Diversifying Risk Parity*

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October 31, 2011

*We are grateful to Attilio Meucci, Carsten Zimmer, and seminar participants at Deka Investment GmbH. Note that this paper expresses the authors' views that do not have to coincide with those of Deka Investment GmbH.

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ABSTRACT

Striving for maximum diversification we follow Meucci (2009) in measuring and managing a multi-asset class portfolio. Under this paradigm the maximum diversification portfolio is equivalent to a risk parity strategy with respect to the uncorrelated risk sources embedded in the underlying portfolio assets. Our paper characterizes the mechanics and properties of this diversified risk parity strategy. Moreover, we explore the risk and diversification characteristics of traditional risk-based asset allocation techniques like minimum-variance, risk parity, or 1/N and demonstrate the diversified risk parity strategy to be quite meaningful when benchmarked against these alternatives.

Keywords: Risk-based Asset Allocation, Risk Parity, Diversification, Entropy

JEL Classification: G11; D81

Diversification pays. This insight is at the heart of most portfolio construction paradigms like the seminal one of Markowitz (1952). Given perfect foresight his mean-variance optimization approach is the rationale of choice to generate efficient portfolios with an optimal risk and return trade-off. However, the corresponding optimal weights are rarely put into practice because of certain limitations. For instance, mean-variance optimization is typically confounded by estimation risk, especially the one embedded in estimates of expected returns, see Chopra and Ziemba (1993). One way to circumvent this problem is to simply refrain from estimating returns and to resort to risk-based allocation techniques. Within the framework of Markowitz (1952) this approach leads to the well-known minimum-variance portfolio. Interestingly, while minimum-variance portfolios indeed are characterized by low volatility they have also been shown to have quite favorable return figures. As a consequence, minimum-variance and related risk-based concepts have become increasingly popular in the asset management industry. However, minimum-variance portfolios are designed to load on low-volatility assets which renders them rather concentrated in a few assets. Thus, minimum-variance portfolios are hardly diversified in terms of a homogenous weights distribution.

In striving for well-diversified portfolios the academic literature offers different concepts for measuring diversification.[1] Early studies of Evans and Archer (1968) or Fisher and Lorie (1970) rely on the number of portfolio assets to measure its degree of diversification. Alternative approaches evaluate the entropy or the concentration of a portfolio's weight distribution, see Woerheide and Persson (1993) or Bera and Park (2008). While all of these concepts are agnostic with respect to the assets' dependence structure Rudin and Morgan (2006) suggest a portfolio diversification index that builds on a principal component analysis of the portfolio assets. Under this paradigm a portfolio is said to be diversified if the extracted principal components evenly contribute to the assets' variability. However, this diversification index merely applies to equally-weighted portfolios and is infeasible for comparing other allocation schemes.

To this end Meucci (2009) constructs a more comprehensive diversification measure. Pursuing principal components analysis of the portfolio assets he extracts the principal components driving the assets' variability. These principal components can be interpreted as principal portfolios

[1] For a comprehensive overview and evaluation of diversification metrics see the recent paper of Frahm and Wiechers (2011).

representing the uncorrelated risk sources inherent in the portfolio assets. For a portfolio to be well-diversified its overall risk should therefore be evenly distributed across these principal portfolios. Condensing this risk decomposition into a single diversification metric Meucci (2009) then opts for the exponential of this risk decomposition's entropy because of its intuitive interpretation as the number of uncorrelated bets.

The contribution of this paper is to apply the framework of Meucci (2009) in a multi-asset allocation study. Under this paradigm the maximum diversification portfolio emerges from a risk parity strategy that is budgeting risk with respect to the extracted principal portfolios rather than the underlying portfolio assets. Therefore, we think of this approach as a diversified risk parity strategy which turns out to be a reasonable alternative when it comes to risk-based asset allocation. Moreover, the framework allows for a litmus test of competing techniques like $1/N$, minimum-variance, and risk parity. Especially, risk parity has been put to the fore recently, see Qian (2006, 2011) and Maillard, Roncalli, and Teiletche (2010). While $1/N$ and minimum-variance strategies are fairly well-known for picking up rather concentrated risks we find the traditional risk parity strategy to be more balanced. However, benchmarking risk parity against diversified risk parity one observes a degeneration in its diversification characteristics over time rendering the traditional risk parity a rather concentrated bet in the current environment.

The paper is organized as follows. Section I reviews the approach of Meucci (2009) for managing and measuring diversification. Section II presents the data and further demystifies the concept of principal portfolios. Section III is devoted to contrasting the diversified risk parity strategy to alternative risk-based asset allocation studies. Section IV concludes.

I. Managing Diversification

In the following we present the approach of Meucci (2009) to measuring and managing a given portfolio's diversification. He considers a portfolio consisting of N assets with return vector $\mathbf{R}$. Given weights $\mathbf{w}$ the resulting portfolio return is $R_w = \mathbf{w}'\mathbf{R}$. Diversification especially pays when combining low-correlated assets. In this vein, Meucci (2009) constructs uncorrelated risk sources by applying a principal component analysis (PCA) to the variance-covariance matrix $\mathbf{\Sigma}$ of the portfolio assets. According to the spectral decomposition theorem the covariance matrix can be expressed as a product

$$\mathbf{\Sigma} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}' \tag{1}$$

where $\mathbf{\Lambda} = diag(\lambda_1, ..., \lambda_N)$ is a diagonal matrix consisting of $\mathbf{\Sigma}$'s eigenvalues that are assembled in descending order, $\lambda_1 \geq ... \geq \lambda_N$. The columns of matrix $\mathbf{E}$ represent the eigenvectors of $\mathbf{\Sigma}$. These eigenvectors define a set of N uncorrelated *principal portfolios*[2] with variance λ_i for $i = 1, ..., N$ and returns $\tilde{R} = \mathbf{E}'\mathbf{R}$. As a consequence, a given portfolio can be either expressed in terms of its weights $\mathbf{w}$ in the original assets or in terms of its weights $\tilde{\mathbf{w}} = \mathbf{E}'\mathbf{w}$ in the principal portfolios. Since the principal portfolios are uncorrelated by design the total portfolio variance emerges from simply computing a weighted average over the principal portfolios' variances λ_i using weights $\tilde{w}_i^2$:

$$Var(R_w) = \sum_{n=1}^{N} \tilde{w}_i^2 \lambda_i \tag{2}$$

Normalizing the principal portfolios' contributions by the portfolio variance then yields the *diversification distribution*:

$$p_i = \frac{\tilde{w}_i^2 \lambda_i}{Var(R_w)}, \qquad n = 1, ..., N \tag{3}$$

Note that the diversification distribution is always positive and that the p_is sum to one. Building on this concept Meucci (2009) conceives a portfolio to be well-diversified when the p_i are "approximately equal and the diversification distribution is close to uniform". This definition of a well-diversified portfolio coincides with allocating equal risk budgets to the principal portfolios. Therefore, we dub this approach *diversified risk parity*. Conversely, portfolios loading on a specific principal portfolio display a peaked diversification distribution. It is thus straightforward to apply a dispersion metric to the diversification distribution for arriving at a single diversification metric. Meucci (2009) chooses the exponential of its entropy[3]:

$$\mathcal{N}_{Ent} = \exp\left(-\sum_{i=1}^{N} p_i \ln p_i\right) \tag{4}$$

[2]Note that Partovi and Caputo (2004) coined the term principal portfolios in their recasting of the efficient frontier in terms of these principal portfolios.

[3]The entropy has been used before in portfolio construction, see e.g. Woerheide and Persson (1993) or more recently Bera and Park (2008). However, these studies consider the entropy of portfolio weights thus disregarding the dependence structure of portfolio assets.

The reason for choosing $\mathcal{N}_{Ent}$ relates to its intuitive meaning as the number of uncorrelated bets. To rationalize this interpretation consider two extreme cases. For a completely concentrated portfolio we have $p_i = 1$ for one i and $p_j = 0$ for $i \neq j$ resulting in an entropy of 0 which implies $\mathcal{N}_{Ent} = 1$. Conversely, $\mathcal{N}_{Ent} = N$ holds for a portfolio that is completely homogenous in terms of uncorrelated risk sources. In this case, $p_i = p_j = 1/N$ holds for all i, j implying an entropy equal to $\ln(N)$.

In the spirit of Markowitz (1952), this framework readily allows for determining a mean-diversification frontier that trades off expected return against a certain degree of diversification. Taking this approach to the extreme we can especially construct maximum diversification portfolios by solving

$$\underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmax}} \, \mathcal{N}_{Ent}(w) \tag{5}$$

where the weights $\mathbf{w}$ may possibly be restricted according to a set of constraints $\mathcal{C}$. Thus, the solution of optimization (5) ultimately results in a diversified risk parity strategy that is potentially subject to some investment constraints.

II. Demystifying Principal Portfolios

A. Data and Descriptive Statistics

In building risk-based asset allocation strategies we focus on five broad asset classes as represented by the following indices: We use the JPM Global Bond Index for government bonds, the MSCI World Total Return Index for developed equities, the MSCI Emerging Markets Total Return Index for emerging equities, the DJ UBS Commodity Index for commodities, and the Barclays U.S. Aggregates Credit Index. All indices are measured in local currency returns and we report total return figures from the perspective of an U.S. investor by employing the 3-months U.S. Treasury Rate.

Table I conveys the descriptive statistics of the above asset classes. Over the whole sample period from December 1987 to September 2011 we observe an annualized bond return of 6.9% at a volatility of 3.8% which happens to be the lowest figure across asset classes. During this period developed equities have fared slightly worse in terms of return (5.9%), however, their volatility is significantly higher (14.5%). For emerging equities, return and volatility figures are higher when compared to developed equities. Conversely, commodities are quite similar to developed equities in terms of volatility and return. Most interestingly, credit exhibits the same return as government bonds. This observation is unexpected given that credit is significantly more volatile than government bonds. However, it is important to note that the bulk of credit volatility is related to the credit crunch in 2008 and the subsequent financial crisis. In terms of risk-adjusted returns we find the high-risk asset classes to rather disappoint given Sharpe Ratios around 0.2. By this metric credit ranks second with a figure of 0.69 still underperforming bonds that exhibit a quite impressive Sharpe Ratio of 0.96.

Further inspecting the asset's dependence structure in Table I we observe bonds to be hardly correlated to equities. Its correlation to commodities is slightly negative while the one to credit amounts to 0.53. All of the remaining correlation coefficients are positive and range from 0.09 (credit vs. commodities) to 0.74 (developed vs. emerging equities). Unsurprisingly, credit is more correlated to both equity indices.

B. Building and Interpreting Principal Portfolios

To foster intuition about the uncorrelated risk sources inherent in our multi-asset time series we investigate the PCA over the whole sample period from December 1987 to September 2011. The economic nature of the principal portfolios is best assessed in terms of the eigenvectors that represent the principal portfolios' weights with respect to the original asset classes. By construction these weights are standardized to lie within the [-1,1]-interval. Given that the correlation of the original assets is generally relatively low the interpretation of the principal portfolios as collected in panel A of Table II is straightforward. The first principal portfolio (PP1) is purely driven by emerging and developed equities with emerging equity having a fairly high weight of 0.87. Therefore, PP1 represents genuine equity risk which is accounting for 69.8% of the overall variance. Conversely, principal portfolio 2 (PP2) reflects the diversification potential of commodities relative to equities as indicated by a commodities weight of -0.96 and has a much smaller variance contributing 19.9% to the overall variance. Principal portfolio 3 (PP3) represents the difference between emerging and developed equities thus capturing the emerging market spread.

Principal portfolio 4 (PP4) mostly loads on credit and government bonds and we interpret it as an interest rate risk factor. Note that PP3 and PP4 explain most of the remaining variance, leaving a minuscule fraction of 0.7% for principal portfolio 5 (PP5). Judging by the weights of PP5 we conceive it to be a mimicking factor of the credit spread. In addition, panel B of Table II gives the principal portfolio weights pertaining to a PCA over the last 60 months of the sample period, that is from October 2006 to September 2011. Despite covering the most turbulent of times these weights prove to be fairly similar to those obtaining for the whole sample period. However, the variance of PP1 is elevated by factor 2 when compared to the results for the whole sample period.

For estimating the principal portfolios over time one has to make a choice with regard to the estimation window. The two most common approaches either rely on an expanding window or a rolling window for estimation. The proponents of expanding window estimation appreciate the fact that building on all available data typically gives rise to a quite robust set of components. On the other hand, rolling window estimation is believed to be more responsive to potential structural breaks. Still, while our main analysis focuses on the discussion of results arising from expanding window estimation we additionally report the results arising from rolling window estimation using a 60 months window.

Further, we plot the principal portfolios' variances over time in Figure 1. Each month, a PCA is performed to extract the principal portfolios embedded in the multi-asset classes and the corresponding principal portfolio variances are stacked in one bar.

[Figure 1 about here.]

Regardless of the estimation method, expanding or rolling, we observe PP1 to be fairly dominant by accounting for at least 60% of the underlying time series' variation at any given point in time. Given that PP2 and PP3 represent some 20% and 10% of the variation, the remaining principal portfolios PP4 and PP5 do only account for a minor fraction. Especially, judging by the rolling window estimation we find PP1 accounting for 80% of the overall variability which bears testimony of the contagion effects emanating from the financial crisis in 2008.

III. Risk-based Asset Allocation

For benchmarking the diversified risk parity strategy we consider three alternative risk-based asset allocation strategies: $1/N$, minimum-variance, and risk parity. First, we implement the $1/N$-strategy that rebalances monthly to an equally weighted allocation scheme, hence, the portfolio weights are

$$\mathbf{w} = \frac{\mathbf{1}}{N} \tag{6}$$

Second, we compute the minimum-variance portfolio either building on an expanding or rolling 60 months window for covariance-matrix estimation. The corresponding weights derive from

$$\min \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \tag{7}$$

subject to to the full investment and positivity constraint, $\mathbf{w}'\mathbf{1} = 1$ and $\mathbf{w} \geq \mathbf{0}$. One can show that the minimum-variance weights obtain as

$$\mathbf{w} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}\boldsymbol{\Sigma}^{-1}\mathbf{1}} \tag{8}$$

Third, we construct the original risk parity strategy by allocating capital such that the asset classes' risk budgets contribute equally to overall portfolio risk. Note that these risk budgets also depend on either expanding or rolling window estimation. Since there are no closed-form solutions available we follow Maillard, Roncalli, and Teiletche (2010) in numerically optimizing

$$\operatorname{argmin} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(w_i(\boldsymbol{\Sigma}\mathbf{w})_i - w_j(\boldsymbol{\Sigma}\mathbf{w})_j\right)^2 \tag{9}$$

which essentially minimizes the variance of the risk contributions. Again, the above full investment and positivity constraints apply.

For constructing the diversified risk parity strategy we first determine the principal portfolios using either expanding or rolling window estimation, as described in Section II. We then optimize portfolios to have maximum diversification following optimization (5), see Section I. To allow for comparison with the other risk-based allocation techniques we also enforce full investment and positivity constraints for the diversified risk parity strategy. Rebalancing of all strategies occurs at a monthly frequency. Given that the first PCA estimation consumes 60 months of data the strategy performance can be assessed from January 1993 to September 2011.

Table III gives performance and risk statistics of the risk-based asset allocation strategies. Across the board we find the strategies to yield rather similar annualized returns. Unsurprisingly, the highest annualized return materializes for the $1/N$-strategy, however, the 7.4% come at the cost of the highest volatility (9.4%). Moreover, the strategy exhibits the highest drawdown among all alternatives (33.5%). Conversely, the minimum-variance strategy provides the lowest return of 6.7%. Given that minimum-variance indeed exhibits the lowest volatility (3.6%) its Sharpe Ratio of 0.97 is the most favorable one. Also, its drawdown statistics are the least severe amounting to a maximum loss of 5.7% during the whole sample period. Paraphrasing Maillard, Roncalli, and Teiletche (2010) we then find the risk parity strategy to be a middle-ground portfolio between $1/N$ and minimum-variance. Its return is 7.0% at a 5.0% volatility thus giving rise to a Sharpe Ratio of 0.75. Also, the maximum drawdown statistics are significantly reduced when compared to the $1/N$-strategy.

While the above strategies' characteristics are fairly well-known it is all the more interesting to investigate the performance of the diversified risk parity strategy. The return of the diversified risk parity strategy coincides with the one of the minimum-variance strategy (6.7%). Because of a 6.4% volatility its Sharpe Ratio is 0.55. Moreover, its maximum drawdown characteristics are halfway in between $1/N$ and the original risk parity strategy. To gauge the strategies' evolution over time we plot their cumulative returns in Figure 2. While the $1/N$-strategy is pursuing a rather rocky path the remaining strategies exhibit a quite steady evolution of performance. In addition it seems as if the strategies' resilience with respect to the financial crisis in 2008 is the main driver in explaining the strategies overall volatility.

[Figure 2 about here.]

While the performance table and figure already give a good grasp of the different strategies we additionally provide mutual tracking errors and mutual correlation coefficients in Table IV. In terms of strategy similarity we find the diversified risk parity strategy to be highly correlated to risk parity and the $1/N$-strategy. Judging by a tracking error of 2.13% it is closest to the risk parity strategy. Unsurprisingly, tracking errors are highest for the $1/N$-strategy with figures ranging from 4.88% (vs. risk parity) to 8.53% (vs. minimum-variance). By and large, these findings continue to hold when using a rolling window estimation. Interestingly, the diversified risk parity strategy benefits the most from switching to rolling window estimation which renders it more responsive to a change in risk structure and thus more resilient to the 2008 financial crisis.

Note that the improved risk-adjusted performance of the diversified risk parity strategy is to be taken with a grain of salt since its monthly turnover is also increased relatively to the other risk-based asset allocation strategies. Switching to rolling window estimation its monthly turnover amounts to 5.3% on average which compares to 2.8% for minimum-variance and 2.3% for risk parity. By construction, the turnover metric for the $1/N$-strategy is 0% (disregarding potential rebalancing because of price movement).

While the above analysis provides a first sense of the strategies' nature we refrain from overstating their mere performance statistics. As argued by Lee (2011) evaluating risk-based portfolio strategies by means of Sharpe Ratios is hard to reconcile with the fact that returns are not entering their respective objective function in the first place. In a vein similar to Lee (2011) we rather resort to contrasting the risk characteristics of these portfolios. Thus, we turn to an in-depth discussion of the risk-based strategies' weights and risk allocation in Figure 4. Especially, risk is being decomposed not only by asset class but also by principal portfolios. Hence, the former analysis provides the well-known percentage risk contributions while the latter analysis performs the very same decomposition with regards to uncorrelated risk sources.

First investigating the $1/N$-strategy we find more than 80% of its overall risk to be driven by equities with the highly volatile emerging equities attracting the highest share of the risk budget. Given that commodities consume most of the remaining risk budget the other asset classes, namely bonds and credit, are close to being irrelevant. Decomposing the strategy's risk by principal portfolios instead reveals the $1/N$-strategy to be budgeting risk mostly to PP1, i.e. equity risk. Even more so, as time progresses the $1/N$-strategy more or less emerges as a single-bet strategy as opposed to an N-bet strategy.

[Figure 3 about here.]

Stepping on to minimum-variance we recover its archetypical weights distribution that is heavily concentrated in the two low-risk asset classes bonds and credit. While equities are hardly entering the minimum-variance portfolio, there is always a diversifying commodities position of some 5% in place. This weights decomposition pretty much serves as a blueprint for the minimum-variance strategy's traditional risk decomposition. Conversely, the decomposition of risk with respect to the principal portfolios demonstrates minimum-variance to be heavily exposed to a single risk source, PP4 representing interest rate risk. When compared to $1/N$ the minimum variance appears to be less concentrated because it is also exhibiting a quite constant exposure to PP3 and PP5.

Third, we examine the risk parity strategy. Its weights decomposition reflects a reasonably smooth allocation over time with global bonds accounting for the highest portfolio fraction; on average, one third is being allocated to this asset class. While the bond share is increasing over time one realizes that this increase is mainly fueled by a decrease of the credit position. This observation relates to the fact that the rising credit volatility induces the strategy to limit its credit exposure for maintaining risk parity. The remaining asset classes, equities and commodities, are characterized by rather constant allocation weights over time that are approximately inversely proportional to their respective time series volatilities. By construction, the traditional risk decomposition exhibits equal weights across asset classes.[4] Interestingly, the decomposition of the risk parity strategy with respect to the principal portfolios is significantly less evenly distributed. At the beginning of the sample period PP1 and PP4 each attract some quarter of the risk budget while PP3 almost absorbs its remainder. However, PP2 and PP3 are constantly losing share giving rise to a 50% risk contribution of PP1 and some 35% of PP4. Hence, the risk parity strategy is rendered highly concentrated in terms of uncorrelated risk sources at the end of the sample period.

Finally, we examine the diversified risk parity strategy. When compared to the other risk-based strategies the weights decomposition is also stable in terms of its allocation over time. It allocates roughly 60% to credit and 0% to bonds reflecting the fact that credit appears to be a "perfect" statistical substitute for government bonds. Besides, there is a quite constant exposure

[4] At some times risk parity does only hold approximately given that the numerical optimization may be tricky, see Maillard, Roncalli, and Teiletche (2010).

to both, equities and commodities, with emerging equities being steadily exchanged for developed equities until the turn of the century. The traditional risk decomposition mirrors this positioning in that one basically recovers a risk-parity portfolio budgeting to only three asset classes, namely equities, commodities, and credit. Moreover, its risk decomposition by principal portfolios is the most homogenous one when compared to the other strategies. Even though facing a long-only and full-investment constraint the objective of risk parity across principal portfolios is fairly well achieved. Notably, this objective turns out to be harder to realize at the end of the sample period. Still, while the other risk-based strategies essentially degenerate in their degree of diversification we find the diversified risk parity strategy to still represent around 4 uncorrelated bets at the end of the sample period.

Except for the diversified risk parity strategy we do not report the weights and risk decomposition for the rolling window estimation, chiefly because the characteristics of $1/N$, minimum-variance, and risk parity are fairly unchanged. Conversely, given the corresponding change in the principal portfolios the diversified risk parity portfolio is relatively more affected in terms of positioning. While the allocation prior to 2008 is similar to the expanding window case there is a drastic change in allocation at the end of 2008: The strategy completely exchanges an 80% allocation in credit for global bonds. Thus, rolling window estimation allows the diversified risk parity strategy to react more timely to the change in risk structure thus realizing that credit is no longer substituting for pure interest rate risk and implicitly captured by the minor position in commodities and equities.

[Figure 4 about here.]

For directly comparing the degree to which the risk-based asset allocation strategies accomplish the goal of diversifying across uncorrelated risk sources we plot the number of uncorrelated bets over time in Figure 5. Reiterating our above interpretation of the associated risk contributions over time we find the $1/N$-strategy to be mostly dominated by the other strategies with minimum-variance faring slightly more favorable. Benchmarking the risk parity strategy against the diversified risk parity strategy we note that the former is constantly losing ground over the sample period.

[Figure 5 about here.]

As a we have argued before the presented diversified risk parity strategy does not fully accomplish the target of equal risk contributions across principal portfolios and time because of the long-only constraint. To demonstrate that the degree of diversification can be easily enhanced by relaxing the latter constraint Figure 6 depicts the risk decomposition by principal portfolios for different sets of weights constraints. Especially, we contrast the long-only risk decomposition to a case that includes minor short exposure: Instead of [0%, 100%] we allow each asset class weight to lie within [-10%, 100%]. Applying these boundaries we observe that the risk-decomposition is more balanced relative to the long-only case for expanding window estimation. For rolling window estimation, this minor relaxation of the long-only constraint is not sufficient to significantly improve the risk profile. The latter goal can nevertheless be accomplished by further relaxing the weights constraint to rather wide boundaries of [-500%, 500%] thus representing a quasi-unconstrained case. Despite some minor "hiccups" during the financial crisis the resulting risk profile is close to uniform. Even more so, the expanding window case is then characterized by total risk parity across the principal portfolios.

[Figure 6 about here.]

IV. Conclusion

Within this paper we embrace the approach of Meucci (2009) to maximizing a portfolio's diversification. His paradigm first stipulates rearranging the portfolio assets into uncorrelated risk sources by means of a simple PCA. Maximum diversification obtains when equally budgeting risk to each of the uncorrelated risk sources prompting us to label the strategy diversified risk parity. Especially, this strategy joins the camp of risk-based asset allocation strategies of which many have become fairly popular given their recent convincing risk-adjusted performance. However, judging these strategies by their returns is at odds with the fact that returns are not part of the strategies' underlying objective function. Following Lee (2011) we rather turn to evaluating their ex-ante risk characteristics, especially with respect to the uncorrelated risk sources. While the diversified risk parity strategy is designed to balance these risk sources it is reassuring to indeed find it meeting this objective well even though facing long-only constraints. Unfortunately, the competing alternatives tend to be rather concentrated in a few bets. While the traditional risk parity strategy appears to be least affected we document a decrease in its degree of diversification over time. Also, the traditional risk parity strategy's nature is critically dependent on the choice of assets for contributing equally to portfolio risk. Conversely, diversified risk parity has a built-in mechanism for tracking the prevailing risk structure thus providing a more robust way for achieving maximum diversification throughout time.

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Table I
Descriptive Statistics: Multi-Assets

The table contains descriptive statistics of the multi-asset classes according to the sample period from December 1987 to September 2011. On the left-hand side, annualized return and volatility figures are reported and the right-hand side gives the corresponding correlation matrix.

	Return p.a.	**Vola** p.a.	**Sharpe Ratio**	**Correlation Matrix**				
				Bonds	Equities Dev.	Equities Emg.	Commodities	Credit
Bonds	6.9%	3.8%	0.96	**1.00**				
Developed Equities	5.9%	14.5%	0.18	0.01	**1.00**			
Emerging Equities	8.3%	24.2%	0.21	-0.05	0.74	**1.00**		
Commodities	5.7%	15.5%	0.16	-0.18	0.18	0.32	**1.00**	
Credit	6.9%	5.3%	0.69	0.53	0.25	0.20	0.09	**1.00**

Table II
Principal Portfolio Weights

The table gives the eigenvectors representing the principal portfolio weights with respect to the underlying asset classes. These eigenvectors arise from a PCA of the multi-asset class covariance matrix over the whole sample period from December 1987 to September 2011. Weights in excess of 0.4 are in bold face, weights in excess of 0.2 are in italics. The principal portfolios' variance is given in absolute terms and relative to the overall data variation. 'Cumulative' represents the fraction of variance being explained by a given number of principal portfolios (with the highest variance contributions). Panel A gives the results for the whole sample period, and Panel B gives the results for the last 60 months of the sample period.

Asset Class	PP1 *Equity*	PP2 *Commodities*	PP3 *EM-Spread*	PP4 *Interest Rate*	PP5 *Credit Spread*
Panel A: December 1987 to September 2011					
JPM Global Bond	-0.01	0.05	-0.04	**0.51**	**0.86**
MSCI World	**0.43**	*0.23*	**-0.86**	-0.13	0.03
MSCI Emerging Markets	**0.87**	0.16	**0.47**	0.02	0.01
DJ UBS Commodities	*0.24*	**-0.96**	-0.14	0.01	0.05
Barclays U.S. Aggr. Credit	0.04	0.01	-0.12	**0.85**	**-0.51**
Variance	7.7%	2.2%	0.8%	0.3%	0.1%
Percent Explained	69.8%	19.9%	6.9%	2.8%	0.7%
Cumulative	69.8%	89.7%	96.5%	99.3%	100.0%
Panel B: October 2006 to September 2011					
JPM Global Bond	-0.03	0.03	*-0.21*	**0.43**	**0.88**
MSCI World	**0.42**	*0.39*	**0.78**	0.22	0.08
MSCI Emerging Markets	**0.76**	*0.33*	**-0.50**	*-0.25*	0.02
DJ UBS Commodities	**0.49**	**-0.86**	0.14	0.06	0.05
Barclays U.S. Aggr. Credit	0.09	0.04	*-0.26*	**0.84**	**-0.47**
Variance	14.4%	2.1%	0.5%	0.4%	0.1%
Percent Explained	82.4%	11.8%	3.0%	2.3%	0.5%
Cumulative	82.4%	94.2%	97.3%	99.5%	100.0%

Table III
Performance and Risk Statistics of Asset Allocation Strategies

The table gives performance and risk statistics of the risk-based asset allocation strategies from January 1993 to September 2011. Annualized return and volatility figures are reported together with the according Sharpe Ratio. Maximum Drawdown is computed over 1 month and over the whole sample period. Turnover is the portfolios' mean monthly turnover over the whole sample period. Gini coefficients are reported using portfolios' weights ('Gini Weights') and risk decomposition with respect to the underlying asset classes ('Gini Risk') or with respect to the principal portfolios ('Gini PP Risk'). The # bets is the exponential of the risk decomposition's entropy when measured against the uncorrelated risk sources. Panel A gives the results for expanding window estimation, and Panel B gives the results for rolling window estimation.

Statistic	**Risk-Based Allocations**			
	1/N	*MV*	*RP*	*DRP*
Panel A: Expanding Window				
Return p.a.	7.4%	6.7%	7.0%	6.7%
Volatility p.a.	9.4%	3.6%	5.0%	6.4%
Sharpe Ratio	0.45	0.97	0.75	0.55
Maximum Drawdown 1M	-15.1%	-2.3%	-7.4%	-11.3%
Maximum Drawdown	-33.5%	-5.7%	-15.5%	-22.5%
Turnover	0.0%	0.9%	0.7%	0.9%
Gini Weights	0.00	0.54	0.31	0.46
Gini Risk	0.51	0.54	0.01	0.31
Gini PP Risk	0.75	0.47	0.48	0.31
# bets	1.85	2.88	3.54	4.38
Panel B: Rolling Window				
Return p.a.	7.4%	6.3%	6.8%	6.7%
Volatility p.a.	9.4%	3.5%	4.7%	5.5%
Sharpe Ratio	0.45	0.86	0.75	0.63
Maximum Drawdown 1M	-15.1%	-2.4%	-5.8%	-9.2%
Maximum Drawdown	-33.5%	-5.1%	-13.1%	-18.6%
Turnover	0.0%	2.8%	2.3%	5.3%
Gini Weights	0.00	0.69	0.33	0.56
Gini Risk	0.49	0.69	0.01	0.46
Gini PP Risk	0.82	0.39	0.51	0.29
# bets	1.61	3.23	3.29	4.36

Table IV
Comparison of Risk-based Asset Allocation Strategies

The table compares the risk-based asset allocation strategies by reporting mutual tracking errors above the diagonal and mutual correlation figures below the diagonal with all figures referring to the sample period January 1993 to September 2011. Panel A gives the results for expanding window estimation, and Panel B gives the results for rolling window estimation.

Tracking Error-Correlation-Matrix

Panel A: Expanding Window

	1/N	*MV*	*RP*	*DRP*
1/N	**1.00**	8.53%	5.30%	4.88%
MV	0.41	**1.00**	3.30%	4.82%
RP	0.90	0.75	**1.00**	2.13%
DRP	0.87	0.67	0.96	**1.00**

Panel B: Rolling Window

	1/N	*MV*	*RP*	*DRP*
1/N	**1.00**	8.58%	5.86%	6.28%
MV	0.39	**1.00**	2.92%	3.99%
RP	0.86	0.78	**1.00**	2.39%
DRP	0.76	0.70	0.90	**1.00**

Figure 1. Variances of the Principal Portfolios

The figure gives the variance of the principal portfolios and its relative decomposition over time. Each month, a PCA is performed to extract the principal portfolios embedded in the multi-asset classes and the corresponding principal portfolio variances are stacked in one bar. The left panel gives results for the expanding window estimation and the right panel gives results for the rolling window estimation using a 60-months window. The results are ranging from January 1993 to September 2011.

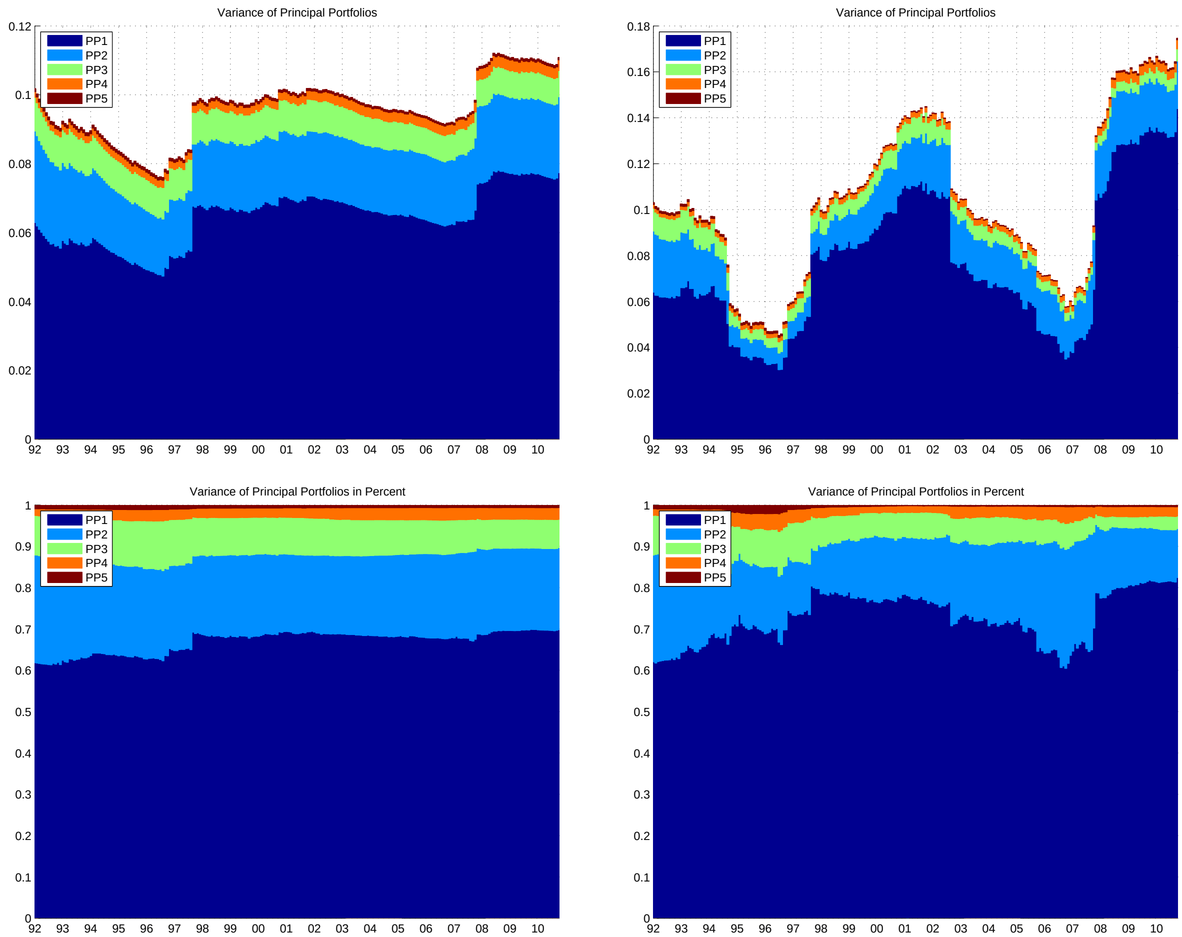

Figure 2. Performance of Risk-Based Asset Allocation

The figure gives the cumulative total return of the risk-based asset allocation strategies over the sample period starting January 1993 to September 2011. The left panel gives results for the expanding window estimation and the right panel gives results for the rolling window estimation using a 60-months window.

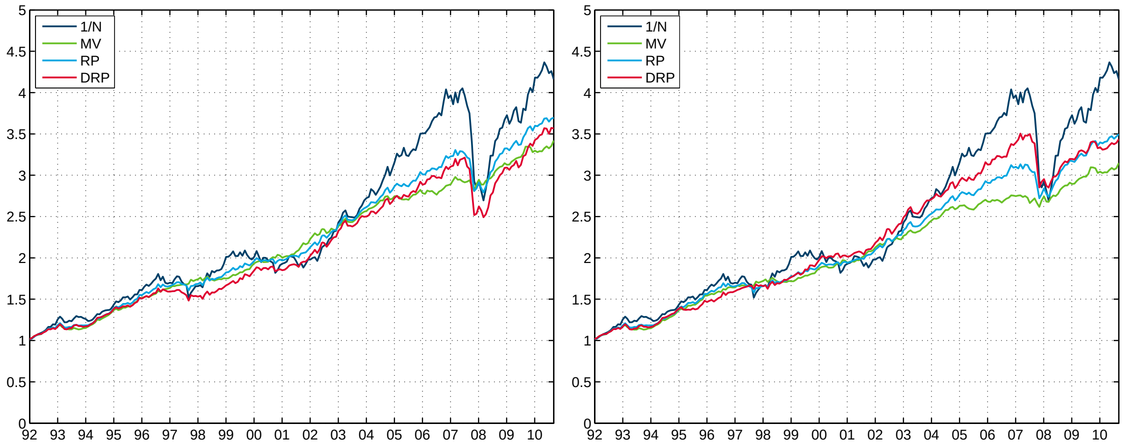

Figure 3. Weights and Risk Decompositions: Expanding Window

The figure gives the decomposition of the risk-based asset allocation strategies in terms of weights and risk. Risk is being decomposed by asset classes and by principal portfolios, respectively. The first row contains the results for the 1/N-strategy, the second row is for minimum variance, the third row for risk parity, and the last row depicts the results for diversified risk parity. The sample period is from January 1993 to September 2011.

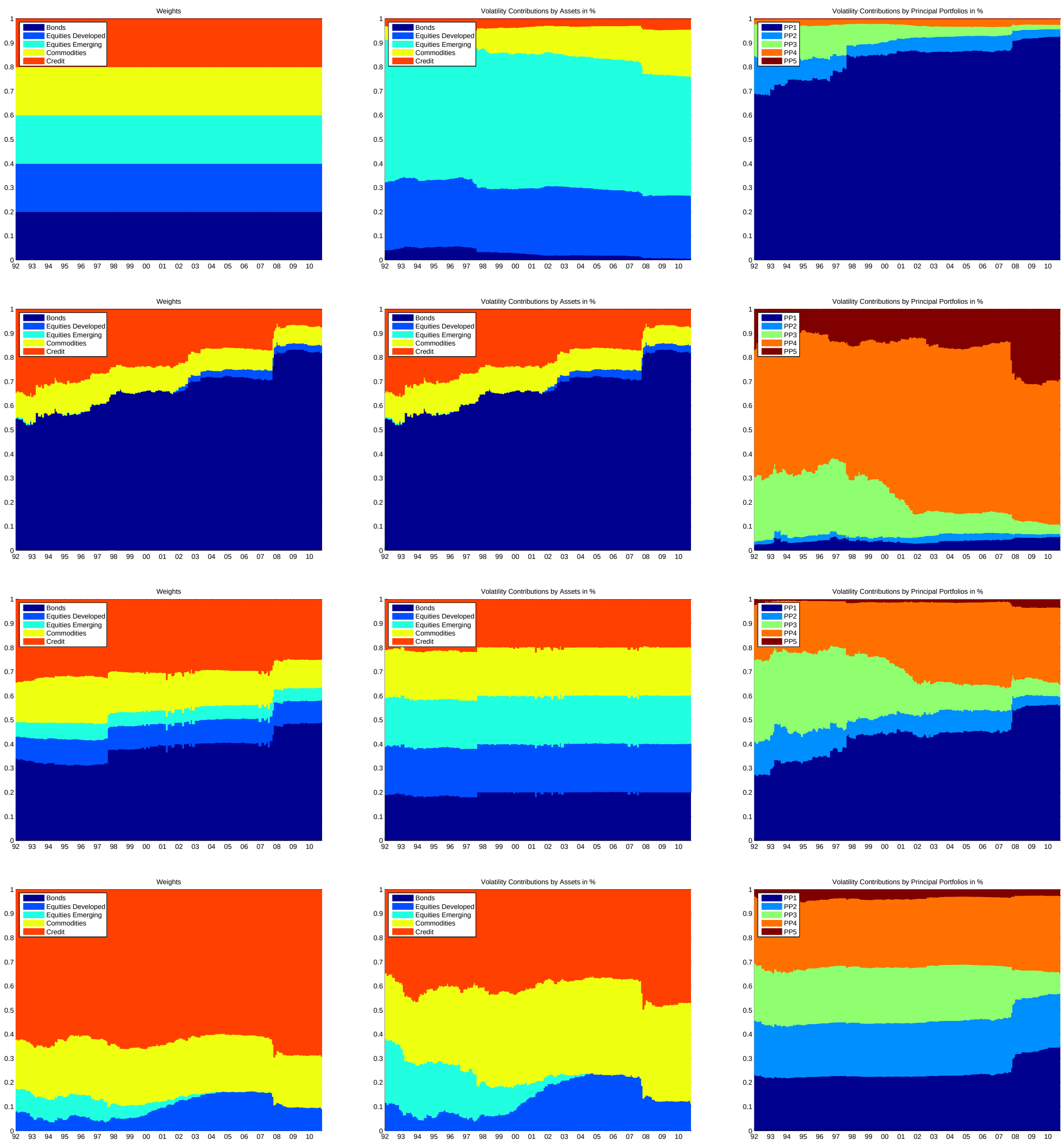

Figure 4. Weights and Risk Decompositions: Rolling Window

The figure gives the decomposition of the diversified risk parity strategies in terms of weights and risk. Risk is being decomposed by asset classes and by principal portfolios, respectively. The sample period is from January 1993 to September 2011.

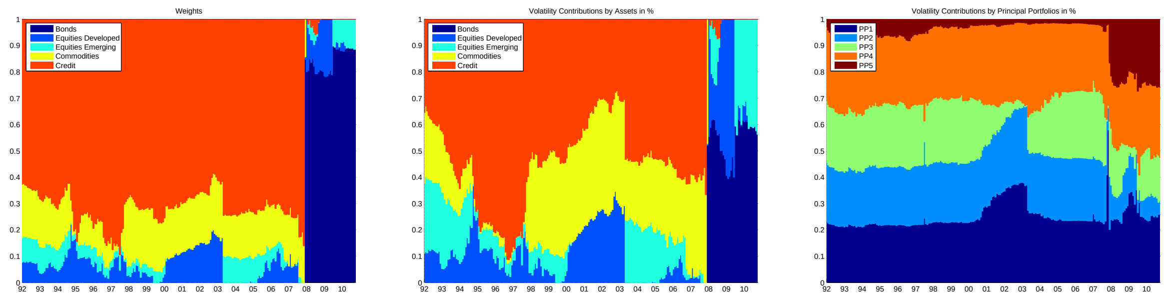

Figure 5. Number of Uncorrelated Bets

We plot the number of uncorrelated bets for the risk-based asset allocation strategies for the sample period January 1993 to September 2011. The left panel gives results for the expanding window estimation and the right panel gives results for the rolling window estimation using a 60-months window.

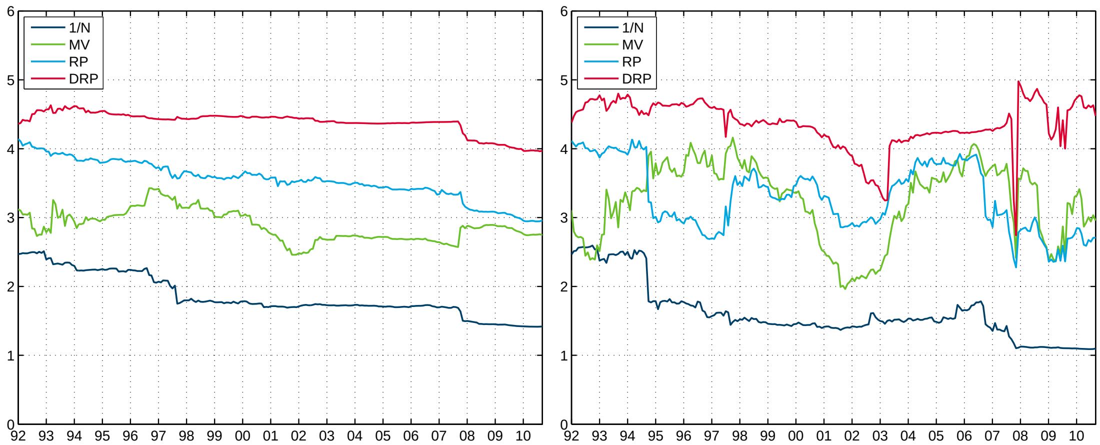

Figure 6. Risk Decompositions: Diversified Risk Parity Portfolios

We plot the risk decomposition with respect to the principal portfolios for various diversified risk parity portfolios. The first row gives the results for expanding window estimation. The second row gives results for the rolling window estimation. The left hand side covers long-only diversified risk parity portfolios, in the middle there are diversified risk parity portfolios with potentially minor short exposure, and the right-hand side covers quasi-unconstrained diversified risk parity portfolios.