Pricing Credit Card Loans with Default Risks

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ABSTRACT

This paper extends the Jarrow and Deventer (1998) model to facilitate the consideration of default risks within the overall evaluation of credit card loans. We derive closed-form solutions within a continuous-time framework, whilst also providing a numerical method for the evaluation of credit card loans within a discrete-time framework. Adopting the market segmentation argument to describe the characteristics of the credit card industry, we find from our simulation results that the shapes of the forward rate and forward spread (default risk premium) term structures play extremely important roles in determining the value of credit card loans.

Keywords: Credit card loans; Closed-form solutions; Default risks.

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1. INTRODUCTION

According to 2003 Federal Reserve statistical reports, the annual growth rate in consumer credit (93 per cent of which is in the form of credit card receivables) averaged out at over 12 per cent between 1980 and 2002, with the average growth rate prior to 1987 having been upwards of 15 per cent; thereafter, securitization became an integral element of growth within the credit card industry. Throughout the capital crunch of the early 1990s, the sector was led by Citicorp, which increased its credit card accounts by 42 per cent between 1990 and 1992, with further growth of 18 per cent in 1994, and 22 per cent in 1995. Securitized credit card receivables subsequently went on to exceed US$180 billion in 1996, by which time credit cards were accounting for 48.4 per cent of the non-mortgage ABS market. By 2001, credit card securitization had grown to US$339.1 billion (Calomris and Mason, 2004).

It is clear, therefore, that the market for credit card loans is a market of extremely rapid growth; however, credit card loans are very difficult to evaluate since the rates charged differ significantly from the market rates for financial securities of equivalent risk. Such major differences have been attributed to markets with imperfect competition, possibly as a result of market friction, regulatory barriers or adverse selection problems under asymmetric information (Hutchison and Pennacchi, 1996).

Another important feature of credit card loans is that such loans have high default risks; indeed, from an overall market perspective, the default rates for credit card loans are, on average, much greater than those for other loans. For example, during Asian financial crisis, the default rate of credit card loans in Thailand is nearly 36%. Even in Singapore, the default rate of credit card loans is 4.3 percent during the same period according to the 2004 Visa credit organization’s estimates. Hence, when constructing a model for the evaluation of credit card loans, it is important to determine the characteristics of such products, including the interest rate differentials and default risks.

With the notable exception of the Jarrow and Deventer (1998) model, the literature on pricing credit card loans is rare. Thus, in this paper, we use the arbitrage-free price method to construct discrete-time and continuous models capable of capturing the respective characteristics of interest rate differentials and default risks for pricing risky credit card loan assets. In order to achieve this, we first modify the Jarrow and Deventer model, extending the Heath-Jarrow-Morton (1990) term-structure model so as to facilitate the consideration of default risks. We then adopt the market segmentation argument to describe the characteristics of the interest rate differentials observed within the credit card industry as a whole.

The model proposed in this study has three distinctive features. First of all, it takes existing spreads as an input into the model, rather than deriving the model from implications on default probabilities and recovery rates. Secondly, rather than working with spot yield curves for default-free and risky debt, we work with 'forward rates' and 'forward spreads'. The specific advantage of using forward rates is that since the current term structure is an input into the model, forward rates can describe short rates, whereas, short rates cannot describe forward rates. Thirdly, on the one hand, we provide a closed-form solution for the evaluation of credit card loans and their securitization products within a continuous-time framework; on the other hand, we offer an applicable lattice approach to the evaluation of credit card loan-related products within a discrete-time framework.

We find from our simulation results that the shape of the forward rate and forward spread (default risk premium) term structures play extremely important roles in determining the value of credit card loans. Furthermore, to the best of our knowledge, this is the first model of its kind to investigate the ways in which the parameters of default risks affect the value of credit card loans.

The remainder of this paper is organized as follows. Following on from this introduction, a brief review of the literature on pricing credit card loans is presented in Section 2. This is followed, in Section 3, by the construction of a discrete-time model for pricing credit card loan assets with default risks. Section 4 derives a closed-form solution for the evaluation of credit card loan-related securities within a continuous-time framework. In Section 5, we set up a lattice approach to the evaluation of credit card loan-related products within a discrete-time framework, and, using the numerical examples, we go on, in Section 6, to investigate the ways in which the key parameters in our model affect the value of credit card loans. The conclusions drawn from this study are presented in Section 7.

2. LITERATURE REVIEW

The existing literature on the evaluation of credit card loans includes Ausubel (1991), O'Brien et al. (1994), the Office of Thrift Supervision (1994), Hutchison and Pennacchi (1996) and Jarrow and Deventer (1998). Ausubel (1991) and the Office of Thrift Supervision (1994) used a model which considered deterministic credit card loan growth, rates paid and interest rates, to compute present values; however, such a method trivializes the problem, since interest rate risk and stochastic growth are the two major confounding factors in determining present values. Although the Office of Thrift Supervision (1994) measured the interest rate risk of credit card loan balances by computing their 'duration', it would seem quite obvious that mixing deterministic and stochastic interest rate analyses in such a way can only generate nonsensical results.

O'Brien et al. (1994) computed present values and sensitivity to interest rates in two ways, the first of which followed the method of the Office of Thrift Supervision, whilst the alternative method was to discount the expected value using stochastic credit card loan balances, credit card loan rates and interest rates. In the latter case, since expectations are computed using a Monte Carlo simulation under an assumption of risk neutrality, such expectations represent a present value only if we have risk-neutral investors. Hutchison and Pennacchi (1996) calculated present values using an equilibrium-based model in an economy within which interest rates followed a square root, mean-reverting process, whilst Jarrow and Deventer (1998) provided an arbitrage-free procedure for computing present values in a stochastic interest rate environment using the Heath et al. (1992) methodology.

3 THE MODEL

3.1 The Model Framework

In this paper, we construct a framework for pricing credit card loans with default risks. Utilizing the risk-neutral pricing methodology, we develop an arbitrage-free model for the evaluation of credit card loans; however, it is clear that such loans are very difficult to evaluate because the rates charged differ significantly from the market rates for financial securities of equivalent risk.

In order to best deal with this characteristic, we adopt the market segmentation argument, with the market having numerous providers of credit cards and no major barriers to entry. Such a market structure leads to competitive performance, with prices adjusting to costs, and issuers earning a normal rate of profit. However, we can see that the credit card interest rates, rather than other rates, are inactive, with the largest issuers fixing their rates at between 18 and 20 per cent. According to Calem and Mester (1995), the imperfect competition existing in the credit card industry arises as a result of search costs, switch costs and adverse selection.

The market segmentation hypothesis proposes that there are just two types of traders; banks (or other financial institutions) and individuals, with the partition between these two types being based upon their ability to issue credit cards. We assume that there are significant regulatory restrictions associated with credit card loans, as well as entry (or mobility) barriers, with only banks (or financial institutions), and not individual investors, having the ability to issue credit cards.

3.2 The Evaluation of Credit Card Loans with Default Risks

Consider an economy on a finite time interval $[0, T^*]$, with the periods being taken to be of length $h > 0$. Thus, a typical time point, t, has the form $k * h$ for integer k. At time point t, $t = k * h$, and it is assumed that at all time points t, a full range of default-free zero-coupon bonds are traded, as are a full range of risky zero-coupon bonds. It is also assumed that the markets are free from arbitrage, so that an equivalent martingale measure Q exists.

For any given pair of time points (t, T) with $0 \leq t \leq T \leq T^* - h$, let $L(t)$ denote the volume of credit card loans to a particular bank at time t, and let $c(t)$ denote the interest rate on credit card loans at time t. The cash flow of credit card loans is as shown in Table 1; although this cash flow shows no apparent risks, there are actually default risks.

<Table 1 is inserted here>

Under a risk-neutral measure, the expected risky cash flow, discounted at risk-free rates, must be equal to the value of the expected risk-free cash flow discounted at risky discount rates. Hence, we use the risky discount rate to calculate the net present value of $L(t)$, and let $V_L(t)$ denote the net present value, to the bank, of $L(t)$, at time t.

For any given pair of time points, (t, T), let $f(t, T)$ denote the forward rate on the default-free bonds applicable to the period $(T, T + h)$; let $r(t)$ denote the short rate; and let $r(t) = f(t,t)$. That is to say, $f(t,T)$ is the rate, as viewed from time t, for a default-free lending or investment transaction over the interval, $(T,T+h)$. The forward rate curve is assumed to evolve according to the process:

$$f(t+h,T) = f(t,T) + \alpha(t,T)h + \sigma_f(t,T)X_1\sqrt{h} \tag{1}$$

where $\alpha(t,T)$ and $\sigma_f(t,T)$ represent the respective forward rate drift term and volatility, and X_1 is a random variable.

Let $\varphi(t,T)$ be the forward rate on the risky bonds implied from the spot yield curve, and let $s(t,T)$ be the forward spread on the risky bonds; this is defined as:

$$s(t,T) = \varphi(t,T) - f(t,T) \tag{2}$$

Assume that the forward spread follows the process given in Equation (3).

$$s(t+h,T) = s(t,T) + \beta(t,T)h + \sigma_{spread}(t,T)X_2\sqrt{h} \tag{3}$$

where $\beta(t,T)$ and $\sigma_{spread}(t,T)$ represent the respective forward spread drift term and volatility, and X_2 is a random variable.

Under a risk-neutral measure, the present value of the expected risk-free cash flow must be discounted at risky discount rates, since the cash flow includes default risks. Using $\varphi(t,T)$ as the discount rate, we can obtain the net present value, $V_L(t)$, of credit card loans at time t, from Equation (2).

$V_L(t)$, the net present value to the financial institution, at time t, of the credit card loans, is as follows:[1]

$$V_L(t) = E_t[-L(t) + J(t) * \sum_{k=t/h}^{T/h-1} \frac{L(kh)*\exp\{c(kh)*h\} - L(kh+h)}{J(kh+h)} + \frac{J(t)*L(T)}{J(T)}] \tag{4}$$

1 Refer to Appendix A for the derivation of Equation (4).

where $J(t)$ denotes the time t value of an account which uses an initial investment of \$1 ($J(0) = 1$), and rolls the proceeds over at rate, φ; that is to say:

$$J(t) = \exp\{\sum_{k=0}^{t/h-1} \varphi(kh, kh) * h\} \tag{5}$$

The overall value of the credit card loan assets to the financial institution, at time t, is denoted by $C_L(t)$, a value which is equal to the initial credit card loans plus their net present value. In other words:

$$C_L(t) \equiv L(t) + V_L(t) \tag{6}$$

3.3 Identifying the Risk-Neutral Drifts

In this section, we derive recursive expressions for the drifts, α and β, in the respective forward rate and forward spread processes, in terms of their volatilities, σ_f and σ_{spread}. We first denote $B(t)$ to be the time t value of a money-market account which uses an initial investment of \$1, and rolls the proceeds over at the default-free short rate; that is,

$$B(t) = \exp\{\sum_{k=0}^{t/h-1} r(kh) * h\} \tag{7}$$

Let $Z(t,T)$ denote the price of a default-free bond discounted using $B(t)$. Under Q (the martingale measure), all asset prices in the economy discounted by $B(t)$ will be martingales.

$$Z(t,T) = \frac{P(t,T)}{B(t)} \tag{8}$$

Since Z is a martingale under Q, we can determine that:

$$Z(t,T) = E^t[Z(t+h,T)] \tag{9}$$

otherwise:

$$E^t[\frac{Z(t+h,T)}{Z(t,T)}] = 1 \tag{10}$$

Under this assumption, we can obtain:

$$\sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} \alpha(t,kh) * h^2 = \ln\{E^t[\exp\{-\sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} \sigma_f(t,kh)X_1 * h^{3/2}\}]\} \tag{11}$$

and

$$\sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} [\alpha(t,kh)+\beta(t,kh)]h^2 = \ln\{E^t[\exp\{-h^{3/2} \sum_{k=\frac{t}{h}+1}^{\frac{T}{h}-1} [\sigma_f(t,kh)X_1 + \sigma_{spread}(t,kh)X_2]\}]\} \tag{12}$$

Using the two equations above, we can obtain α and β, in terms of σ_f and σ_{spread}, and under the Heath-Jarrow-Morton term-structure model, we can use the forward rate and forward spread volatilities to describe the forward rate and forward spread drift terms, with this method potentially reducing the inputs and simplifying the whole model.

4. A CONTINUOUS-TIME MODEL

This section considers a continuous-time economy with trading horizon $[0,\tau]$, and begins with a redefinition of the notations of the last section. Let $f(t,T)$ be the instantaneous forward rate at time t for a default-free transaction at time T. The instantaneous forward rate on the risky bonds with maturity T is denoted as $\varphi(t,T)$, with the instantaneous forward spread, $s(t,T)$, on the risky bonds being defined in Equation (2). The forward rate curve and forward spread processes are assumed to follow:

$$df(t,T) = \alpha(t,T)dt + \sigma_f(t,T)dW_r(t), \tag{13}$$

$$ds(t,T) = \beta(t,T)dt + \sigma_s(t,T)dW_s(t), \tag{14}$$

where $\alpha(t,T)$, $\beta(t,T)$ are the drift terms; $\sigma_f(t,T)$, $\sigma_s(t,T)$ are the volatility coefficients; $W_r(t)$, $W_s(t)$ is a two-dimensional Brownian motion with instantaneous correlation ρ; and $-1 \leqq \rho \leqq 1$. In order to evaluate the credit card loan in the continuous-time economy, we rewrite Equation (4) and evaluate it at time 0.[2]

[2] Refer to Appendix B for the derivation of Equation (15).

$$V_L(0) = E_0\left[\sum_{i=0}^{T/h-1}\frac{L(ih)\left[\exp\left(c(ih)h\right)-\exp\left(\varphi(ih,ih)h\right)\right]}{\exp\left(\sum_{j=0}^{i}\varphi(jh,jh)h\right)}\right] \tag{15}$$

By analogy with Equation (15), the net present value of the credit card loan, at time 0, is given by:

$$V_L(0) = E_0\left[\int_0^{\tau}\frac{L(t)\left[\exp\left(c(t)\right)-\exp\left(\varphi(t,t)\right)\right]}{\exp\left(\int_0^t\varphi(u,u)du\right)}dt\right]. \tag{16}$$

In order to obtain a closed-form solution for Equation (16), we follow Jarrow and Deventer (1998) to consider the stochastic process for $L(t)$ and $c(t)$, as follows:

$$d\log L(t) = \left[\alpha_0+\alpha_1 t+\alpha_2 r(t)\right]dt+\alpha_3 dr(t), \tag{17}$$

$$dc(t) = \left[\beta_0+\beta_1 r(t)\right]dt+\beta_2 dr(t). \tag{18}$$

The solutions for the differential Equations of (17) and (18) are presented as:

$$L(t) = L(0)\exp\left[\alpha_0 t+\alpha_1 t^2/2+\alpha_2\int_0^t r(u)du+\alpha_3\left(r(t)-r(0)\right)\right], \tag{19}$$

$$c(t) = c(0)+\beta_0 t+\beta_1\int_0^t r(u)du+\beta_2\left(r(t)-r(0)\right). \tag{20}$$

Substituting Equations (19) and (20) into Equation (16), we obtain

$$V_L(0)=E_0\left(\int_0^{\tau}\exp\left(-\int_0^t\varphi(u,u)du\right)\cdot L(0)\cdot\exp\left[\alpha_0 t+\alpha_1 t^2/2+\alpha_2\int_0^t r(u)du+\alpha_3\left(r(t)-r(0)\right)\right]\cdot\right.$$
$$\left.\left[\exp\left(c(0)+\beta_0 t+\beta_1\int_0^t r(u)du+\beta_2\left(r(t)-r(0)\right)\right)-\exp\left(r(t)+s(t)\right)\right]dt\right).$$

After simplifying the above expression, we can rewrite $V_L(0)$ as follows:

$$V_L(0) = L(0)\cdot\exp\left(c(0)-\left(\alpha_3+\beta_2\right)r(0)\right)\cdot\int_0^{\tau}\exp\left((\alpha_0+\beta_0)t+\alpha_1 t^2/2\right)\cdot$$

$$E_0\left(\exp\left((\alpha_2+\beta_1-1)\cdot\int_0^t r(u)du+(\alpha_3+\beta_2)r(t)-\int_0^t s(u)du\right)\right)dt$$

$$-L(0)\cdot\exp\left(-\alpha_3 r(0)\right)\cdot\int_0^\tau \exp\left(\alpha_0 t+\alpha_1 t^2/2\right)\cdot \tag{21}$$

$$E_0\left(\exp\left((\alpha_2-1)\cdot\int_0^t r(u)du+(\alpha_3+1)r(t)-\int_0^t s(u)du+s(t)\right)\right)dt$$

In order to obtain a closed-form solution for the value of $V_L(0)$, we consider the case of a Gaussian economy within which the process of spot rate and spot spread under a risk-neutral probability measure are as follows:

$$dr(t)=a_r\left[\bar{r}(t)-r(t)\right]dt+\sigma_r d\widetilde{W}_r(t) \tag{22}$$

$$ds(t)=a_s\left[\bar{s}(t)-s(t)\right]dt+\sigma_s d\widetilde{W}_s(t), \tag{23}$$

where $r(t)=f(t,t)$ is the instantaneous spot rate; $s(t)=s(t,t)$ is the instantaneous spot spread; α_r and α_s are constants; σ_r (σ_s) is spot rate (spot spread) volatility; and $\bar{r}(t)$ and $\bar{s}(t)$ are the deterministic functions to fit the initial forward rate curve $\{f(0,T),\ 0\leqq T\leqq\tau\}$ and forward spread curve $\{s(0,T),\ 0\leqq T\leqq\tau\}$. So as to avoid arbitrage and to match the initial curve, $\bar{r}(t)$ and $\bar{s}(t)$ must satisfy the following conditions:

$$\bar{r}(t)=f(0,t)+\left[\partial f(0,t)/\partial t+\sigma_r^2(1-e^{-2a_r t})/2a_r\right]/a_r \tag{24}$$

$$\bar{s}(t)=s(0,t)+\left[\partial s(0,t)/\partial t+\sigma_s^2(1-e^{-2a_s t})/2a_s\right]/a_s. \tag{25}$$

The solutions for Equations (24) and (25) are then obtained as follows:

$$r(t)=f(0,t)+\sigma_r^2(e^{-a_r t}-1)^2/(2a_r^2)+\int_0^t \sigma_r e^{-a_r(t-u)}d\widetilde{W}_r(u) \tag{26}$$

$$s(t)=s(0,t)+\sigma_s^2(e^{-a_s t}-1)^2/(2a_s^2)+\int_0^t \sigma_s e^{-a_s(t-u)}d\widetilde{W}_s(u). \tag{27}$$

$$\text{Let } X \equiv \begin{bmatrix} \int_0^t r(u)du \\ r(t) \\ \int_0^t s(u)du \\ s(t) \end{bmatrix}, \quad \mu \equiv \begin{bmatrix} \mu_1(t) \\ \mu_2(t) \\ \mu_3(t) \\ \mu_4(t) \end{bmatrix}, \quad \Sigma \equiv \begin{bmatrix} \sigma_1^2(t) & \sigma_{12}(t) & \sigma_{13}(t) & \sigma_{14}(t) \\ \sigma_{21}(t) & \sigma_2^2(t) & \sigma_{23}(t) & \sigma_{24}(t) \\ \sigma_{31}(t) & \sigma_{32}(t) & \sigma_3^2(t) & \sigma_{34}(t) \\ \sigma_{41}(t) & \sigma_{42}(t) & \sigma_{43}(t) & \sigma_4^2(t) \end{bmatrix}, \quad \gamma \equiv \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix},$$

where X is a vector of normal random variables with mean, μ, and covariance matrix, Σ; and γ is a vector of constants.

Using Equations (21), (24) and (25), along with the above definitions, we can obtain the closed-form solution for $V_L(0)$, as follows:

$$\begin{aligned} V_L(0) = L(0)\exp\left(c(0) - (\alpha_3 + \beta_2) r(0)\right) \int_0^\tau \exp\left((\alpha_0 + \beta_0)t + \alpha_1 t^2 / 2\right) \cdot \\ M\left(t, \alpha_2 + \beta_1 - 1, \alpha_3 + \beta_2, -1, 0\right) dt - L(0)\exp\left(-\alpha_3 r(0)\right) \int_0^\tau \exp\left(\alpha_0 t + \alpha_1 t^2 / 2\right) \cdot \quad (28) \\ M\left(t, \alpha_2 - 1, \alpha_3 + 1, -1, 1\right) dt. \end{aligned}$$

where $M(t, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \equiv E_0\ (e^{\gamma^T X}) = \exp\ (\gamma^T \mu + \gamma^T \Sigma \gamma)$ is the moment-generating function of the normal random vector, X.[3]

$$\mu_1(t) \equiv \int_0^t f(0,u)du + \int_0^t \left[\sigma_r^2 \left(\exp(-a_r u) - 1 \right)^2 / (2a_r^2) \right] du$$

$$\mu_2(t) \equiv f(0,t) + \sigma_r^2 \left(\exp(-a_r t) - 1 \right)^2 / (2a_r^2)$$

$$\mu_3(t) \equiv \int_0^t s(0,u)du + \int_0^t \left[\sigma_s^2 \left(\exp(-a_s u) - 1 \right)^2 / (2a_s^2) \right] du$$

$$\mu_4(t) \equiv s(0,t) + \sigma_s^2 \left(\exp(-a_s t) - 1 \right)^2 / (2a_s^2)$$

$$\sigma_1^2(t) \equiv \int_0^t \left[\sigma_r^2 \left(1 - \exp(-a_r (t-u)) \right)^2 / a_r^2 \right] du$$

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[3] Refer to Appendix C for the derivation of these integers.

$$\sigma_2^2(t) \equiv \int_0^t \sigma_r^2 \exp\left(-2a_r(t-u)\right) du$$

$$\sigma_3^2(t) \equiv \int_0^t \left[\sigma_s^2 \left(1-\exp\left(-a_s(t-u)\right)\right)^2 / a_s^2 \right] du$$

$$\sigma_4^2(t) \equiv \int_0^t \sigma_s^2 \exp\left(-2a_s(t-u)\right) du$$

$$\sigma_{12}(t) \equiv \left(\sigma_r^2 /(2a_r^2)\right)\left(1-\exp\left(-a_r t\right)^2\right)$$

$$\sigma_{13}(t) \equiv \int_0^t (\sigma_r \sigma_s \rho)\left[1-\exp\left(-a_r(t-u)\right)\right]\left[1-\exp\left(-a_s(t-u)\right)\right]/(a_r a_s) du$$

$$\sigma_{14}(t) \equiv \int_0^t (\sigma_r \sigma_s \rho)\left[1-\exp\left(-a_r(t-u)\right)\right]\exp\left(-a_s(t-u)\right)/ a_r du$$

$$\sigma_{23}(t) \equiv \int_0^t (\sigma_r \sigma_s \rho)\left[1-\exp\left(-a_s(t-u)\right)\right]\exp\left(-a_r(t-u)\right)/ a_s du$$

$$\sigma_{24}(t) \equiv \int_0^t \sigma_r \sigma_s \rho \exp\left(-a_r(t-u)-a_s(t-u)\right) du$$

$$\sigma_{34}(t) \equiv \left(\sigma_s^2 /(2a_s^2)\right)\left(1-\exp\left(-a_s t\right)^2\right)$$

It is easy to show that the closed-form solution derived by Jarrow and Deventer (1998) is a special case in our study when the default risk parameters are set as zero. Hence we contribute to the literature by adding in the components of default risks for credit card loans, which are an important characteristic observed in such loans.

5. NUMERICAL PROCEDURES

5.1 The Process

In this section we adopt a lattice approach to the evaluation of credit card loans, an approach which is easily implemented, and describe the procedures involved in constructing this lattice in the following sub-sections.

5.1.1 Random variables

There are two random variables in the above model, X_1 and X_2; we assume that these are binominal random variables, and that each respectively takes on the value +1 and −1 with probability ½. Let ρ denote the correlation between these two variables and note that ρ may be neither equal to zero, nor constant. It is also assumed that the joint distribution of (X_1, X_2) is:

$$(X_1, X_2) = \begin{Bmatrix} (+1,+1), w.p.(1+\rho)/4 \\ (+1,-1), w.p.(1-\rho)/4 \\ (-1,+1), w.p.(1-\rho)/4 \\ (-1,-1), w.p.(1+\rho)/4 \end{Bmatrix} \tag{29}$$

5.1.2 Forward rate term structure and volatility

A forward rate term may be one of three types, downward sloping, upward sloping or flat, with a number of different theories having previously been proposed. The simplest is the 'expectations theory', which suggests that long-term interest rates should reflect expected future short-term interest rates, whilst 'segmentation theory' argues that there need be no relationship between short-term, medium-term and long-term interest rates, since short-term interest rates are determined by supply and demand in the short-term market, medium-term interest rates are determined by supply and demand in the medium-term market, and so on. Based upon the underlying assumption that investors prefer to preserve their liquidity, 'liquidity preference theory' argues that long-term interest rates should always be higher than short-term interest rates. This leads to a situation in which the shape of the curve is upward sloping. For implementation reasons, we assume that forward rate volatility is of the form:

$$\sigma_f(t,T) = \sigma * \exp\{-\lambda(T-t)\} \tag{30}$$

where $\sigma > 0$ is a positive constant and $\lambda \geq 0$ is a non-negative constant. This structure, which is obtained by permitting volatility to depend upon the maturity of the forward rate, $(T-t)$ is a more realistic volatility structure for forward rates. If $\lambda = 0$, then forward rate volatility is constant, $\sigma_f(t,T) = \sigma$. If $\lambda > 0$, then this implies that with a decrease in maturity $(T-t)$ there is an increase in forward rate volatility. This exponentially-dampened volatility structure exploits the fact that near-term forward rates are more volatile than distant forward rates.

5.1.3 Forward spread term structure and volatility

According to the experiments undertaken by Zhou (2001), the term structure of credit spreads can generate various shapes, including upward-sloping, downward-sloping, flat and hump-shaped. Thus, we can set the form of the forward spread term structure to be of different types. We also assume, for the purpose of implementation, that forward spread volatility is of the form:

$$\sigma_{spread}(t,T) = \sigma_s * \exp\{-\lambda_s (T-t)\} \tag{31}$$

where $\sigma_s > 0$ is a positive constant.

The forward spread term structure is obtained by permitting volatility to depend upon the maturity of the forward spread $(T-t)$. If $\lambda_s = 0$, then forward spread volatility is constant, $\sigma_{spread}(t,T) = \sigma_s$. If $\lambda_s > 0$, then this implies that with a decrease in maturity $(T-t)$, there is an increase in forward spread volatility. Conversely, if $\lambda_s < 0$, then this implies that with a decrease in maturity $(T-t)$, there is a corresponding decrease in forward spread volatility. That is to say, there are three possible shapes of forward spread volatility.

5.2 The Implementation of our Model

We need data on forward rate and forward spread in order to implement the model; based upon the above assumptions, this may be easily implemented on a lattice. The double-binomial structure, as described above, results in a branching process with four branches emanating from each node. The branching lattice is illustrated in Figure 1.

<Figure 1 is inserted here>

We achieve the risk-neutral drifts, α and β, by forward rate volatility and forward spread volatility. Once the risk-neutral drifts have been computed, the possible value of forward rates and forward spreads are obtained for one period out. The process is as follows.

Let F_u and F_d respectively refer to the forward rates resulting from F, if X_1 equals $+1$ and -1, and let S_u and S_d respectively refer to the forward spreads resulting from S, if X_2 equals $+1$ and -1. The probability of each branching, which is shown in Equation (13), is dependent upon the joint distribution of (X_1, X_2).

6. NUMERICAL RESULTS

6.1 Demonstration of the Model

In order to demonstrate our model, we implement a simple example. Consider an economy on a finite time interval [0,2], with periods taken to be of length h, and $h = 0.5$ (half-year). Details of the cash flow of credit card loans, for five periods, are presented in Table 2, with the process subsequently being described below.

<Table 2 is inserted here>

According to the whole model described above, some inputs are required for implementation. We set the volume of credit card loans, $L(t)$, as an increasing volume with a fixed rate, g; the initial credit card loan amounts to NT$100 billion; and the half-year growth rate, g, as 5 per cent. According to Calem and Mester (1995), we can determine that credit card interest rates are sticky; thus, at time t, we set the credit card interest rate, $c(t)$, as being constantly equal to 19 per cent.

We can determine the risk-neutral drifts, α and β, by forward rate volatility and forward spread volatility. Once the risk neutral drifts have been computed, the possible values of the forward rates and forward spreads are obtained for one period out. The first of these, the forward rate volatility term structure, is as shown in Table 3.

<Table 3 is inserted here>

Assuming that forward rate volatility, σ_f, follows Equation (14), then volatility, σ, is equal to 2 per cent, and the volatility reduction factor, λ, is equal to 0.1. That is to say, forward rate volatility is:

$$\sigma_f(t,T) = 0.02 * \exp\{-0.1(T-t)\} \tag{32}$$

We also set forward spread volatility, σ_{spread}, with the same form, following Equation (15). The volatility, σ_s, is equal to 2 per cent and the volatility reduction factor, λ_s, is equal to 0.1. That is to say, forward spread volatility is:

$$\sigma_{spread}(t,T) = 0.02 * \exp\{-0.1(T-t)\} \tag{33}$$

The forward spread volatility term structure is as shown in Table 4.

<Table 4 is inserted here>

Using Mathematica (Wolfram 1988), the forward rate and forward spread term structures are as shown in Table 5.

<Table 5 is inserted here>

In order to make use of the data in Table 5, and the forward rate and forward spread volatility term structures, we achieve the double-binomial structure results in a branching lattice with four branches emanating from each node. We now see that, at time 0, the net present value of credit card loans, $V_L(0)$, is NT$23.072 billion, whilst the value of credit card loan assets to the financial institution, $C_L(0)$, is NT$123.072 billion.

Next, by performing a sensitivity analysis, we further discuss the major factors of forward rate, forward spread, forward rate volatility and forward spread volatility, within this model, and the ways in which they affect the values of $V_L(0)$ and $C_L(0)$.

6.2 The Effects of Changes in the Forward Rate Term Structure

The forward rate term structure may be of different shapes, including downward sloping, flat and upward sloping (Figure 2); we calculate the net present value, at time 0, of credit card loans, $V_L(0)$, along with the value of credit card loan assets, $C_L(0)$, under these three types. The results are presented in Table 6.

<Figure 2 is inserted here>

<Table 6 is inserted here>

We can clearly see that $C_L(0)$ has the smallest value when the forward rate term structure is upward sloping; conversely, $C_L(0)$ has the largest value when the forward rate term structure is downward sloping. The credit card interest rate has already been set as fixed and equal to 19 per cent; therefore, if the forward rate term structure is upward sloping, this indicates that the capital cost is greater in the future than in the present. Under this term structure, when the earnings rate is fixed, $C_L(0)$ will achieve the minimum value of these three situations.

6.3 The Effects of Changes in the Forward Spread Term Structure

According to the experiments undertaken by Zhou (2001), the credit spread term structure may generate various shapes. We discuss here the effects of three basic types of forward spread on the net present value, at time 0, of credit card loans, $V_L(0)$, and on the value of credit card loan assets, $C_L(0)$, assuming that the forward spread term structures are as shown in Figure 3. The results are presented in Table 7.

<Figure 3 is inserted here>

<Table 7 is inserted here>

The spread determines the default risks in credit card loans, and as we can clearly see, $C_L(0)$ has the smallest value when the forward spread term structure is upward sloping; conversely, $C_L(0)$ has the largest value when the forward spread term structure is downward sloping. If the forward spread term structure is upward sloping, this indicates that the default risks are greater in the future than in the present; thus, more premiums are required to compensate for the risks. However, given a situation in which the earning rate is fixed, $C_L(0)$ will have the smallest value.

6.4 The Effects of Changes in Forward Rate Volatility

As noted in section 3, we assume that forward rate volatility abides by Equation (14). According to the empirical performance of the single factor proposed by Flesaker (1993) (the constant volatility version of the interest rate contingent claims valuation model of Heath, Jarrow and Morton (1992), which used a generalized method of moments (GMM) and which tested three-year daily data for Eurodollar futures and futures options), the results show that the estimated volatility is extremely close to 0.02. The minimum value is 0.196846, the maximum value is 0.202845, and the mean value is 0.0200070. Following the results of the empirical performance, we initially set forward rate volatility, σ, as being equal to 2 per cent, and the volatility reduction factor, λ, as being equal to 0.1. Next, we examine the effects of changes in forward rate volatility, and discuss this under two topics, the effects of volatility, and the effects of the volatility reduction factor.

6.4.1 The effects of volatility

We set volatility, σ, as being equal to 2 per cent, and subsequently change the value to examine the effects, at time 0, on the net present value of credit card loans, $V_L(0)$, and on the value of credit card loan assets, $C_L(0)$. The results are presented in Table 8.

<Table 8 is inserted here>

As shown in Figure 4, we can determine that with an increase in volatility, σ, there is a rapid decrease, with maturity, in the forward rate volatility term structure, σ_f, and a corresponding reduction in the value of $C_L(0)$.

<Figure 4 is inserted here>

This exponentially-dampened volatility structure exploits the fact that near-term forward rates are more volatile than distant forward rates. An increase in volatility indicates that the forward rate is more uncertain, thereby reducing the value of $C_L(0)$.

6.4.2 The effects of the volatility reduction factor

We also change the value of the forward rate volatility reduction factor and discuss the results; these are presented in Table 9.

<Table 9 is inserted here>

We find that with an increase in the volatility reduction factor, λ , there is a corresponding decrease in the slope of the forward rate volatility term structure, making forward rate volatility more stable, and also increasing the value of $C_L(0)$.

6.5 The Effects of Changes in Forward Spread Volatility

For the purpose of implementation, we assume that the forward spread volatility term structure abides by Equation (15), and discuss the effects of the volatility reduction factor, λ_s , under settings of positive, zero and negative values (Figure 5). The results are presented in Table 10.

<Figure 5 is inserted here>

<Table 10 is inserted here>

We find that if $\lambda_s = 0$, then the forward spread volatility term structure is flat; that is to say, $\sigma_{spread}(t, T) = \sigma_s$. If $\lambda_s > 0$, this implies that with an increase in maturity, $(T - t)$, there is a corresponding decrease in forward spread volatility. Conversely, if $\lambda_s < 0$, this implies that with an increase in maturity $(T-t)$, there is also an increase in forward spread volatility. When $\lambda_s < 0$, forward spread volatility is greater, leading to $C_L(0)$ having the smallest value.

6.5.1 The effects of volatility

We further discuss the effects of forward spread volatility under three types of λ_s. Firstly, if $\lambda_s > 0$, this implies that forward spread volatility decreases with an increase in maturity, $(T-t)$. We change the volatility, σ_s, and examine the effects, at time 0, on the net present value of credit card loans, $V_L(0)$, and on the value of credit card loan assets, $C_L(0)$. We find that with an increase in volatility, σ_s, there is a rapid decrease, with maturity, in the forward spread volatility term structure, σ_{spread} , and a reduction in the value of $C_L(0)$. This exponentially-dampened volatility structure exploits the fact that near-term forward spreads are more volatile than distant forward spreads. An increase in volatility indicates that the forward spread is more uncertain, thereby reducing the value of $C_L(0)$. The results are presented in Table 11.

<Table 11 is inserted here>

Secondly, if $\lambda_s = 0$, this indicates that the forward spread volatility term structure is flat; that is to say, $\sigma_{spread}(t, T) = \sigma_s$. We change the volatility, σ_s, and examine the effects, at time 0, on the net present value of credit card loans, $V_L(0)$, and on the value of credit card loan assets, $C_L(0)$. We find that with an increase in volatility, σ_s , the forward spread volatility term structure, σ_{spread}, remains flat, but that there is a corresponding reduction in the value of $C_L(0)$. This volatility structure exploits the fact that near-term and distant forward spread volatilities are the same. An increase in volatility indicates that the forward spread is more uncertain, thereby reducing the value of $C_L(0)$. The results are presented in Table 12.

<Table 12 is inserted here>

Thirdly, if $\lambda_s < 0$, then this implies that with an increase in maturity, $(T-t)$, there is a corresponding increase in forward spread volatility. We change the volatility, σ_s, and examine the effects, at time 0, on the net present value of credit card loans, $V_L(0)$, and on the value of credit card loan assets, $C_L(0)$. We find that with an increase in volatility, σ_s, there is a rapid increase, with maturity, in the forward spread volatility term structure, σ_{spread}, and a corresponding reduction in the value of $C_L(0)$. This exponentially-upward volatility structure exploits the fact that near-term forward spreads are more stable than distant forward spreads. An increase in volatility indicates that the forward spread is more uncertain, thereby reducing the value of $C_L(0)$. The results are presented in Table 13.

<Table 13 is inserted here>

We then change volatility, σ_s, under the three types of forward spread volatility term structure. Comparing Tables 11, 12 and 13, we find that when $\lambda_s < 0$, the value of $C_L(0)$ changes more than other values, indicating that if the slope of the forward spread volatility term structure is upward, then changes in volatility will affect the value of $C_L(0)$ more than other values.

6.5.2 The effects of the volatility reduction factor

First of all, under a situation of $\lambda_s > 0$, we change the volatility reduction factor and examine the effects, at time 0, on the net present value of credit card loans, $V_L(0)$, and on the value of credit card loan assets, $C_L(0)$ (Figure 6). We find that with an increase in the volatility reduction factor, λ_s, there is a decrease in the slope of the forward spread volatility term structure, making forward spread volatility more stable, and thereby increasing the value of $C_L(0)$. The results are presented in Table 14.

<Figure 6 is inserted here>

<Table 14 is inserted here>

Secondly, under a situation of $\lambda_s < 0$, we change the volatility reduction factor and examine the effects, at time 0, on the net present value of credit card loans, $V_L(0)$, and on the value of credit card loan assets, $C_L(0)$ (Figure 7). We find that with an increase in the volatility reduction factor, λ_s, there is a corresponding increase in the slope of the forward spread volatility term structure, making forward spread volatility more volatile, and also reducing the value of $C_L(0)$. The results are presented in Table 15.

<Figure 7 is inserted here>

<Table 15 is inserted here>

In summarizing this section, we should point out that the application of our pricing model to the evaluation of credit card asset-backed securities with default risk is quite straightforward. The findings on the ways in which the parameters of default risks affect the value of different tranches in credit card asset-backed securities are quite similar to those for credit card loans; therefore, for the purpose of conciseness, we do not report the results for pricing credit card asset-backed securities.

7. CONCLUSIONS

Using risk-neutral pricing methodology, we present an arbitrage-free model for the evaluation of credit card loans and credit card loan ABS. The model is based upon an extension of the Heath-Jarrow-Morton (1990) term-structure model, in order to facilitate the consideration of default risks. Since there is imperfect competition in the credit card industry, with constantly sticky interest rates, we therefore use the market segmentation argument to describe this characteristic.

Our model has three distinctive features. First of all, it takes existing spreads as an input into the model, rather than deriving the model from implications on default probabilities and recovery rates. Secondly, rather than working with spot yield curves for default-free and risky debt, we work with 'forward rates' and 'forward spreads'. The specific advantage of using forward rates is that since the current term structure is an input into the model, forward rates can describe short rates, whereas short rates cannot describe forward rates. Thirdly, on the one hand, we provide a closed-form solution for the evaluation of credit card loans and their securitization products within a continuous-time framework; on the other hand, we offer an applicable lattice approach for evaluating credit card loan-related products within a discrete-time framework.

We find that the shapes of the forward rate and forward spread (default risk premium) term structures play extremely important roles in determining the value of credit card loans.

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Table 1 Cash-flow of credit card loans

t	$t+h$	$t+2h$	$T-h$	T
$-L(t)$	$+L(t)$*exp $\{c(t)\}$ $-L(t+h)$	$+L(t+h)$*exp $\{c(t+h)\}$ $-L(t+2h)$	$+L(T-2h)$ *exp $\{c(T-2h)\}$ $-L(T-h)$	$+L(T-h)$ *exp $\{c(T-h)\}$

Table 2 Cash-flow of credit card loans for five periods

0	0.5	1.0	1.5	2.0
$-L(0)$	$+L(0)$*exp $\{c(0)\}$ $-L(0.5)$	$+L(0.5)$*exp {c(0.5)} $-L(1)$	$+L(1)$*exp $\{c(1)\}$ $-L(1.5)$	$+L(1.5)$*exp $\{c(1.5)\}$

Table 3 Forward rate volatility term structure

$(T-t)$	0	0.5	1.0	1.5	2.0
$\sigma_f\ (T-t)$	0.0200	0.0190246	0.0180967	0.0172142	0.0163746

Table 4 Forward spread volatility term structure

$(T-t)$	0	0.5	1.0	1.5	2.0
$\sigma_{spread}\ (T-t)$	0.0200	0.0190246	0.0180967	0.0172142	0.0163746

Table 5 Forward rate and forward spread term structures

Period	T	$(0, T)$	$f(0, T)$	$s(0, T)$
0	0	(0, 0)	0.05	0.008
1	0.5	(0, 0.5)	0.06	0.010
2	1.0	(0, 1.0)	0.07	0.015
3	1.5	(0, 1.5)	0.08	0.020
4	2.0	(0, 2.0)	0.09	0.022
ρ	-0.074			

Table 6 The value of $V_L(0)$ and $C_L(0)$ under three types of forward rate term structures

Forward rate term structure	Upward sloping	Flat	Downward sloping
$f(0,0)$	0.05	0.05	0.05
$f(0,0.5)$	0.06	0.05	0.04
$f(0,1.0)$	0.07	0.05	0.03
$f(0,1.5)$	0.08	0.05	0.02
$V_L(0)$	23071.80641	25891.17073	29808.70709
$C_L(0)$	123071.80641	125891.17073	129808.70709
% Change	–	2.29	5.47

Table 7 The value of $V_L(0)$ and $C_L(0)$ under three types of forward spread term structures

Forward spread term structure	Upward sloping	Flat	Downward sloping
$s(0,0)$	0.008	0.008	0.008
$s(0,0.5)$	0.010	0.008	0.006
$s(0,1.0)$	0.015	0.008	0.003
$s(0,1.5)$	0.020	0.008	0.001
$V_L(0)$	23071.80641	23748.61399	24094.39675
$C_L(0)$	123071.80641	123748.61399	124094.39675
% Change	–	0.55	0.83

Table 8 The value of $V_L(0)$ and $C_L(0)$ under different forward rate volatility levels

$\sigma_f\ (T-t)$	σ = 1%	σ = 2%	σ = 3%	σ = 4%	σ = 5%
0	0.01	0.0200	0.03	0.04	0.05
0.5	0.009512	0.019025	0.028537	0.038049	0.047562
1.0	0.009048	0.018097	0.027145	0.036194	0.045242
1.5	0.008607	0.017214	0.025821	0.034428	0.043035
2.0	0.008187	0.016375	0.024562	0.032749	0.040937
$V_L(0)$	23072.5635	23071.80641	23070.6711	23068.8847	22958.45694
$C_L(0)$	123072.56335	123071.80641	123070.6711	123068.8847	122958.45694
% Change	0.00615	–	-0.00923	-0.02374	-0.92100

Table 9 The value of $V_L(0)$ and $C_L(0)$ with different forward rate volatility reduction factor levels

$\sigma_f\ (T-t)$	$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.3$	$\lambda = 0.5$
0	0.0200	0.0200	0.0200	0.0200
0.5	0.019506	0.0190246	0.017214	0.015576
1.0	0.019025	0.0180967	0.014816	0.012131
1.5	0.018554	0.0172142	0.012753	0.009447
2.0	0.018097	0.0163746	0.010976	0.007358
$V_L(0)$	23071.71402	23071.80641	23072.18038	23072.36517
$C_L(0)$	123071.71402	123071.80641	123072.18038	123072.36517
% Change	0.00075	–	0.00304	0.00454

Table 10 The value of $V_L(0)$ and $C_L(0)$ under three types of forward spread volatility reduction factors

$\sigma_{spread}\ (T-t)$	$\lambda_S = 0.1$	$\lambda_S = 0$	$\lambda_S = -0.1$
0	0.0200	0.0200	0.0200
0.5	0.0190246	0.0200	0.0210254
1.0	0.0180967	0.0200	0.0221034
1.5	0.0172142	0.0200	0.0232367
2.0	0.0163746	0.0200	0.0244281
$V_L(0)$	23071. 80641	22972.86488	22855.19066
$C_L(0)$	123071. 80641	122972.86488	122855.19066
% Change	0.0805	–	–0.0957

Table 11 The value of $V_L(0)$ and $C_L(0)$ with different forward spread volatility levels under $\lambda_S > 0$

$\sigma_{spread}\ (T-t)$	$\sigma_S = 1\%$	$\sigma_S = 2\%$	$\sigma_S = 3\%$	$\sigma_S = 4\%$	$\sigma_S = 5\%$
0	0.01	0.0200	0.03	0.04	0.05
0.5	0.009512	0.019025	0.028537	0.038049	0.047562
1.0	0.009048	0.018097	0.027145	0.036194	0.045242
1.5	0.008607	0.017214	0.025821	0.034428	0.043035
2.0	0.008187	0.016375	0.024562	0.032749	0.040937
$V_L(0)$	23493.1088	23071.80641	22406.56576	21607.65253	20797.23996
$C_L(0)$	123493.1088	123071.80641	122406.56576	121607.65253	120797.23996
% Change	0.3423	–	-0.5405	-1.1897	-1.8482

Table 12 The value of $V_L(0)$ and $C_L(0)$ with different forward spread volatility levels under $\lambda_S = 0$

$\sigma_{spread}\ (T-t)$	$\sigma_S = 1\%$	$\sigma_S = 2\%$	$\sigma_S = 3\%$	$\sigma_S = 4\%$	$\sigma_S = 5\%$
0	0.01	0.02	0.03	0.04	0.05
0.5	0.01	0.02	0.03	0.04	0.05
1.0	0.01	0.02	0.03	0.04	0.05
1.5	0.01	0.02	0.03	0.04	0.05
2.0	0.01	0.02	0.03	0.04	0.05
$V_L(0)$	23484.8248	22972.86488	22213.80386	21329.84318	20452.46576
$C_L(0)$	123484.8248	122972.86488	122213.80386	121329.84318	120452.46576
% Change	0.41632	–	-0.61726	-1.33608	-2.04956

Table 13 The value of $V_L(0)$ and $C_L(0)$ with different forward spread volatility levels under $\lambda_S < 0$

$\sigma_{spread}\ (T-t)$	$\sigma_S = 1\%$	$\sigma_S = 2\%$	$\sigma_S = 3\%$	$\sigma_S = 4\%$	$\sigma_S = 5\%$
0	0.01	0.0200	0.03	0.04	0.05
0.5	0.010513	0.021025	0.031538	0.042051	0.052564
1.0	0.011052	0.022103	0.033155	0.044207	0.055259
1.5	0.011618	0.023237	0.034855	0.046473	0.058092
2.0	0.012214	0.024428	0.036642	0.048856	0.061070
$V_L(0)$	23465.8728	22855.19066	21974.86713	21013.82813	20060.53263
$C_L(0)$	123465.8728	122855.19066	121974.86713	121013.82813	120060.53263
% Change	0.49707	–	-0.71656	-1.49881	-2.27476

Table 14 The value of $V_L(0)$ and $C_L(0)$ with different forward spread volatility reduction factors under $\lambda_S > 0$

$\sigma_{spread}\ (T-t)$	$\lambda_S = 0.05$	$\lambda_S = 0.1$	$\lambda_S = 0.3$	$\lambda_S = 0.5$
0	0.0200	0.0200	0.0200	0.0200
0.5	0.019506	0.019025	0.017214	0.015576
1.0	0.019025	0.018097	0.014816	0.012131
1.5	0.018554	0.017214	0.012753	0.009447
2.0	0.018097	0.016375	0.010976	0.007358
$V_L(0)$	23023.53637	23071.80641	23240.47368	23354.93278
$C_L(0)$	123023.53637	123071.80641	123240.47368	123354.93278
% Change	-0.03922	–	0.13705	0.23005

Table 15 The value of $V_L(0)$ and $C_L(0)$ with different forward spread volatility reduction factors under $\lambda_S < 0$

$\sigma_{spread}\ (T-t)$	$\lambda_S = 0.05$	$\lambda_S = 0.1$	$\lambda_S = 0.3$	$\lambda_S = 0.5$
0	0.0200	0.0200	0.0200	0.0200
0.5	0.020506	0.021025	0.023237	0.025681
1.0	0.021025	0.022103	0.026997	0.032974
1.5	0.021558	0.023237	0.031366	0.042340
2.0	0.022103	0.024428	0.036442	0.054366
$V_L(0)$	22919.78508	22855.19066	22524.63045	22052.6728
$C_L(0)$	122919.78508	122855.19066	122524.63045	122052.6728
% Change	0.05258	–	-0.26906	-0.65326

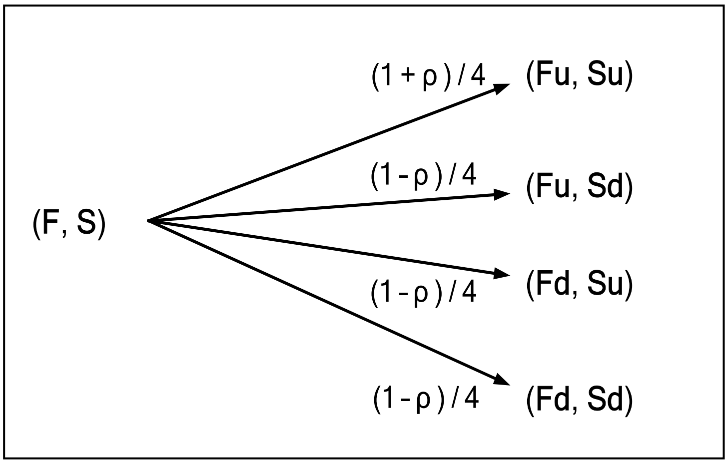

Figure 1 The branching lattice

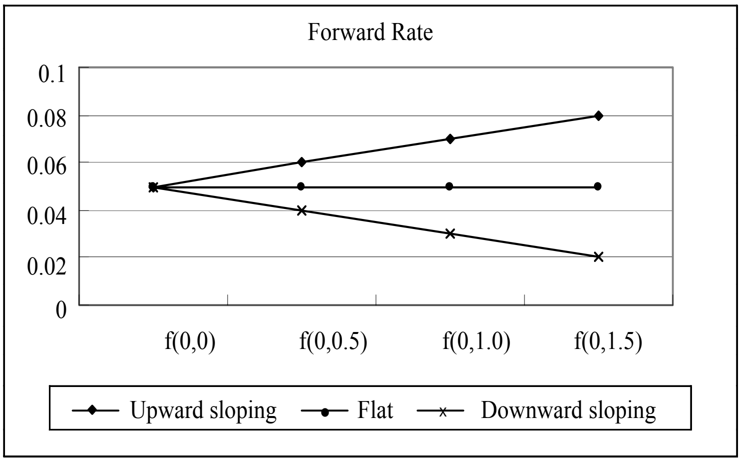

Figure 2 Three types of forward rate term structure

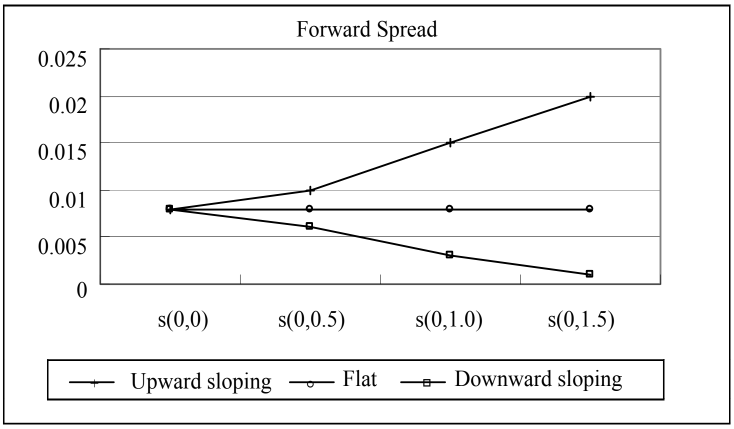

Figure 3 Three types of forward spread term structure

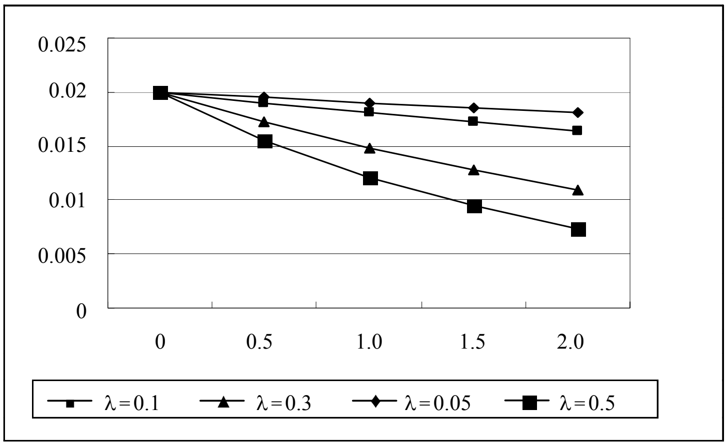

Figure 4 Forward rate volatility term structure with different volatility reduction factors

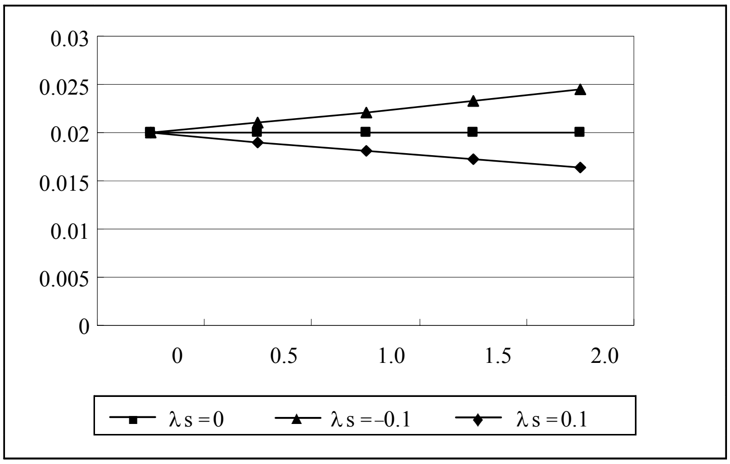

Figure 5 Forward spread volatility term structure with different volatility reduction factors

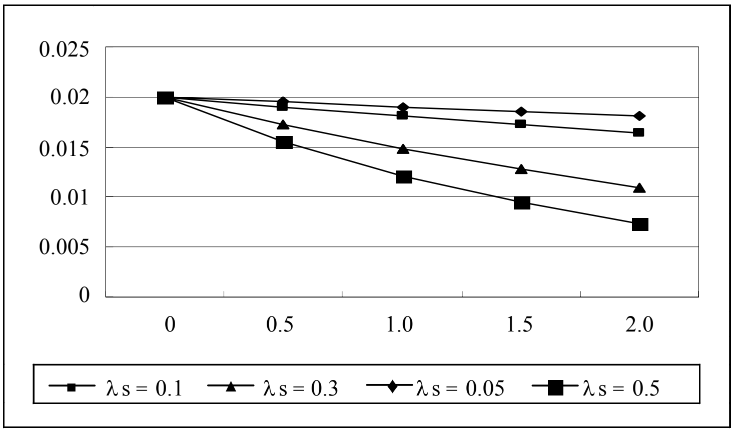

Figure 6 Forward spread volatility term structure with different volatility reduction factors under $\lambda_s > 0$

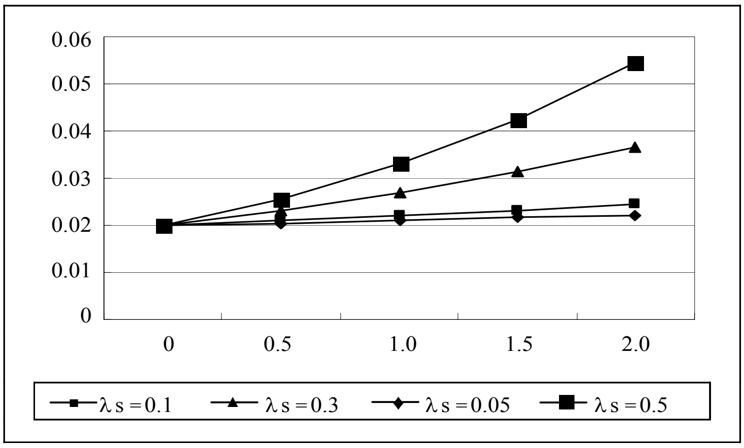

Figure 7 Forward spread volatility term structure with different volatility reduction factors under $\lambda_s < 0$

Appendix A

The Derivation of Equation (4)

$$V_L(t) = E_t[-L(t) + \frac{1}{\exp\{\varphi(t,t)*h\}}[L(t)*\exp\{c(t)*h\} - L(t+h)]$$

$$+\frac{1}{\exp\{[\varphi(t,t)+\varphi(t+h,t+h)]*h\}}[L(t+h)*\exp\{c(t+h)*h\} - L(t+2h)]$$

$$+\frac{1}{\exp\{[\varphi(t,t)+\varphi(t+h,t+h)+\varphi(t+2h,t+2h)]*h\}}[L(t+2h)*\exp\{c(t+2h)*h\} - L(t+3h)]$$

$$+\cdots\cdots+\cdots\cdots+\cdots\cdots+\frac{1}{\exp\{\sum_{i=\frac{t}{h}}^{\frac{T}{h}-2}\varphi(ih,ih)*h\}}[L(T-2h)*\exp\{c(T-2h)*h\} - L(T-h)]$$

$$+\frac{1}{\exp\{\sum_{i=\frac{t}{h}}^{\frac{T}{h}-1}\varphi(ih,ih)*h\}}[L(T-h)*\exp\{c(T-h)*h\}]]$$

$$= E_t[-L(t) + \frac{J(t)}{J(t+h)}[L(t)*\exp\{c(t)*h\} - L(t+h)] + \frac{J(t)}{J(t+2h)}[L(t+h)*\exp\{c(t+h)*h\} - L(t+2h)]$$

$$+\frac{J(t)}{J(t+3h)}[L(t+2h)*\exp\{c(t+2h)*h\} - L(t+3h)] + \cdots\cdots$$

$$+\frac{J(t)}{J(T-h)}[L(T-2h)*\exp\{c(T-2h)*h\} - L(T-h)] + \frac{J(t)}{J(T)}[L(T-h)*\exp\{c(T-h)*h\}]]$$

$$= E_t[-L(t)$$

$$+[\frac{J(t)}{J(t+h)}*L(t)*\exp\{c(t)*h\} + \frac{J(t)}{J(t+2h)}*L(t+h)*\exp\{c(t+h)*h\} + \frac{J(t)}{J(t+3h)}*L(t+2h)*\exp\{c(t+2h)*h\}$$

$$+\cdots\cdots+\frac{J(t)}{J(T-h)}*L(T-2h)*\exp\{c(T-2h)*h\} + \frac{J(t)}{J(T)}*L(T-h)*\exp\{c(T-h)*h\}]$$

$$-[\frac{J(t)}{J(t+h)}*L(t+h) + \frac{J(t)}{J(t+2h)}*L(t+2h) + \frac{J(t)}{J(t+3h)}*L(t+3h) + \cdots\cdots + \frac{J(t)}{J(T-h)}*L(T-h)]]$$

$$= E_t[-L(t) + J(t)[\sum_{k=t/h}^{T/h-1}\frac{L(kh)*\exp\{c(kh)*h\}}{J(kh+h)} - \sum_{k=t/h}^{T/h-2}\frac{L(kh+h)}{J(kh+h)}]]$$

$$= E_t[-L(t) + J(t)[\sum_{k=t/h}^{T/h-1}\frac{L(kh)*\exp\{c(kh)*h\} - L(kh+h)}{J(kh+h)} + \frac{L(T)}{J(T)}]]$$

$$= E_t[-L(t) + J(t)*\sum_{k=t/h}^{T/h-1}\frac{L(kh)*\exp\{c(kh)*h\} - L(kh+h)}{J(kh+h)} + \frac{J(t)*L(T)}{J(T)}]$$

Appendix B

The Derivation of Equation (15)

From Equation (4), the net present value of the credit card loan can be expressed as:

$$V_L(0) = E_0\left[-L(0) + \sum_{i=0}^{T/h-2} \frac{L(ih)\exp\left(c(ih)h\right) - L((i+1)h)}{\exp\left(\sum_{j=0}^{i}\varphi(jh, jh)h\right)} + \frac{L(T-h)\exp\left(c(T-h)h\right)}{\exp\left(\sum_{j=0}^{T/h-1}\varphi(jh, jh)h\right)}\right]$$

$$= E_0\left[\sum_{i=0}^{T/h-2} \frac{L(ih)\exp\left(c(ih)h\right)}{\exp\left(\sum_{j=0}^{i}\varphi(jh, jh)h\right)} + \frac{L(T-h)\exp\left(c(T-h)h\right)}{\exp\left(\sum_{j=0}^{T/h-1}\varphi(jh, jh)h\right)}\right.$$

$$\left.- \frac{L(0)\exp\left(\varphi(0,0)h\right)}{\exp\left(\varphi(0,0)h\right)} - \sum_{i=0}^{T/h-2} \frac{L((i+1)h)\exp\left(\varphi\left((i+1)h,(i+1)h\right)h\right)}{\exp\left(\sum_{j=0}^{i}\varphi(jh, jh)h\right)\exp\left(\varphi\left((i+1)h,(i+1)h\right)h\right)}\right]$$

$$= E_0\left[\sum_{i=0}^{T/h-1} \frac{L(ih)\left[\exp\left(c(ih)h\right) - \exp\left(\varphi(ih,ih)h\right)\right]}{\exp\left(\sum_{j=0}^{i}\varphi(jh, jh)h\right)}\right].$$

Appendix C

The Derivation of Expression (28)

From Equation (26), we have:

$$r(t) = f(0,t) + \sigma_r^2 (e^{-a_r t} - 1)^2 / (2a_r^2) + \int_0^t \sigma_r e^{-a_r(t-u)} d\widetilde{W}_r(u), \tag{C.1}$$

We can then write:

$$\begin{aligned} \int_0^t r(u)du &= \int_0^t f(0,u)du + \int_0^t \sigma_r^2 (e^{-a_r u} - 1)^2 / (2a_r^2) du + \int_0^t \int_0^v \sigma_r e^{-a_r(v-u)} d\widetilde{W}_r(u) dv \\ &= \int_0^t f(0,u)du + \int_0^t \sigma_r^2 (e^{-a_r u} - 1)^2 / (2a_r^2) du + \int_0^t \int_u^t \sigma_r e^{-a_r(v-u)} dv d\widetilde{W}_r(u) \qquad \text{(C.2)} \\ &= \int_0^t f(0,u)du + \int_0^t \sigma_r^2 (e^{-a_r u} - 1)^2 / (2a_r^2) du + \int_0^t (\sigma_r / a_r) \left[1 - e^{-a_r(t-u)}\right] d\widetilde{W}_r(u). \end{aligned}$$

In a similar way, we can restate Equation (27) as:

$$s(t) = s(0,t) + \sigma_s^2 (e^{-a_s t} - 1)^2 / (2a_s^2) + \int_0^t \sigma_s e^{-a_s(t-u)} d\widetilde{W}_s(u), \tag{C.3}$$

Hence we obtain

$$\int_0^t s(u)du = \int_0^t f(0,u)du + \int_0^t \sigma_s^2 (e^{-a_s u} - 1)^2 / (2a_s^2) du + \int_0^t (\sigma_s / a_s) \left[1 - e^{-a_s(t-u)}\right] d\widetilde{W}_s(u). \tag{C.4}$$

Equations (C.1) to (C.4) are four normal random variables with the following respective means, variances and covariances.

$$\mu_1(t) \equiv E\left(\int_0^t r(u)du\right) = \int_0^t f(0,u)du + \int_0^t \left[\sigma_r^2 \left(\exp(-a_r u) - 1\right)^2 / (2a_r^2)\right] du$$

$$\mu_2(t) \equiv E\left(r(t)\right) = f(0,t) + \sigma_r^2 \left(\exp(-a_r t) - 1\right)^2 / (2a_r^2)$$

$$\mu_3(t) \equiv E\left(\int_0^t s(u)du\right) = \int_0^t s(0,u)du + \int_0^t \left[\sigma_s^2 \left(\exp(-a_s u) - 1\right)^2 / (2a_s^2)\right] du$$

$$\mu_4(t) \equiv E\left(s(t)\right) = s(0,t) + \sigma_s^2 \left(\exp(-a_s t) - 1\right)^2 / (2a_s^2)$$

$$\sigma_1^2(t) \equiv Var\left(\int_0^t r(u)du\right)$$

$$= Var\left(\int_0^t (\sigma_r / a_r)\left[1-\exp\left(-a_r(t-u)\right)\right]d\widetilde{W}_r(u)\right)$$

$$= \int_0^t \left[\sigma_r^2 \left(1-\exp\left(-a_r(t-u)\right)\right)^2 / a_r^2\right]du$$

$$\sigma_2^2(t) \equiv Var\left(r(t)\right)$$

$$= Var\left(\int_0^t \sigma_r \left(-a_r(t-u)\right)d\widetilde{W}_r(u)\right)$$

$$= \int_0^t \sigma_r^2 \exp\left(-2a_r(t-u)\right)du$$

$$\sigma_3^2(t) \equiv Var\left(\int_0^t s(u)du\right)$$

$$= Var\left(\int_0^t (\sigma_s / a_s)\left[1-\exp\left(-a_s(t-u)\right)\right]d\widetilde{W}_s(u)\right)$$

$$= \int_0^t \left[\sigma_s^2 \left(1-\exp\left(-a_s(t-u)\right)\right)^2 / a_s^2\right]du$$

$$\sigma_4^2(t) \equiv Var\left(s(t)\right)$$

$$= Var\left(\int_0^t \sigma_s \left(-a_s(t-u)\right)d\widetilde{W}_s(u)\right)$$

$$= \int_0^t \sigma_s^2 \exp\left(-2a_s(t-u)\right)du$$

$$\sigma_{12}(t) \equiv \text{cov}\left(\int_0^t r(u)du,\ r(t)\right)$$

$$= \text{cov}\left(\int_0^t (\sigma_r / a_r)\left[1-\exp\left(-a_r(t-u)\right)\right]d\widetilde{W}_r(u),\ \int_0^t \sigma_r \exp\left(-a_r(t-u)\right)d\widetilde{W}_r(u)\right)$$

$$= \int_0^t (\sigma_r^2 / a_r)\exp\left(-a_r(t-u)\right)\left[1-\exp\left(-a_r(t-u)\right)\right]du$$

$$= \left(\sigma_r^2 /(2a_r^2)\right)\left(1-\exp\left(-a_r t\right)^2\right)$$

$$\sigma_{13}(t) \equiv \operatorname{cov}\left(\int_0^t r(u)du,\ \int_0^t s(u)du\right)$$

$$= \operatorname{cov}\left(\int_0^t \frac{\sigma_r}{a_r}\left[1-\exp\left(-a_r(t-u)\right)\right]d\widetilde{W}_r(u),\ \int_0^t \frac{\sigma_s}{a_s}\left[1-\exp\left(-a_s(t-u)\right)\right]d\widetilde{W}_s(u)\right)$$

$$= \int_0^t (\sigma_r\sigma_s\rho)\left[1-\exp\left(-a_r(t-u)\right)\right]\left[1-\exp\left(-a_s(t-u)\right)\right]/(a_r a_s)du$$

$$\sigma_{14}(t) \equiv \operatorname{cov}\left(\int_0^t r(u)du,\ s(t)\right)$$

$$= \operatorname{cov}\left(\int_0^t (\sigma_r / a_r)\left[1-\exp\left(-a_r(t-u)\right)\right]d\widetilde{W}_r(u),\ \int_0^t \sigma_s \exp\left(-a_s(t-u)\right)d\widetilde{W}_s(u)\right)$$

$$= \int_0^t (\sigma_r\sigma_s\rho)\left[1-\exp\left(-a_r(t-u)\right)\right]\exp\left(-a_s(t-u)\right)/ a_r du$$

$$\sigma_{23}(t) \equiv \operatorname{cov}\left(r(t),\ \int_0^t s(u)du\right)$$

$$= \operatorname{cov}\left(\int_0^t \sigma_r \exp\left(-a_r(t-u)\right)d\widetilde{W}_r(u),\ \int_0^t (\sigma_s / a_s)\left[1-\exp\left(-a_s(t-u)\right)\right]d\widetilde{W}_s(u)\right)$$

$$= \int_0^t (\sigma_r\sigma_s\rho)\left[1-\exp\left(-a_s(t-u)\right)\right]\exp\left(-a_r(t-u)\right)/ a_s du$$

$$\sigma_{24}(t) \equiv \operatorname{cov}\left(r(t), s(t)\right)$$

$$= \operatorname{cov}\left(\int_0^t \sigma_r \exp\left(-a_r(t-u)\right)d\widetilde{W}_r(u),\ \int_0^t \sigma_s \exp\left(-a_s(t-u)\right)d\widetilde{W}_s(u)\right)$$

$$= \int_0^t (\sigma_r\sigma_s\rho)\exp\left(-a_r(t-u) - a_s(t-u)\right)du$$

$$\sigma_{34}(t) \equiv \operatorname{cov}\left(\int_0^t s(u)du,\ s(t)\right)$$

$$= \operatorname{cov}\left(\int_0^t (\sigma_s / a_s)\left[1-\exp\left(-a_s(t-u)\right)\right]d\widetilde{W}_s(u),\ \int_0^t \sigma_s \exp\left(-a_s(t-u)\right)d\widetilde{W}_s(u)\right)$$

$$= \int_0^t (\sigma_s^2 / a_s)\exp\left(-a_s(t-u)\right)\left[1-\exp\left(-a_s(t-u)\right)\right]du$$

$$= \left(\sigma_s^2 /(2a_s^2)\right)\left(1-\exp\left(-a_s t\right)^2\right)$$