

Cyclical risk exposure of pension funds: a theoretical framework

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Abstract

We study the asset allocation problem for a pension fund which operates in a PAYG system and periodically revises its investment strategies. If the optimal amount of wealth invested in risky assets is always positive, then during the management period the optimal portfolio is constantly riskier (less risky) than Merton's portfolio when the growth rate of workers is higher (lower) than the growth rate of pensioners. In particular, there exists a time when the risk exposure is a maximum (minimum).

JEL classification: G11, G23.

Key words: pension funds, PAYG, stochastic dynamic programming.

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1 Introduction and conclusion

In this paper we take into account the asset allocation problem for a pension fund which behaves according to a pay-as-you-go (PAYG) rule. The subscribers to the fund are assumed to be both workers and pensioners. Workers pay contributions to the fund while pensioners receive their pensions which are outflows from the fund wealth. We assume that both the total number of workers and pensioners are stochastic variables. In order to be able to present a closed form solution for the optimal portfolio we assume that these stochastic variables can be spanned on a complete financial market. In other words, there does not exist any non-hedgeable risk.

Our model can deal with both the defined contribution (DC) and defined benefit (DB) systems. In particular, the pension fund sets either contributions (in the DB case) or pensions (in the DC case) in order to guarantee the equilibrium of the fund. We define this equilibrium condition by imposing that the expected value of all the future contributions equates the expected present value of all the future pensions. Similar conditions are assumed in Josa-Fombellida and Rincón-Zapatero (2001) and Battocchio *et al.* (2003).

The financial market we deal with is described by: (i) a riskless asset paying an instantaneously stochastic interest rate, (ii) a set of risky assets following general diffusion processes (without specifying any particular functional form for the drift and diffusion terms), (iii) a set of stochastic state variables (so-called investment opportunities).

In such a framework we show an exact solution for the asset allocation problem of a pension fund which maximizes the expected utility of its final wealth. In particular, we assume that the fund periodically revises the structure of the stochastic model describing the financial market and the demographic risk.

We show that our model accounts for a cyclical behaviour of the pension fund risk exposure (i.e. the amount of wealth invested in risky assets). In a simplified framework we show that if the amount of wealth invested in risky assets is always positive, then the pension fund strategy is always riskier (less risky) than the Merton's (1969, 1971) one when the growth rate of workers is higher (lower) than the growth rate of pensioners.

The literature that analyses the asset allocation of a pension fund generally takes into account the fully funded system. Some examples can be found in Blake *et al.* (2000), Josa-Fombellida and Rincón-Zapatero (2001), Charupat and Milevsky (2002), Battocchio *et al.* (2003), Battocchio and Menoncin (2004). While the main part of this literature concentrate either on the accumulation phase or on the distribution phase of a fully funded pension fund, Battocchio *et al.* (2003) show that the optimal asset allocation during the management period presents a riskiness which decreases during the accumulation phase while it decreases during the distribution phase. In our framework we are able to replicate this kind of behaviour and we show that for both PAYG and fully funded systems a cyclical exposure to risk is optimal for pension funds.

Through this work we consider agents trading continuously in a frictionless arbitrage-free market.

The paper is structured as follows. The framework is outlined in Section 2. First we describe the financial market. Then we present the demographic risk and we compute the equilibrium condition on contributions and pensions. Eventually, we present the dynamic budget constraint on pension fund wealth. In Section 3 we compute the optimal portfolio and give an allocation rule in closed quasi-explicit form. Section 4 shows the solution to the optimal asset allocation in a very simple framework where we are able to explicitly compute the difference between the Merton's (1969, 1971) portfolio and the pension fund optimal portfolio. We show that this difference behaves cyclically.

2 The model

2.1 The financial market

On the financial market there are d assets whose values (S) follow the stochastic differential equation

$$\underset{d \times 1}{dS} = \underset{d \times 1}{\mu}(S, X, t)dt + \underset{d \times k}{\Sigma}(S, X, t)' \underset{k \times 1}{dW}, \quad S(t_0) = S_0, \quad (1)$$

where W is a k -dimensional Wiener process, and the prime denotes transposition. The drift and diffusion terms μ and Σ are supposed to satisfy the usual Lipschitz conditions guaranteeing that Equation (1) has a unique strong solution (see Karatzas and Shreve, 1991). Furthermore, μ and Σ are \mathcal{F}_t -measurable, where \mathcal{F}_t is the σ -algebra through which the Wiener processes are measured on the complete probability space $(\Theta, \mathcal{F}, \mathbb{P})$. All processes below are supposed to satisfy the same properties as those stated for Equation (1). Values of all variables are known at the initial date t_0 and are equal to the non-stochastic variable S_0 .

The variable vector X contains all the state variables affecting the asset prices. It is assumed to satisfy the stochastic differential equation

$$\underset{s \times 1}{dX} = \underset{s \times 1}{\mu_X}(X, t)dt + \underset{s \times k}{\Sigma_X}(X, t)' \underset{k \times 1}{dW}, \quad X(t_0) = X_0. \quad (2)$$

Finally, there exists a riskless asset whose value G follows

$$dG = Gr(X, t) dt, \quad G(t_0) = 1.$$

The financial market is assumed to be complete ($\exists! \Sigma^{-1} \Rightarrow d = k$). Thus, there exists only one market price of risk given by

$$\xi = \Sigma'^{-1}(\mu - rS), \quad (3)$$

through which we can define the martingale equivalent measure

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left(- \int_{t_0}^H \xi' dW_t - \frac{1}{2} \int_{t_0}^H \|\xi\|^2 dt \right). \quad (4)$$

Furthermore, according to Girsanov Theorem, the stochastic process

$$dW^{\mathbb{Q}} = \xi dt + dW, \quad (5)$$

is a Wiener process with respect to \mathbb{Q} .

2.2 The participants to the fund

At each time both the number of workers and the number of pensioners who are into the fund are stochastic variables. When the number of total participants to the fund is sufficiently high, then these stochastic variables can be described through a diffusion process. In particular, if we call n the number of workers and m the number of pensioners then we can assume

$$\begin{aligned} dn &= \mu_n(n, t) dt + \sigma_n(n, t)' dW, \\ dm &= \mu_m(m, t) dt + \sigma_m(m, t)' dW, \end{aligned}$$

with $n(t_0)$ and $m(t_0)$ known constants. We underline that μ_n and μ_m can be both positive and negative. In fact, during the management period the total number of workers (or pensioners) may either increase or decrease. Instead, the signs of the elements of σ_n and σ_m depends on the correlation between the total number of workers (pensioners) and the financial market. For instance, it is likely that during a recession (boom) period the total number of workers decreases (increases).

We underline that we have used for both n and m the same risk sources of asset prices. This formulation, together with the hypothesis of a complete financial market implies that the numbers of workers and pensioners can be perfectly spanned on the market (i.e. there exists a portfolio whose value exactly behaves as either n or m). We leave to future extensions of this model the case where some risk sources on n and m cannot be spanned through a suitable portfolio. Here, we just stress that this case is akin to that of an incomplete market for which an exact solution to the asset allocation problem is very difficult to find.

Since workers pay a contribution $c(t)$ while pensioners receive a pension $p(t)$, then the total net inflow at time t ($\Phi(t)$) into the fund wealth is given by

$$\Phi(t) = \Phi(0) + \int_0^t c(s) dn(s) - \int_0^t p(s) dm(s),$$

and, accordingly

$$\begin{aligned} d\Phi(t) &= c(t) dn(t) - p(t) dm(t) \\ &= (c(t) \mu_n(n, t) - p(t) \mu_m(m, t)) dt \\ &\quad + (c(t) \sigma_n(n, t)' - p(t) \sigma_m(m, t)') dW. \end{aligned}$$

Now, let us assume that the fund decides to ask for constant contribution and pay constant pensions for a given length of time (let us for H periods).

This means that the fund managers revise both the contributions and pensions each H periods. This seems to be necessary in order to take into account structural changes in the stochastic variables n and m . Then, from t to $t + H$, contributions and pensions must be set in order to guarantee a balance between the total amount of contributions received and the total amount of pensions paid.

Accordingly, in a complete market, the "fair" relationship that must link contributions and pensions can be written as

$$\mathbb{E}_0^{\mathbb{Q}} \left[\int_0^H G(s)^{-1} d\Phi(s) \right] = 0, \quad (6)$$

which implies¹

$$\mathbb{E}_0^{\mathbb{Q}} \left[\int_0^H G(s)^{-1} c(s) (\mu_n - \sigma'_n \xi) ds \right] = \mathbb{E}_0^{\mathbb{Q}} \left[\int_0^H G(s)^{-1} p(s) (\mu_m - \sigma'_m \xi) ds \right].$$

Now, in order to ease the following computations, we assume that, during a length of H periods the fund does not change p and c . Thus, since c and p are constant, then the previous relationship can be simplified as

$$\frac{c^*}{p^*} = \frac{\mathbb{E}_0^{\mathbb{Q}} \left[\int_0^H G(s)^{-1} (\mu_m - \sigma'_m \xi) ds \right]}{\mathbb{E}_0^{\mathbb{Q}} \left[\int_0^H G(s)^{-1} (\mu_n - \sigma'_n \xi) ds \right]}. \quad (7)$$

As it can be easily understood from this relationship, the subscriber to the fund can freely choose either his contribution c^* or his pension p^* while the other will be set by the fund. It is easy to see that a subscriber can, for instance, chose a dynamic structure of pensions (i.e. $p^*(t)$) but, also in this case, there will exist a constant c^* such that (6) is satisfied. In other words, the mean value theorem allows us to take constant values of both c^* and p^* without loss of generality.

Let us underline that this framework allows us to take into account both the defined contribution (DC) and defined benefit (DB) schemes. In fact, in a DC scheme the contribution c^* can be considered as a constant during all the working life of a subscriber to the fund while the pension p^* is periodically adjusted (with periodicity H). On the contrary, with a DB scheme, the pension p^* is kept constant while the contributions are periodically set in order to satisfy the equilibrium condition (6).

¹We have used the transformation

$$dW^{\mathbb{Q}} = \xi dt + dW.$$

2.3 The fund wealth

Let $w \in \mathbb{R}^{n \times 1}$ be the vector containing the number of risky assets held in the portfolio and $w_G \in \mathbb{R}$ the number of riskless asset held. At any time t the total managed wealth $R(t)$ is thus given by

$$R(t) = w'S + w_G G, \quad (8)$$

whose differential is

$$dR = \underbrace{w'dS + w_G dG}_{dR_1} + \underbrace{(dw)'(S + dS) + Gdw_G}_{dR_2}.$$

The self-financing condition asks for dR_2 to be financed by the net in-flows to the managed fund. Thus, since we have called Φ these net in-flows, we can write

$$dR_2 = d\Phi,$$

and, after plugging both this condition and the value of w_G obtained from (8) in dR we finally have

$$dR = (Rr + w'(\mu - Sr) + \mu_\Phi) dt + (w'\Sigma' + \Sigma'_\Phi) dW,$$

where

$$\begin{aligned} \mu_\Phi &\equiv c^* \mu_n - p^* \mu_m, \\ \Sigma_\Phi &\equiv c^* \sigma_n - p^* \sigma_m. \end{aligned}$$

3 The optimal portfolio

Given the structure of the model we have exposed in the previous section, we can formulate the problem of the pension fund as:

$$\left\{ \begin{array}{l} \max_w \mathbb{E}_{t_0} \left[\frac{1}{1-\delta} R(H)^{1-\delta} \right] \\ \left[\begin{array}{l} dz \\ dR \end{array} \right] = \left[\begin{array}{l} \mu_z \\ Rr + w'M + \mu_\Phi \end{array} \right] dt + \left[\begin{array}{l} \Omega' \\ w'\Sigma' + \Sigma'_\Phi \end{array} \right] dW, \\ \bar{R}(t_0) = R_0, \quad z(t_0) = z_0, \quad \forall t_0 < t < H \end{array} \right. \quad (9)$$

where he have taken into account a Constant Relative Risk Aversion utility function (the risk aversion is given by δ). The deterministic variable H is the time horizon of the investor and

$$\begin{aligned} \underset{(s+n+1) \times 1}{z} &\equiv \left[X' \quad S' \quad \Phi \right]', \\ \underset{(s+n+1) \times 1}{\mu_z} &\equiv \left[\mu'_X \quad \mu' \quad \mu_\Phi \right]', \\ \underset{k \times (s+n+1)}{\Omega} &\equiv \left[\Sigma_X \quad \Sigma \quad \Sigma_\Phi \right]. \end{aligned}$$

From the first order condition we obtain the optimal portfolio in term of the value function as follows (see Appendix A):

$$w^* = \underbrace{-(\Sigma'\Sigma)^{-1}\Sigma'\Sigma\Phi}_{w_{(1)}^*} - \underbrace{\frac{J_R}{J_{RR}}(\Sigma'\Sigma)^{-1}M}_{w_{(2)}^*} - \underbrace{\frac{1}{J_{RR}}(\Sigma'\Sigma)^{-1}\Sigma'\Omega J_{zR}}_{w_{(3)}^*}. \quad (10)$$

This formula is identical to the one obtained in Menoncin (2002). Thus, the reader is referred to the author for a comment of the portfolio components. Here, we just recall that:

1. the first component $w_{(1)}^*$ minimizes the instantaneous variance of the real wealth differential (see Menoncin, 2002, Proposition 2);
2. the second component $w_{(2)}^*$ is the typical Merton-Markowitz term;
3. the third component $w_{(3)}^*$ hedges the portfolio against the changes into the state variables z .

In the following subsection we compute the functional form of the value function solving the Hamilton-Jacobi-Bellman equation. As it has already been highlighted in Menoncin (2002) the hypothesis of a complete financial market is crucial for obtaining a closed form solution of the optimal portfolio w^* .

Proposition 1 *The optimal portfolio solving Problem (9) is*

$$w^* = -\Sigma^{-1}\Sigma\Phi + \frac{R + \Delta(z, t)}{\delta}(\Sigma'\Sigma)^{-1}M + \frac{R + \Delta(z, t)}{h(z, t)}\Sigma^{-1}\Omega\frac{\partial h(z, t)}{\partial z} - \Sigma^{-1}\Omega\frac{\partial \Delta(z, t)}{\partial z},$$

where

$$\begin{aligned} \Delta(z, t) &= \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^H \frac{G(z, s)}{G(z, t)} d\Phi(z, s) \right], \\ h(z, t) &= \mathbb{E}_t \left[\exp \left\{ \int_t^H \left(r(Z, s) + \frac{1}{2} \frac{1-\delta}{\delta} \frac{1-\delta}{\delta} \xi(Z, s)' \xi(Z, s) \right) ds \right\} \right], \\ dZ &= \left(\mu_z(Z, s) + \frac{1-\delta}{\delta} \Omega(Z, s)' \xi(Z, s) \right) ds + \Omega(Z, s)' dW, \quad Z(t) = z. \end{aligned}$$

Proof. See Appendix A. ■

Proposition 1 shows that the function $\Delta(z, t)$ plays a major role in determining the optimal portfolio composition. In particular, we must investigate more carefully its behaviour in order to understand how the portfolio riskiness

varies across time. Next section is dedicated to this aim. We just stress that we define riskier a portfolio containing a higher number of risky assets.

In order to guarantee the balance of the pension fund we have set $\Delta(z, 0) = 0$ while the final value $\Delta(z, H)$ equals zero by construction. In particular, in $t = 0$ either the pensions or the contributions are set in order to keep the fund in balance and so there is no need for further hedging. Furthermore, when the time t reaches the horizon H when the structure of the model is reset, the hedging need falls against to zero since there is no more "future" to hedge against. Our model is accordingly able to explain a cyclical behaviour of pension funds risk exposure.

In the following section we present a very simple framework where the behaviour of $\Delta(z, t)$ can be computed in a closed form.

4 A simple framework

We now specify a simple model in order to explicitly compute the solution presented in Proposition 1. In particular, the framework we take into account is as follows:

1. the instantaneously riskless interest rate follows the process

$$\begin{aligned} dr &= \alpha(\beta - r) dt - \sigma_r dW_r^{\mathbb{Q}}, \\ r(0) &= r_0, \end{aligned} \quad (11)$$

where \mathbb{Q} is the martingale equivalent measure defined in (4) while α , β , and σ_r are positive constant parameters;

2. the riskless asset value $G(t)$ follows

$$\frac{dG}{G} = r(t) dt, \quad G(0) = 1,$$

3. there exists a (long run) zero coupon bond with a fixed maturity date $T > H$ whose value $B(t)$ follows (see Appendix B)

$$\frac{dB(t)}{B(t)} = (r(t) + C(t, T) \sigma_r \xi_r) dt + C(t, T) \sigma_r dW_r, \quad (12)$$

where ξ_r is the (constant) market price of the interest rate risk source and

$$C(t, T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha}.$$

The differential equation for the bond is derived in the appendix by considering the bond as an interest rate derivative. The function $C(t, T)$ measures the risk of the bond which is positively correlated with its maturity (the further the maturity the riskier the bond). Since the maturity

of the bond if fixed, then its riskiness becomes lower and lower while time goes on. This is the effect which implies a major reallocation of the portfolio aimed at keeping its riskiness at the "suitable" level;

4. there exists a risky asset (a stock) whose price S is assumed to follow

$$\begin{aligned}\frac{dS(t)}{S(t)} &= (r(t) + \sigma_{Sr}\sigma_r\xi_r + \sigma_S\xi_S) dt + \sigma_{Sr}\sigma_r dW_r + \sigma_S dW_S, \\ S(0) &= S_0,\end{aligned}$$

where ξ_S is the (constant) market price of the stock risk source;

5. finally, in order to keep a very simple structure for the demographic part we assume that the total number of workers and pensioners follow a deterministic path

$$\begin{aligned}\frac{dn(t)}{n(t)} &= \mu_n dt, & n(0) &= n_0, \\ \frac{dm(t)}{m(t)} &= \mu_m dt, & m(0) &= m_0.\end{aligned}$$

Such a framework can be traced back to our more general setting via the following matrices

$$\begin{aligned}\Sigma' &= \begin{bmatrix} B(t)C(t,T)\sigma_r & 0 \\ S(t)\sigma_{Sr}\sigma_r & S(t)\sigma_S \end{bmatrix}, \\ M &= \begin{bmatrix} B(t)C(t,T)\sigma_r\xi_r \\ S(t)\sigma_{Sr}\sigma_r\xi_r + S(t)\sigma_S\xi_S \end{bmatrix}, \\ \Omega' &= \begin{bmatrix} -\sigma_r & 0 \end{bmatrix},\end{aligned}$$

and the optimal portfolio, as stated in Proposition 1, is

$$\begin{aligned}w^* &= \frac{R + \Delta(r,t)}{\delta} (\Sigma'\Sigma)^{-1} M \\ &+ \frac{R + \Delta(r,t)}{h(r,t)} \Sigma^{-1} \Omega \frac{\partial h(r,t)}{\partial r} - \Sigma^{-1} \Omega \frac{\partial \Delta(r,t)}{\partial r},\end{aligned}$$

and, after substituting for the matrices we have shown above we can write

$$\begin{aligned}\begin{bmatrix} B(t) & 0 \\ 0 & S(t) \end{bmatrix} w^* &= \frac{R + \Delta(r,t)}{\delta\sigma_S} \begin{bmatrix} \frac{\sigma_S\xi_r - \sigma_{Sr}\xi_S}{C(t,T)} \\ \xi_S \end{bmatrix} \\ &- \frac{R + \Delta(r,t)}{h(r,t)} \begin{bmatrix} \frac{1}{C(t,T)} \\ 0 \end{bmatrix} \frac{\partial h(r,t)}{\partial r} + \begin{bmatrix} \frac{1}{C(t,T)} \\ 0 \end{bmatrix} \frac{\partial \Delta(r,t)}{\partial r},\end{aligned}$$

where we still have to compute functions Δ and h (and their derivatives).

The optimal portfolio thus contains both the bond and the stock but their role is different. The stock just plays a speculative role while the bond is also

used for hedging the portfolio against the variations in the interest rate r which is the only stochastic state variable. This result is already well known in the financial literature since it is demonstrated that the only asset that is used in order to hedge the portfolio against a given risk is the most correlated with such a risk. In fact, in our case, the most correlated asset with the interest rate risk is the bond (which is a derivative on the interest rate).

Furthermore, we stress that $C(t, T)$ measures the reaction of the bond price to the changes in the riskless interest rate (in fact, $\frac{\partial B}{\partial r} \frac{1}{B} = C$). Accordingly, $C(t, T)$ is often used as a measure of the bond riskiness. In fact, we can immediately see that this function appears in the denominator of the optimal portfolio component for the bond: the higher $C(t, T)$ the lower the amount of bond that is hold.

As it is shown in the appendix, the function $h(r, t)$ has the following value:

$$\begin{aligned} h(r, t) &= e^{\frac{1}{2} \frac{1-\beta}{\beta} (\xi_r^2 + \xi_S^2)(H-t)} \mathbb{E}_t^{\mathbb{Q}_\beta} \left[e^{\int_t^H r(s) ds} \right] \\ &= e^{\frac{1}{2} \frac{1-\beta}{\beta} (\xi_r^2 + \xi_S^2)(H-t)} e^{-A(t, H, \hat{\beta}) + r(t)C(t, H)}, \end{aligned}$$

where (see Appendix B)

$$\hat{\beta} \equiv \beta + \sigma_r \frac{1 - 2\delta}{\alpha \delta} \xi_r,$$

and so we have

$$\frac{\partial h(r, t)}{\partial r} \frac{1}{h(r, t)} = C(t, H). \quad (13)$$

The optimal portfolio can thus be furtherly simplified as

$$\begin{aligned} \begin{bmatrix} B(t) & 0 \\ 0 & S(t) \end{bmatrix} w^* &= \frac{R + \Delta(r, t)}{\delta \sigma_S} \begin{bmatrix} \frac{\sigma_S \xi_r - \sigma_{Sr} \xi_S}{C(t, T)} - \delta \sigma_S \frac{C(t, H)}{C(t, T)} \\ \xi_S \end{bmatrix} \\ &+ \begin{bmatrix} \frac{1}{C(t, T)} \\ 0 \end{bmatrix} \frac{\partial \Delta(r, t)}{\partial r}. \end{aligned} \quad (14)$$

The value of function $\Delta(r, t)$ as in Proposition 1 is

$$\begin{aligned} \Delta(r, t) &= c^* \mu_n \int_t^H n(s) \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^s r(u) du} \right] ds \\ &- p^* \mu_m \int_t^H m(s) \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^s r(u) du} \right] ds, \end{aligned}$$

and

$$\begin{aligned} \Delta(r, t) &= c^* \mu_n n(t) \int_t^H e^{\mu_n(s-t) - A(t, s, \beta) - r(t)C(s, t)} ds \\ &- p^* \mu_m m(t) \int_t^H e^{\mu_m(s-t) - A(t, s, \beta) - r(t)C(s, t)} ds, \end{aligned} \quad (15)$$

where the values of c^* and p^* satisfy $\Delta(r, 0) = 0$.

The optimal wealth is given by

$$dR = \left(Rr + \frac{R + \Delta}{\delta} \xi' \xi + \frac{R + \Delta}{h} h_r \Omega' \xi - \Delta_r \Omega' \xi + \mu_{\Phi} \right) dt + \left(\frac{R + \Delta}{\delta} \xi' + \frac{R + \Delta}{h} h_r \Omega' - \Delta_r \Omega' \right) dW,$$

and given the result of Equation (13)

$$\begin{aligned} dR = & R \left(r + \frac{1}{\delta} (\xi_r^2 + \xi_S^2) - C(t, H) \sigma_r \xi_r \right) dt & (16) \\ & + \left(\frac{\Delta}{\delta} (\xi_r^2 + \xi_S^2) - \Delta C(t, H) \sigma_r \xi_r + \Delta_r \sigma_r \xi_r + c^* n \mu_n - p^* m \mu_m \right) dt \\ & + R \left(\frac{1}{\delta} [\xi_r \quad \xi_S] - [C(t, H) \sigma_r \quad 0] \right) dW \\ & + \left(\frac{\Delta}{\delta} [\xi_r \quad \xi_S] - \Delta [C(t, H) \sigma_r \quad 0] + \Delta_r [\sigma_r \quad 0] \right) dW \end{aligned}$$

5 A numerical example

The numerical example we are going to present will follow three steps:

1. given the numerical values of the variables the feasible c^* and p^* are computed from the condition $\Delta(r_0, 0) = 0$ (with the value of $\Delta(r, t)$ as in Equation (15));
2. the value of both $\Delta(r, t)$ and $\frac{\partial \Delta(r, t)}{\partial r}$ are computed;
3. the optimal wealth path is obtained from Equation (16);
4. the optimal asset allocations $w_B^* B$ and $w_S^* S$ are computed as in Equation (14).

Let us start with the choice of the suitable values for the parameters.

For the interest rate that follows a Vasicek structure a complete estimation of the parameters (i.e. the volatility, the mean interest rate, and the strength of the mean reverting effect) can be found in Babbs and Nowman (1998, 1999) who construct zero-coupon yields. Given the results exposed in their works, we have chosen the values $\beta = 0.05$, $\alpha = 0.2$, and $\sigma_r = 0.01$. The initial value for the interest rate r_0 is assumed to coincide with β (i.e. its equilibrium level) since we do not want to study how misalignment of interest rate affects the asset allocation.

Once the interest rate parameters have been estimated, the bond drift and diffusion terms can be obtained by fixing a market price of risk ξ_r and a maturity

Table 1: Values of parameters

Interest rate		Rolling-maturity bond	
Mean reversion, α	0.2	Maturity, T	10
Mean rate, β	0.05	Market price of risk, ξ_r	0.46
Volatility, σ_r	0.01		
Initial rate, r_0	0.05	Demography	
		Number of workers (n_0)	10
Stock		Growth rate μ_n	0.15
Market price of risk ξ_S	0.31	Number of pensioners (m_0)	10
Interest rate source risk, σ_{Sr}	0.06	Growth rate μ_m	0.1
Stock own volatility, σ_S	0.19	Pension rate p^*	1

T . The longest maturity taken into account in Babbs and Nowman (1998, 1999) is 10 years and so we put $T = 10$. Now, since the interest rate on a bond with 10 years maturity is around 7%, then we can compute ξ_r by solving

$$0.07 = \beta + \frac{1 - e^{-\alpha T}}{\alpha} \sigma_r \xi_r,$$

which immediately gives $\xi_r = 0.46$. The initial value of the bond is assumed to be $B(0) = 1$.

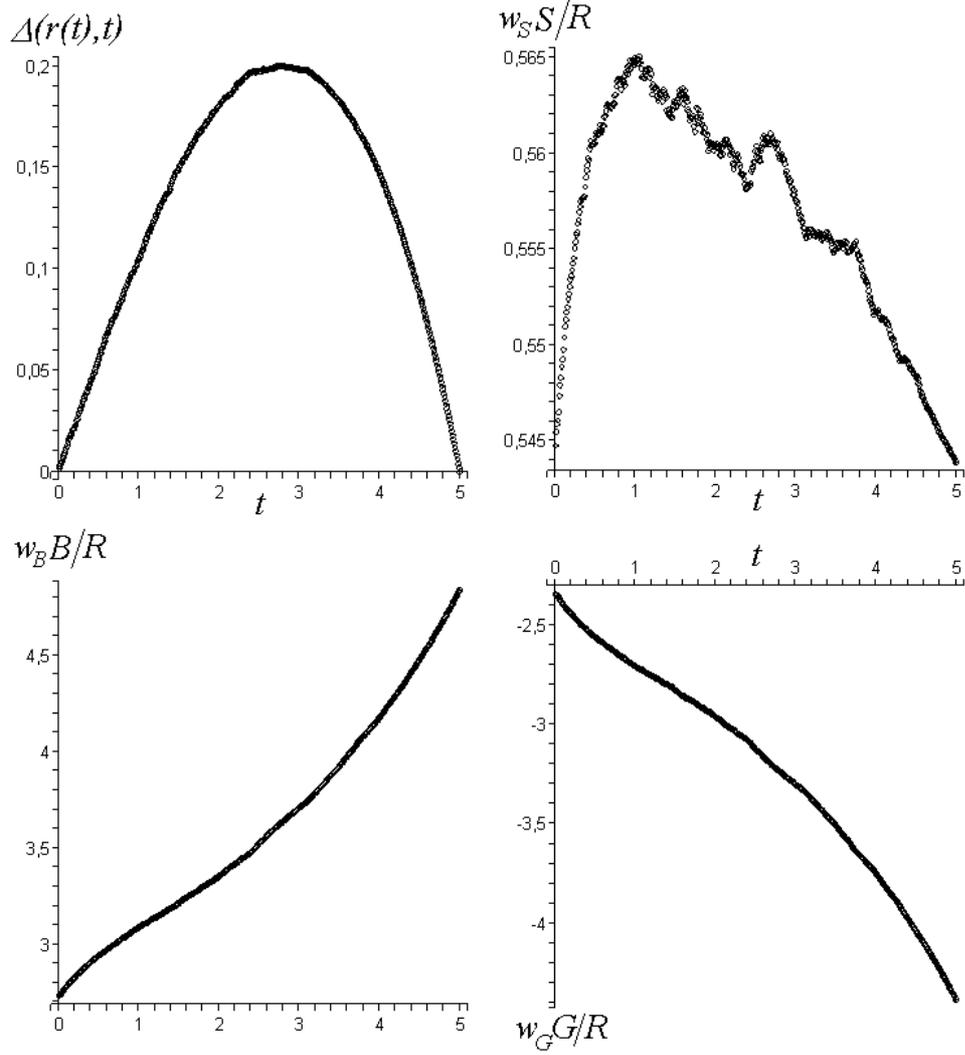
The risk premium on the stock ($\sigma_{Sr} \sigma_r \xi_r + \sigma_S \xi_S$) is chosen after the analysis of Mehra and Prescott (1985). They found that the risk premium was approximately 0.06 for the United States of America during the period 1889-1978. The standard deviation of the market return was about 0.2 for the same period. Hence, we put $\sqrt{\sigma_{Sr}^2 + \sigma_S^2} = 0.2$. In this case we have one degree of freedom. To the best of our knowledge, there are no works dedicated to disentangle σ_{Sr} and σ_S , that is the stock own volatility component and the stock volatility due to the changes in the riskless interest rate. We have decided to give more weight to the σ_S component by putting $\sigma_S = 0.19$ and $\sigma_{Sr} = 0.06$. Accordingly, the market price for asset risk is given by

$$\xi_S = \frac{0.06 - \sigma_{Sr} \sigma_r \xi_r}{\sigma_S} \simeq 0.31.$$

Parameter values (which are also consistent with the numerical analysis presented by Boulier *et al.*, 2001) are gathered in Table 1. The initial wealth is set to $R(t_0) = 1$ in order to take into account the case when contributions and pensions mainly contribute to the wealth (i.e. np and mc are relative high with respect to R). The risk aversion is set to $\delta = 3$. Finally, we assume that the fund is managed for 5 periods (i.e. $H = 5$).

The graphs in Figure 1 we can check the behaviours of the optimal portfolio components and of the function $\Delta(r, t)$. In particular we see what follows:

Figure 1: Behaviours of the optimal portfolio components ($w_S S/R$, $w_B B/R$, and $w_G G/R$) and of the function $\Delta(r, t)$.



1. $\Delta(r, t)$: this function has the behaviour we have already underlined in the theoretical framework (since $\mu_n > \mu_m$); thus it has a concave shape which first increases and then decreases.
2. $w_S S/R$: the stock part in the optimal portfolio follows the behaviour of $\Delta(r, t)$ as it can be seen in the (14); we can check that the function Δ does not dramatically affect the amount of wealth invested in the stock which remains between 54% and 56%;
3. $w_B B/R$: the bond part in the optimal portfolio is less affected by the concave behaviour of function $\Delta(r, t)$ since it is suit for hedging the optimal portfolio exactly against the changes in the function $\Delta(r, t)$; the continuously increasing behaviour depends on the function $C(t, T)$ which tends to zero when $t = T$. Accordingly, if the bond has a maturity (T) which is close to H then while t becomes closer and closer to H , the wealth invested in the bond becomes higher and higher (and it tends towards infinity when $H = T$). This result is very intuitive and it strongly depends on the stochastic process of B in (12). The bond has a higher return with respect to the riskless interest rate but while the time approaches the maturity date, then two effects arise: (i) the bond return becomes closer and closer to the riskless interest rate and (ii) the bond volatility becomes closer and closer to zero. This means that while time goes on it must be optimal to invest higher and higher amount of money in the bond (by borrowing money at the rate r);
4. $w_G G/R$: the riskless asset in the optimal portfolio symmetrically behaves with respect to the bond and so the comments are the same as in the previous point.

6 Conclusion

In this paper we have studied the asset allocation problem for a pension fund working in a PAYG system. In a general stochastic framework we demonstrate that the demographic risk affects the optimal portfolio in the following way: during a management period lasting for a finite length, the riskiness increases (decreases) and then decreases (increases) when the growth rate of workers is higher (lower) than the growth rate of pensioners.

We carry out a numerical simulation on a simple model allowing for a closed form solution to the optimal portfolio. This simulation suggests that the cyclical behaviour above cited does not crucially affect the optimal asset allocation. In particular, the cyclical behaviour arises for the stock while the bond and the riskless asset follow a monotonic behaviour. Since the bond volatility approaches zero while the maturity becomes closer and closer, then the weight of the bond constantly increases and the money invested in it is borrowed at the riskless rate.

A Stochastic dynamic programming

The Hamiltonian of Problem (9) is

$$\begin{aligned} \mathcal{H} = & \mu'_z J_z + J_R (Rr + w'M + \mu_\Phi) + \frac{1}{2} \text{tr} (\Omega' \Omega J_{zz}) \\ & + (w' \Sigma' + \Sigma'_\Phi) \Omega J_{zR} + \frac{1}{2} J_{RR} (w' \Sigma' + \Sigma'_\Phi) (\Sigma w + \Sigma_\Phi), \end{aligned} \quad (17)$$

where $J(R, z, t)$ is the value function solving the Hamilton-Jacobi-Bellman partial differential equation, verifying:

$$J(R, z, t) = \sup_w \mathbb{E}_t \left[\frac{1}{1-\delta} R(H)^{1-\delta} \right],$$

and the subscripts on J indicate the partial derivatives.

The system of the first and second order conditions is

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial w} &= J_R M + \Sigma' \Omega J_{zR} + J_{RR} \Sigma' \Sigma w + J_{RR} \Sigma' \Sigma_\Phi = 0, \\ \frac{\partial^2 \mathcal{H}}{\partial w' \partial w} &= J_{RR} \Sigma' \Sigma \text{ neg. def.} \end{aligned}$$

Since $\Sigma' \Sigma$ is a positive definite matrix, then we can conclude that the second order conditions are satisfied since it is well known that a strictly concave objective function leads to a concave value function.

The Hamilton-Jacobi-Bellman equation for Problem (9) is defined as follows:

$$\begin{cases} J_t + \mathcal{H}^* = 0, \\ J(R, z, H) = \frac{1}{1-\delta} R(H)^{1-\delta}, \end{cases}$$

where \mathcal{H}^* is the Hamiltonian (17) containing w^* instead of w . So, under the market completeness hypothesis, we have

$$\begin{aligned} 0 = & J_t + \mu'_z J_z + J_R (Rr + \mu_\Phi - \Sigma'_\Phi \xi) - \frac{1}{2} \frac{J_R^2}{J_{RR}} \xi' \xi \\ & - \frac{J_R}{J_{RR}} \xi' \Omega J_{zR} + \frac{1}{2} \text{tr} (\Omega' \Omega J_{zz}) - \frac{1}{2} \frac{1}{J_{RR}} J'_{zR} \Omega' \Omega J_{zR}. \end{aligned}$$

Since Merton (1969, 1971) a separability condition on the value function is checked. Now, we try the following functional form

$$J(R, z, t) = \frac{1}{1-\delta} F(z, t) (\Delta(z, t) + R)^{1-\delta},$$

which is not separable by product in R and z and where $F(z, t)$ and $\Delta(z, t)$ are two functions that must be determined. The boundary condition equating the value function in H with the utility function, is transformed into the following boundary conditions:

$$\begin{cases} F(z, H) = 1, \\ \Delta(z, H) = 0. \end{cases}$$

After substituting the guess function for J into the HJB equation we obtain

$$\begin{aligned}
0 &= F_t (\Delta + R)^{1-\delta} + F(1-\delta) (\Delta + \gamma R)^{-\delta} \Delta_t + F (\Delta + R)^{-\delta} Rr \\
&\quad + (\Delta + R)^{1-\delta} \left(\mu'_z + \frac{1-\delta}{\delta} \Omega' \xi \right) F_z \\
&\quad + (1-\delta) F (\Delta + R)^{-\delta} (\mu'_z - \Omega' \xi) \Delta_z + (1-\delta) F (\Delta + R)^{-\delta} (\mu_\Phi - \Sigma'_\Phi \xi) \\
&\quad + \frac{1}{2} \frac{1-\delta}{\delta} (\Delta + R)^{1-\delta} F \xi' \xi + \frac{1}{2} (\Delta + R)^{1-\delta} \text{tr} (\Omega' \Omega F_{zz}) \\
&\quad + \frac{1}{2} (1-\delta) F (\Delta + R)^{-\delta} \text{tr} (\Omega' \Omega \Delta_{zz}) + \frac{1}{2} \frac{1-\delta}{\delta} (\Delta + R)^{1-\delta} \frac{1}{F} F'_z \Omega' \Omega F_z.
\end{aligned}$$

In this equation there are two sets of terms: (i) one containing the function of R given by $(\Delta + R)^{1-\delta}$, and (ii) one containing the function of R given by $(\Delta + R)^{-\delta}$.² After collecting these two terms and simplifying, we obtain the following system of two partial differential equations:

$$\begin{cases} F_t + (\mu'_z + \frac{1-\delta}{\delta} \xi' \Omega) F_z + (r + \frac{1}{2} \frac{1-\delta}{\delta} \xi' \xi) F + \frac{1}{2} \text{tr} (\Omega' \Omega F_{zz}) \\ \quad + \frac{1}{2} \frac{1-\delta}{\delta} \frac{1}{F} F'_z \Omega' \Omega F_z = 0, \\ \Delta_t + (\mu'_z - \Omega' \xi) \Delta_z + \frac{1}{2} \text{tr} (\Omega' \Omega \Delta_{zz}) - \Delta r + (\mu_\Phi - \Sigma'_\Phi \xi) = 0. \end{cases}$$

Now, for simplifying the first equation we use the method exposed in Zariphopoulou (2001) and we put

$$F(z, t) = h^\delta(z, t).$$

So, after dividing by $\delta h^{\delta-1}$, the two previous equations become

$$\begin{cases} h_t + (\mu'_z + \frac{1-\delta}{\delta} \xi' \Omega) h_z + \frac{1}{2} \text{tr} (\Omega' \Omega h_{zz}) + (r + \frac{1}{2} \frac{1-\delta}{\delta} \xi' \xi) h = 0, \\ \Delta_t + (\mu'_z - \xi' \Omega) \Delta_z + \frac{1}{2} \text{tr} (\Omega' \Omega \Delta_{zz}) - \Delta r + (\mu_\Phi - \Sigma'_\Phi \xi) = 0. \end{cases}$$

The solutions of both equations can be represented thanks to the Feynman-Kač theorem.³ In particular, given the boundary conditions

$$\begin{cases} h(z, H) = 1, \\ \Delta(z, H) = 0, \end{cases}$$

we can write

$$h(z, t) = \mathbb{E}_t^Z \left[\exp \left\{ \int_t^H \left(r(Z, s) + \frac{1}{2} \frac{1-\delta}{\delta} \xi(Z, t)' \xi(Z, t) \right) ds \right\} \right], \quad (18)$$

where

$$dZ = \left(\mu_z(Z, t) + \frac{1-\delta}{\delta} \Omega(Z, t)' \xi(Z, t) \right) ds + \Omega(Z, t)' dW, \quad Z(t) = z,$$

²We underline that the term $F(G+R)^{-\beta} Rr$ can be written as $F(G+R)^{1-\beta} r - F(G+R)^{-\beta} Gr$.

³For a complete exposition of the Feynman-Kač theorem the reader is referred to Duffie (1996), Björk (1998) and Øksendal (2000).

and

$$\Delta(z, t) = \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^H (\mu_{\Phi} - \Sigma'_{\Phi} \xi) e^{-\int_t^s r(u) du} ds \right], \quad (19)$$

where \mathbb{Q} is the martingale equivalent measure already presented in (4).

Accordingly, the optimal portfolio can be written as

$$w^* = -\Sigma^{-1} \Sigma_{\Phi} + \frac{1}{\delta} (\Delta + R) (\Sigma' \Sigma)^{-1} M + \Sigma^{-1} \Omega \left((\Delta + R) \frac{1}{h} h_z - \Delta_z \right),$$

and, after substituting the values of functions $\Delta(z, t)$ and $h(z, t)$, the result in the text follows.

B Zero coupon bond

Under the martingale equivalent measure, the value of the zero coupon is given by

$$B(r, t, T) = \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\chi \int_t^T r(s) ds} \right],$$

where χ will be useful for disentangle the case with $\chi = 1$ and $\chi = -1$. After Itô's lemma, we can write

$$dB = \left(\frac{\partial B}{\partial t} + \frac{\partial B}{\partial r} \alpha(\beta - r) + \frac{1}{2} \frac{\partial B}{\partial r^2} \sigma_r^2 \right) dt - \frac{\partial B}{\partial r} \sigma_r dW_r^{\mathbb{Q}}.$$

Now, because of arbitrage reasons, the following partial differential equation must hold

$$\frac{\partial B}{\partial t} + \frac{\partial B}{\partial r} \alpha(\beta - r) + \frac{1}{2} \frac{\partial B}{\partial r^2} \sigma_r^2 = \chi B r, \quad (20)$$

with the natural boundary condition

$$B(T, T) = 1.$$

Let the guess function be

$$B(r, t, T) = e^{-\chi A(t, T) - \chi r(t) C(t, T)},$$

and so Equation (20) can be written as

$$-\chi A_t - \chi r C_t - \chi C \alpha(\beta - r) + \frac{1}{2} \chi^2 C^2 \sigma_r^2 - \chi r = 0,$$

which is rewritten as a system of two ordinary differential equations:

$$\begin{cases} A_t + C \alpha \beta - \frac{1}{2} \chi C^2 \sigma_r^2 = 0, \\ C_t - C \alpha + 1 = 0, \end{cases}$$

with boundary conditions

$$\begin{aligned} A(T, T) &= 0, \\ C(T, T) &= 0, \end{aligned}$$

and whose solutions are

$$C(t, T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha}, \quad (21)$$

$$\begin{aligned} A(t, T, \beta) &= \int_t^T \left(C(s) \alpha \beta - \frac{1}{2} \chi C(s)^2 \sigma_r^2 \right) ds \\ &= \frac{3\chi\sigma_r^2 - 4\beta\alpha^2}{4\alpha^3} + \frac{2\beta\alpha^2 - \chi\sigma_r^2}{2\alpha^2} (T-t) \\ &\quad + \frac{\beta\alpha^2 - \chi\sigma_r^2}{\alpha^3} e^{-\alpha(T-t)} + \chi \frac{\sigma_r^2}{4\alpha^3} e^{-2\alpha(T-t)}. \end{aligned} \quad (22)$$

It can be easily obtained from Equation (18) in Appendix A that

$$dW_r^{\mathbb{Q}_\beta} = \frac{1-\delta}{\delta} \xi_r dt + dW_r,$$

and, given Equation (5), we have

$$dW_r^{\mathbb{Q}_\beta} = \frac{1-2\delta}{\delta} \xi_r dt + dW_r^{\mathbb{Q}}.$$

This means that the interest rate, under the probability \mathbb{Q}_β , follows

$$\begin{aligned} dr &= \alpha(\beta - r) dt - \sigma_r \left(dW_r^{\mathbb{Q}_\beta} - \frac{1-2\delta}{\delta} \xi_r dt \right) \\ &= \alpha \left(\beta + \frac{1-2\delta}{\alpha\delta} \sigma_r \xi_r - r \right) dt - \sigma_r dW_r^{\mathbb{Q}_\beta}, \end{aligned}$$

and thus, after defining

$$\hat{\beta} \equiv \beta + \frac{1-2\delta}{\alpha\delta} \sigma_r \xi_r,$$

we can write the interest rate process as

$$dr = \alpha \left(\hat{\beta} - r \right) dt - \sigma_r dW_r^{\mathbb{Q}_\beta},$$

and the previous computations keep valid; it is just sufficient to put $\hat{\beta}$ in the function $A(t, T, \beta)$.

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