for $\bar{N}_2^l < N_2$, in particular for all $P_2 < N_2$. Therefore, $E[L_{\text{VC}}(P_2, N_2) I_{\text{liq}}] = \lambda P_1 \text{Prob}[\text{liquidation}] < \lambda \text{Prob}[P_2 < N_2]$ and $E[\{P_2 - N_2\} I_{\text{fin}}] \leq E[\max\{P_2 - N_2, 0\}]$. Overall, we find

$$\begin{aligned} & N_1 - e^{-r\frac{T}{2}} E[L_{\text{VC}}(P_2, N_2) I_{\text{liq}} + \{P_2 - N_2\} I_{\text{fin}}] \\ > \quad & N_1 - \lambda P_1 \text{Prob}[P_2 < N_2] - e^{-r\frac{T}{2}} E[\max\{P_2 - N_2, 0\}]. \end{aligned}$$

The last term can be interpreted as an exchange option (a.k.a. outperformance option) and Margrabe's formula gives $e^{-r\frac{T}{2}} E[\max\{P_2 - N_2, 0\}] = P_1 \mathcal{N}(d_1) - N_1 \mathcal{N}(d_2)$ where $d_{1/2} = (\ln(P_1/N_1) \pm \sigma_{PN}^2 T)/(\sigma_{PN}\sqrt{T/2})$ with $\sigma_{PN} = \sigma_P^2 + \sigma_N^2 - 2\sigma_P \sigma_N \kappa_{PN}$. We also know that $\text{Prob}[P_2 < N_2] = \mathcal{N}(d_2)$ and so

$$N_1 - e^{-r\frac{T}{2}} E[L_{\text{VC}}(P_2, N_2) I_{\text{liq}} + \{P_2 - N_2\} I_{\text{fin}}] > N_1 - \lambda P_1 \mathcal{N}(d_2) - P_1 \mathcal{N}(d_1) + N_1 \mathcal{N}(d_2).$$

When $N_1 = P_1 = 1$ then $d_{1/2} = \pm\sigma\sqrt{T/2}$ and the right-hand side becomes $1 - \mathcal{N}(d_1) + (1 - \lambda)\mathcal{N}(d_2) = \mathcal{N}(-d_1) + (1 - \lambda)\mathcal{N}(d_2) = (2 - \lambda)\mathcal{N}(d_2) > 0$. Because the right-hand side is a continuous function in N_1 it will be larger than zero also for some interval of N_1 before P_1 and consequently on that interval $\alpha_1(P_1, N_1) > 1$. This proves the statement about the liquidation cutoff. ■

Proof of proposition 8. First, we study the first VC's stake with full ratchet for N_1 close to zero. Using equation (6) we get

$$\begin{aligned} \alpha_1 \quad = \quad & \frac{N_1 - e^{-r\frac{T}{2}} E\Big[L_{\text{VC}}(P_2, N_2) I_{\text{liq}} + \{BS(P_2, M_2 N_2) - BS(P_2, M_1 N_1 + M_2 N_2)\} I_{\text{fin}}\Big]}{e^{-r\frac{T}{2}} E\Big[\{BS(P_2, M_1 N_1 + M_2 N_2) - BS(P_2, M_2 N_2) + P_2 - N_2\} I_{(\text{fin, no rat})}\Big]} \\ & - \frac{E\Big[\frac{N_1}{N_2} \alpha_2 BS(P_2, M_1 N_1 + M_2 N_2) I_{(\text{fin, rat})}\Big]}{e^{-r\frac{T}{2}} E\Big[\{BS(P_2, M_1 N_1 + M_2 N_2) - BS(P_2, M_2 N_2) + P_2 - N_2\} I_{(\text{fin, no rat})}\Big]}. \end{aligned}$$

When the company is financed, it must be that $N_2 < P_2$; then $BS(P_2, M_1 N_1 + M_2 N_2) - BS(P_2, M_2 N_2) > 0$ and so also $BS(P_2, M_1 N_1 + M_2 N_2) - BS(P_2, M_2 N_2) + P_2 - N_2 > 0$. The probability of the joint event that the company is not liquidated and the ratchet does not apply is positive. This implies that the denominator in both fractions is strictly positive. Next we note that $\alpha_2 \geq 0$ when the second VC contracts. Therefore, the numerator in the second fraction and therefore the entire fraction is positive. Finally, we noted also in the proof of proposition 7 that the numerator $N_1 - e^{-r\frac{T}{2}} E\Big[L_{\mathrm{VC}}(P_2, N_2) I_{\mathrm{liq}} + \{BS(P_2, M_2N_2) - BS(P_2, M_1N_1 + M_2N_2)\} I_{\mathrm{fin}}\Big]$ in the first fraction gets negative for N_1 close to 0, when $M_1 > \exp(-rT/2)/\mathrm{Prob[financing]}$. The contribution from the second fraction is *always* negative; overall, we can then conclude that α_1 gets negative for N_1 close to 0, if

$$M_1 > \frac{\exp(rT/2)}{\mathrm{Prob[financing]}}.$$

Finally, it remains to study the limiting behavior of α_1 for N_1 near P_1. Using equation (6) we get

$$\begin{aligned}
\alpha_1 \quad = \quad & \frac{N_1 - e^{-r\frac{T}{2}} E\Big[L_{\mathrm{VC}}(P_2, N_2) I_{\mathrm{liq}} + \{BS(P_2, M_2N_2) - BS(P_2, M_1N_1 + M_2N_2)\} I_{\mathrm{fin}}\Big]}{e^{-r\frac{T}{2}} E\Big[\{BS(P_2, M_1N_1 + M_2N_2) - BS(P_2, M_2N_2) + P_2 - N_2\} I_{(\mathrm{fin,\ no\ rat})}\Big]} \\
& - \frac{E\Big[\frac{N_1}{N_2}\alpha_2 BS(P_2, M_1N_1 + M_2N_2) I_{(\mathrm{fin,\ rat})}\Big]}{e^{-r\frac{T}{2}} E\Big[\{BS(P_2, M_1N_1 + M_2N_2) - BS(P_2, M_2N_2) + P_2 - N_2\} I_{(\mathrm{fin,\ no\ rat})}\Big]} .
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\alpha_1 \quad = \quad & 1 + \frac{N_1 - e^{-r\frac{T}{2}} E\Big[L_{\mathrm{VC}}(P_2, N_2) I_{\mathrm{liq}} + (P_2 - N_2) I_{\mathrm{fin}}\Big]}{e^{-r\frac{T}{2}} E\Big[\{BS(P_2, M_1N_1 + M_2N_2) - BS(P_2, M_2N_2) + P_2 - N_2\} I_{(\mathrm{fin,\ no\ rat})}\Big]} \\
& + \frac{-E\Big[\{BS(P_2, M_2N_2) + \Big(\frac{N_1}{N_2}\alpha_2 - 1\Big) BS(P_2, M_1N_1 + M_2N_2)\} I_{(\mathrm{fin,\ rat})}\Big]}{e^{-r\frac{T}{2}} E\Big[\{BS(P_2, M_1N_1 + M_2N_2) - BS(P_2, M_2N_2) + P_2 - N_2\} I_{(\mathrm{fin,\ no\ rat})}\Big]} .
\end{aligned}$$

As before, the denominator in both fractions is strictly positive. Therefore $\alpha_1 > 1$ when the sum of the numerators in the two fractions is positive.

We make use of $\alpha_1^a = N_1/N_2\alpha_2$, see equation (3), and note that $\alpha_1^a + \alpha_2$ must be less than 100%; otherwise the ratchet will have to be waived. Therefore, $\alpha_1^a \leq 1 - \alpha_2$; Using equation (6) we then find that $-(\alpha_1^a - 1) BS(P_2, M_1N_1 + M_2N_2) \geq \alpha_2 BS(P_2, M_1N_1 + M_2N_2) = N_2 - P_2 + BS(P_2, M_2N_2)$.

This implies

$$
\begin{aligned}
& -E\left[\{BS(P_2, M_2N_2) + \left(\frac{N_1}{N_2}\alpha_2 - 1\right) BS(P_2, M_1N_1 + M_2N_2)\} I_{(\text{fin, rat})}\right] \\
\geq \quad & E\left[(N_2 - P_2) I_{(\text{fin, rat})}\right],
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\alpha_1 \quad \geq \quad & 1 + \frac{N_1 - e^{-r\frac{T}{2}} E\Big[L_{\text{VC}}(P_2, N_2) I_{\text{liq}} + (P_2 - N_2) I_{\text{fin}} + (N_2 - P_2) I_{(\text{fin, rat})}\Big]}{e^{-r\frac{T}{2}} E\Big[\{BS(P_2, M_1N_1 + M_2N_2) - BS(P_2, M_2N_2) + P_2 - N_2\} I_{(\text{fin, no rat})}\Big]} \\
= \quad & 1 + \frac{N_1 - e^{-r\frac{T}{2}} E\Big[L_{\text{VC}}(P_2, N_2) I_{\text{liq}} + (P_2 - N_2) I_{(\text{fin, no rat})}\Big]}{e^{-r\frac{T}{2}} E\Big[\{BS(P_2, M_1N_1 + M_2N_2) - BS(P_2, M_2N_2) + P_2 - N_2\} I_{(\text{fin, no rat})}\Big]}.
\end{aligned}
$$

Exactly as in the proof of proposition 7 we can now show that for N_1 near P_1 the numerator is larger than $N_1 - \lambda P_1 \text{Prob}[P_2 < N_2] - e^{-rT/2} E[\max\{P_2 - N_2, 0\}]$, and that this is strictly positive on some interval for N_1 near P_1. ■

References

Admati, A. R., Pfleiderer, P., 1994. Robust financial contracting and the role of venture capitalists. The Journal of Finance 49, 371–401.

Amit, R., Glosten, L., Muller, E., 1989. Entrepreneurial ability, venture investments, and risk sharing. Management Science 36 (10), 1232–1245.

Bagley, C. E., Dauchy, C., 2002. The Entrepreneur's Guide to Business Law. South-Western Publishing.

Bergemann, D., Hege, U., 1998. Venture capital financing, moral hazard, and learning. Journal of Banking and Finance 22, 703–735.

Boyle, G. W., Guthrie, G. A., 2003. Investment, uncertainty, and liquidity. Journal of Finance 58 (5), 2143–2166.

Cochrane, J. H., 2005. The risk and return of venture capital. Journal of Financial Economics 75, 3–52.

Cornelli, F., Yosha, O., 2003. Stage financing and the role of convertible debt. Review of Economic Studies 70, 1–32.

Cossin, D., Leleux, B., Saliasi, E., 2002. Understanding the economic value of legal covenants in investment contracts: A real-options approach to venture equity contracts. FAME Working Paper.

Cumming, D. J., 2005. Agency costs, institutions, learning, and taxation in venture capital contracting. Journal of Business Venturing 20, 573–622.

Cumming, D. J., 2008. Contracts and exits in venture capital finance. The Review of Financial Studies 21 (5), 1947–82.

Cumming, D. J., Johan, S. A., 2006. Venture Capital and Private Equity Contracting. Academic Press.

Cumming, D. J., Walz, U., 2010. Private equity returns and disclosure around the world. Journal of International Business Studies 41 (4), 727–54.

de Bettignies, J.-E., 2008. Financing the entrepreneurial venture. Management Science 54 (1), 151–166.

Dixit, A. K., Pindyck, R. S., 1996. Investment under Uncertainty, 1st Edition. Princeton University Press.

Duffie, D., 2001. Dynamic Asset Pricing Theory, 3rd Edition. Princeton University Press.

Fenn, G. W., Liang, N., Prowse, S., 1995. The economics of the private equity market. Federal Reserve Board Staff Studies 168.

Fenwick, West, 2010. Trends in terms of venture financings in the san francisco bay area. Quarterly Venture Capital Survey, http://www.fenwick.com.

Gilson, R., Schizer, D., 2003. Venture capital structure: A tax explanation for convertible preferred stock. Harvard Law Review 116, 875–916.

Gompers, P. A., 1995. Optimal investment, monitoring, and the staging of venture capital. The Journal of Finance 50, 1461–1488.

Gompers, P. A., Lerner, J., 1999. The Venture Capital Cycle. MIT Press.

Hellmann, T., 2006. IPOs, acquisitions and the use of convertible securities in venture capital. Journal of Financial Economics 81 (3), 649–679.

Kaplan, S. N., Martel, F., Strömberg, P., 2007. How do legal differences and experience affect financial contracts? Journal of Financial Intermediation 16, 273–311.

Kaplan, S. N., Strömberg, P., 2003. Financial contracting theory meets the real world: An empirical analysis of venture capital contracts. Review of Economic Studies 70 (2), 281–315.

Kerr, W. R., Nanda, R., 2009. Financing constraints and entrepreneurship. Working Paper Harvard Business School.

Lerner, J., Schoar, A., 2005. Does legal enforcement affect financial transactions? The contractual channel in private equity. The Quarterly Journal of Economics 120, 223–46.

Metrick, A., 2007. Venture Capital and the Finance of Innovation. Wiley.

Peng, L., 2001. Building a venture capital index. Yale University ICF Working Paper No. 00-51.

Quigley, J. M., Woodward, S. E., 2003. An index for venture capital. Working Paper University of California Berkeley.

Sahlman, W. A., 1990. The structure and governance of venture-capital organizations. Journal of Financial Economics 27, 473–521.

Sahlman, W. A., Stevenson, H. H., Roberts, M. J., Bhidé, A., 1999. The Entrepreneurial Venture, 2nd Edition. Harvard Business School Press.

Schmidt, K., 2003. Convertible securities and venture capital finance. The Journal of Finance 58 (3), 1139–1166.

Trigeorgis, L., 1996. Real Options. MIT Press.