

The Immunization Performance of Traditional and Stochastic Durations: A Mean-Variance Analysis*

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Abstract

This paper provides a mean-variance analysis of immunization strategies that trade off coupon reinvestment risk with resale price risk. For static immunization strategies, neither traditional nor stochastic durations fall in the set of efficient horizons. This finding is robust across various interest rate environments and bond characteristics, and explains the poor immunization results obtained by the comparative study of Gultekin and Rogalski (1984). When dynamic portfolio rebalancing is allowed, traditional and stochastic durations induce efficient strategies with similar performance. We therefore obtain that immunization performance is more driven by strategy sophistication rather than by the choice of duration, which corroborates the empirical finding of Agca (2005). Our results still hold under the two-factor term structure model with stochastic volatility of Longstaff and Schwartz (1992).

JEL Codes: G10, G11.

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1 Introduction

Immunizing bond portfolios against interest rate fluctuations is a major challenge in fixed-income management. Since the work of Macaulay, duration has been viewed as a relevant tool for immunization purposes. Yet, extending the definition of duration to account for stochastic interest rates is a non trivial exercise. Several papers propose new definitions of stochastic duration: Cox, Ingersoll and Ross (1979) and Wu (2000) for equilibrium single factor models, Au and Thurston (1995) and Frühwirth (2002) for Heath-Jarrow-Morton models, and Munk (1999) for multi-factor models. Despite these theoretical contributions, empirical studies on immunization performance do not conclude on the superiority of stochastic durations over traditional ones. Gultekin and Rogalski (1984) find that all of the seven durations they study induce disappointing results. Studies edited by Kaufman, Bierwag and Toevs (1983) as well as the more recent works of Wu (2000) and Agca (2005) reach similar conclusions. All this statistical evidence calls for investigating the immunization performance that one can expect from traditional and stochastic durations.

This paper undertakes a mean variance analysis of returns generated by a couple of stylized immunization strategies. The basic strategy captures the fundamental issue in duration-based immunization, that is, trading off coupon reinvestment risk and resale price risk (see e.g. Sundaresan, 2002). The other strategy is a dynamic extension that allows for frequent rebalancing. We characterize the set of coupon reinvestment horizons yielding mean-variance efficiency. As a by-product, we highlight the duration that achieves the lowest variance in returns. Neither traditional nor stochastic durations fall in the set of efficient horizons. This finding is robust across various interest rate environments and bond characteristics. Our findings shed a new light on prior evidence. The *ex ante* inefficiency of both traditional and stochastic duration-based strategies could explain the poor immunization results obtained by Gultekin and Rogalski (1984) in their comparative study of *ex post* performance.

Moreover, as we move from the basic to the dynamic strategy, efficiency is retrieved and, although the same performance ranking across duration measures holds, differences

in immunization performance are narrowed. This is consistent with Agca (2005) who concludes that immunization performance is more driven by strategy sophistication rather than by the choice of duration. Our finding is robust when we alternatively work under the two-factor term structure model with stochastic volatility of Longstaff and Schwartz (1992), suggesting the choice of the term structure model does not affect the ranking among immunization performance.

The paper proceeds as follows. Section 2 presents the term structure model and the set of durations under study. Section 3 describes the basic immunization strategy. Section 4 undertakes the mean variance analysis and shows the existence of an optimal set of reinvestment horizons. Section 5 extends the analysis to a dynamic immunization strategy where the reinvestment horizon is recalculated and the portfolio is rebalanced periodically. We conclude in section 6.

2 Term structure model and durations

The mean-variance analysis is carried out under the Vasicek (1977) term structure model. This setting allows for explicitly characterizing the mean and the variance of strategy returns.¹ The term structure is driven by the instantaneous risk-free rate r_t which follows a Gaussian mean-reverting process under the risk-neutral measure

$$dr_t = \alpha(\beta + \lambda - r_t) dt + \eta dZ_t. \quad (1)$$

In equation (1), the (physical) process r_t reverts to the long-term mean β , α is the reversion speed, λ is the market price of risk, and random increments are driven by the Brownian motion Z_t with volatility η . The time- t value of the default-free discount bond with principal \$1 and time to maturity τ , denoted by $P(t, \tau)$, is given by

$$P(t, \tau) = a(\tau) \exp(-b(\tau) r_t),$$

¹Korn and Koziol (2006) conduct a mean-variance analysis of bond returns in the same framework. Their focus is on portfolio optimization, and not on risk management issues like immunization.

with

$$\begin{aligned}
 a(\tau) &= \exp\left(\frac{(b(\tau) - \tau)(\alpha^2(\beta + \lambda) - \eta^2/2)}{\alpha^2} - \frac{\eta^2 b^2(\tau)}{4\alpha}\right) \\
 b(\tau) &= \frac{1 - \exp(-\alpha\tau)}{\alpha}.
 \end{aligned}$$

Let $P_c(t, T - t)$ denote the time- t value of the coupon bond. Without loss of generality, we assume a continuous coupon stream c so that

$$P_c(t, T - t) = c \int_t^T P(t, k - t) dk + P(t, T - t).$$

One denotes by $\Omega = \{r_0, \alpha, \beta, \lambda, \eta\}$ the set of parameters. Typical shapes of the zero-coupon yield curves generated by the Vasicek model are: increasing, decreasing and humped. We work under three sets of parameters to capture different interest rate environments. Figure 1 details the parameterizations and shows the corresponding term structures.

We consider three different duration measures: the Macaulay and the Fisher-Weil durations among the so-called “traditional” measures, and the Cox-Ingersoll-Ross stochastic duration. For a coupon bond with face value 1 and continuous coupon c , the Macaulay duration is defined by

$$\theta^{mac} = \frac{c \int_0^T k e^{-yk} dk + T e^{-yT}}{P_c(0, T)},$$

where y is the continuously compounded yield-to-maturity on the coupon bond $P_c(0, T)$.

The Fisher-Weil duration is

$$\theta^{fw} = \frac{c \int_0^T k P(0, k) dk + T P(0, T)}{P_c(0, T)}.$$

These two traditional durations are widely used in the industry. On a theoretical ground, Ingersoll, Skelton and Weil (1978) show that these durations are valid risk measures when changes in the term structure are limited to parallel shifts. Cox, Ingersoll and Ross (1979) propose to define the stochastic duration as the maturity, expressed in units of time, of a discount bond with same basis risk as the initial instrument. Therefore, in the context of

the Vasicek model²

$$\theta^{sto} = b^{-1} \left(\frac{c \int_0^T b(k) P(0, k) dk + b(T) P(0, T)}{P_c(0, T)} \right)$$

where

$$b^{-1}(t) = -\frac{1}{\alpha} \ln(1 - \alpha t), t < \frac{1}{\alpha}.$$

3 The basic immunization strategy

We introduce a stylized strategy that captures the trade-off between coupon reinvestment risk and resale price risk. Consider an investor who is currently long in one coupon bond with face value 1, maturity T and continuous coupon c . The strategy with horizon θ ($0 \leq \theta \leq T$) consists in (i) holding the coupon bond between dates 0 and θ , (ii) re-investing each time- t coupon in the discount bond $P(t, \theta - t)$, and (iii) closing the positions at time θ .³

The value π_θ of the investor's portfolio at date θ is

$$\pi_\theta = P_c(\theta, T - \theta) + c \int_0^\theta \frac{ds}{P(s, \theta - s)}. \quad (2)$$

Setting $\theta = 0$, the strategy reduces to the instantaneous resale of the coupon bond. Setting $\theta = T$, the strategy reduces to the buy and hold of the coupon bond. As the investor selects a longer θ , his or her portfolio gets more exposed to coupon reinvestment risk and less exposed to resale price risk. The immunization strategy looks for the best trade-off.

Proposition 1 *For any horizon θ , the random value of the basic immunization strategy*

²It is well known that the extended Vasicek model is equivalent to the one-factor Heath-Jarrow-Morton model with exponentially decaying volatility. Consequently, the duration θ^{cir} of this paper is analogous to the measure used by Au and Thurston (1995) example 2, and Agca (2005), equation (8).

³We implicitly assume a perfect bond market where discount bonds of all maturities are traded. Empirical papers (see Agca, 2005, and references therein) study specific portfolio formation strategies (such as the bullet and the barbell portfolios) to cope with restrictions on available bonds. In this paper, we abstract from these constraints.

has mean and variance respectively equal to

$$E(\pi_\theta) = c \int_\theta^T A(\theta, k, -) dk + A(\theta, T, -) + c \int_0^\theta A(s, \theta, +) ds.$$

and

$$\text{var}(\pi_\theta) = \Sigma_P^2 + \Sigma_C^2 + 2\Sigma_{PC}$$

where

$$\begin{aligned} \Sigma_P^2 &= 2c^2 \int_\theta^T \int_j^T A(\theta, k, -) A(\theta, j, -) B(t, u, v, \pm) B(\theta, k, \theta, j, +) dk dj \\ &\quad + A(\theta, T, -)^2 B(\theta, T, \theta, T, +) \\ &\quad + 2c \int_\theta^T A(\theta, k, -) A(\theta, T, -) B(\theta, k, \theta, T, +) dk, \\ \Sigma_C^2 &= 2c^2 \int_0^\theta \int_s^\theta A(s, \theta, +) A(u, \theta, +) B(s, \theta, u, \theta, +) ds du, \\ \Sigma_{PC} &= c^2 \int_\theta^T \int_0^\theta A(\theta, k, -) A(s, \theta, +) B(s, \theta, \theta, k, -) ds dk \\ &\quad + c \int_0^\theta A(\theta, T, -) A(s, \theta, +) B(s, \theta, \theta, T, -) ds, \end{aligned}$$

with

$$\begin{aligned} A(t, u, \pm) &= a(u-t)^{-(\pm 1)} \exp\left(\pm b(u-t)m_t + \frac{1}{2}b^2(u-t)v_t\right) \\ B(t, u, v, w, \pm) &= \exp(\pm b(u-t)b(w-v)g_{t,v}) - 1 \end{aligned}$$

and

$$\begin{aligned} m_t &: = E(r_t) = r_0 e^{-\alpha t} + \beta(1 - e^{-\alpha t}), \\ v_t &: = \text{var}(r_t) = \frac{\eta^2}{2\alpha}(1 - e^{-2\alpha t}), \\ g_{t,u} &: = \text{cov}(r_t, r_u) = \eta^2 \exp(-\alpha(t+u)) \frac{\exp(2\alpha t) - 1}{2\alpha}. \end{aligned}$$

Proof: see appendix.

Proposition 1 allows for a variance decomposition of the basic immunization strategy returns.⁴ The variances corresponding to resale price risk and coupon reinvestment risk

⁴Without loss of generality, the analysis will focus on annualized returns defined by $R_\theta = (\pi_\theta - P_c(0, T)) / (\theta P_c(0, T))$.

involve Σ_P^2 and Σ_C^2 , respectively. The immunization strategy also induces a covariance term involving Σ_{PC} . In line with intuition, Figure 2 shows that the basic immunization strategy with a longer horizon decreases the exposure to resale price risk and increases the risk of coupon reinvestment. The covariance component typically reduces the total risk and its effect is the strongest around half time to maturity. Similar patterns (not reported) hold for different interest rate environments.

Proposition 2 *For the instantaneous resale strategy, the return has mean and standard deviation*

$$\begin{aligned}\lim_{\theta \rightarrow 0^+} E(R_\theta) &= r \\ \lim_{\theta \rightarrow 0^+} \sigma(R_\theta) &= \eta \cdot b(\theta^{sto}).\end{aligned}$$

Proof: see appendix.

At the limit, buying and immediately selling the coupon bond yields a return equal to the instantaneous spot rate. The volatility of this strategy is that of the instantaneous spot rate multiplied by the sensitivity of the coupon bond (since $b(\theta^{sto}) = P'_c(t, T-t)/P_c(t, T-t)$).

4 Mean-variance analysis

In this section, we characterize the set of efficient basic immunization strategies. Then we assess the performance of basic immunization strategies using the duration measures introduced in section 2.

Figure 3 plots, for every horizon θ , the returns of the basic immunization strategy in the mean-standard deviation space $(0, \sigma(R_\theta), E(R_\theta))$. Inspection of Figure 3 shows that there exists a minimum variance horizon θ^* . This is the horizon that achieves the lowest variance trade-off between coupon reinvestment risk and resale price risk. The same plot is represented in Figure 4 for the other interest rate environments Ω_2 (decreasing yield curve) and Ω_3 (humped yield curve), as well as in Figure 5 for different bond characteristics. The

returns of the basic immunization strategy display a robust pattern across various interest rate parameters and bond characteristics: A minimum variance is attained for $\theta = \theta^*$ and the most “north-western” returns are obtained with θ ranging from θ^* to T . In the absence of coupons, $\theta^* = T$. Hence θ^* can be interpreted as a duration. For the remainder of the paper, it will be referred to as the *minimum variance duration*.⁵

Proposition 3 *The basic immunization strategy with horizon θ is mean-variance efficient if and only if*

$$\theta \geq \theta^* = \arg \min_{\theta} [\sigma (R_{\theta})].$$

Corollary *The buy and hold strategy ($\theta = T$) is mean-variance efficient.*

Using Propositions 1 and 3, we can gauge the performance of immunization strategies based on traditional and stochastic durations. Results are displayed in Figure 6 using the base case parameters of Figure 3. As a robustness check, Tables 1, 2 and 3 show the means and standard deviations of returns for different interest rate environments (Table 1), and different coupon rates and maturities (Tables 2 and 3). Sharpe ratios are also reported.⁶ Our findings can be summarized in three main points.

First, both traditional and stochastic durations are always (except for one case) below the minimum variance duration, and therefore induce inefficient immunization strategies. Oddly enough, all these durations induce an underexposure to the coupon reinvestment risk. Second, The Cox-Ingersoll-Ross duration is systematically the lowest among the three measures. Hence immunization strategies based on traditional durations outperform those based on the stochastic duration, especially for long bonds. This finding helps explain why previous empirical studies fail to prove the superiority of stochastic durations. Third, the strategy based on Macaulay duration and that based on Fisher-Weil duration yield very

⁵Note that, just like traditional and stochastic durations, the minimum variance duration need not be a strictly increasing function of maturity.

⁶In our context, the Sharpe ratio is a relevant measure of immunization performance because (i) bond returns are Gaussian in the Vasicek model, and (ii) for a given target excess return, the Sharpe ratio penalizes dispersion (measured as the variance) around the mean.

similar immunization results.

5 Dynamic immunization strategies

In this section, we check consider a dynamic extension of the basic immunization strategy. Specifically, the coupon reinvestment horizon is not determined once at initial date, but rather recalculated periodically. As the spot rate changes over time, all the accumulated coupons that were previously invested at date t in the zero coupon bond with maturity θ_t are then reinvested in the zero coupon bond with maturity $\theta_{t+\Delta t}$.⁷

As in the former section, we use the traditional and stochastic durations to determine the reinvestment horizons $\{\theta_t\}_{t \geq 0}$. We then challenge these strategies with the one consisting in minimizing the variance of the investment strategy returns (this minimization is now performed periodically).

5.1 Strategy description

We discretize the time interval $[0, T]$ and we denote by Δt the (arbitrarily small) time step. This time step represents the investor's rebalancing frequency. The dynamic immunization strategy is described as follows.

1. Buy the coupon bond with maturity T at time 0,
2. At any date $i\Delta t$, the accumulated capitalized coupons (acc_i) are defined as

$$acc_i = \sum_{k=1}^i c\Delta t \left(\prod_{j=k}^i \frac{P(j\Delta t, \theta_{(j-1)}) - j\Delta t}{P((j-1)\Delta t, \theta_{(j-1)}) - (j-1)\Delta t} \right),$$

and they will be invested in the zero coupon bond with maturity $\theta_{(i)}$,

3. At the same date, the next reinvestment horizon $\theta_{(i)}$ is calculated along the chosen duration measure (see the appendix for details),

⁷Obviously, this stylized strategy would entail significant transaction costs. Note however, that every strategy based on each duration would face the same amount of transaction costs. Hence, these costs do not affect the comparison between strategies.

4. Accumulated capitalized coupons are reinvested until date $\Theta = I\Delta t$ where

$$I = \inf \{i \geq 0 : \theta_{(i)} < \Delta t\}.$$

Note that the starting point of the dynamic strategy coincides with the basic immunization strategy defined in the above section. That is, we have that $\pi_{0,\theta} = \pi_\theta$ and hence $\theta_{(0)}^x = \theta^x$, where x characterizes the duration measure. Then basic and dynamic strategies differ as the spot rate follows a particular path and the investor updates the reinvestment horizon accordingly.

5.2 Monte Carlo results

We carry out a mean-variance analysis of dynamic strategies by simulating a large number of scenarios. Without any loss in generality, we assume the investor's rebalancing frequency to be monthly ($\Delta t = 1/12$). The Vasicek process for the spot rate is simulated using the Euler scheme. To reduce the bias caused by time discretization, the Euler scheme uses a daily time step ($\Delta t = 1/360$). Then the monthly interest rate process is simply obtained by taking one every thirty realizations. The same set of sample paths is used to run all dynamic strategies. The number of sample paths is set to 1,000, which is found to be sufficient for the level of accuracy reported in our tables.

Figure 7 and Table 4 report the mean-variance analysis. Not surprisingly, Sharpe ratios are higher than in the basic case (Table 3), especially for long maturities, because of the value created by updating information.

As in the basic case, strategies based on traditional and stochastic durations fail to beat that based on the minimum variance. Furthermore, the performance ranking across durations remains the same. The strategy based on the stochastic duration exhibits the poorest performance. Thus, the degree of strategy sophistication does not alter our previous conclusion: Both traditional and stochastic durations imply mean-variance inefficient immunization strategies, and stochastic durations fail to yield a superior performance.

Most importantly, the differences in Sharpe ratios across durations become small when the strategy gets dynamic. Let RS denote the relative spread in Sharpe ratios between

the best performing strategy (based on the minimum variance duration) and the worst performing strategy (based on the stochastic duration)

$$RS = \frac{\text{Sharpe ratio}(\theta^*) - \text{Sharpe ratio}(\theta^{cir})}{\text{Sharpe ratio}(\theta^{cir})}.$$

In the basic case: $RS = 173\%$, 336% , and 518% for $T = 1, 5$, and 10 , respectively. By contrast, with the dynamic strategy: $RS = 0.9\%$, 4.7% , and 12.7% for $T = 1, 5$, and 10 , respectively. Thus, we obtain that immunization performance is more driven by strategy sophistication rather than by the choice of duration. The empirical study of Agca (2005) reaches a similar conclusion.

5.3 Alternative interest rate model

We check the extent to which our results could be driven by the chosen interest rate model. Simulations of the dynamic immunization strategies are performed under a different interest rate model. Specifically, we choose to work under the Longstaff and Schwartz (1992) two factor model where

$$\begin{aligned} dr_t = & \left(\alpha\gamma + \beta\eta - \frac{\beta\delta - \alpha\xi}{\beta - \alpha}r_t - \frac{\xi - \delta}{\beta - \alpha}V_t \right) dt \\ & + \alpha\sqrt{\frac{\beta r_t - V_t}{\alpha(\beta - \alpha)}}dW_t^1 + \beta\sqrt{\frac{V_t - \alpha r_t}{\beta(\beta - \alpha)}}dW_t^2 \end{aligned}$$

and

$$\begin{aligned} dV_t = & \left(\alpha^2\gamma + \beta^2\eta - \frac{\alpha\beta(\delta - \xi)}{\beta - \alpha}r_t - \frac{\beta\xi - \alpha\delta}{\beta - \alpha}V_t \right) dt \\ & + \alpha^2\sqrt{\frac{\beta r_t - V_t}{\alpha(\beta - \alpha)}}dW_t^1 + \beta^2\sqrt{\frac{V_t - \alpha r_t}{\beta(\beta - \alpha)}}dW_t^2, \end{aligned}$$

where W^1 and W^2 are two independent Brownian motions and $\alpha, \beta, \gamma, \delta, \eta$ and ξ are positive constant parameters.

Simulating the strategy described in subsection 5.1 requires (i) knowledge of the discount bond value function and (ii) determination of the coupon reinvestment horizon. The Longstaff and Schwartz (1992) model allows for a closed-form solution to $P(t, \tau)$. As

for durations, Macaulay and Fisher-Weil measures keep the same definitions. Moreover, Munk (1999) solves for the stochastic (Cox-Ingersoll-Ross type) duration in the Longstaff and Schwartz (1992) model. He finds that θ^{sto} is the solution to

$$0 = \frac{\alpha(\beta r - V)}{\varphi^2} \left(\left(e^{\varphi \theta^{sto}} - 1 \right)^2 a_m (\theta^{sto})^2 - K_a^2 \right) + \frac{\beta(V - \alpha r)}{\psi^2} \left(\left(e^{\psi \theta^{sto}} - 1 \right)^2 b_m (\theta^{sto})^2 - K_b^2 \right),$$

with $\varphi = \sqrt{2\alpha + \delta^2}$, $\psi = \sqrt{2\beta + (\xi + \lambda)^2}$, λ is the market price of risk, and

$$\begin{aligned} a_m(x) &= \frac{2\varphi}{(\delta + \varphi)(e^{\varphi x} - 1) + 2\varphi} \\ b_m(x) &= \frac{2\psi}{(\xi + \lambda + \psi)(e^{\psi x} - 1) + 2\psi} \\ K_a &= \frac{c \int_t^T P(t, k-t) (e^{\varphi(k-t)} - 1) a_m(k-t) dk}{P_c(t, T-t)} \\ &\quad + \frac{P(t, T-t) (e^{\varphi(T-t)} - 1) a_m(T-t)}{P_c(t, T-t)} \\ K_b &= \frac{c \int_t^T P(t, k-t) (e^{\psi(k-t)} - 1) b_m(k-t) dk}{P_c(t, T-t)} \\ &\quad + \frac{P(t, T-t) (e^{\psi(T-t)} - 1) b_m(T-t)}{P_c(t, T-t)}. \end{aligned}$$

Simulations results are reported in Table 5. Again, we use the Euler scheme with a daily time step ($\Delta t = 1/360$) to discretize the interest rate process. Then we use one every thirty realizations of the process to work with monthly values. When simulating processes with stochastic volatility, the standard Euler scheme can generate negative variance with non-zero probability, which causes the time stepping scheme to fail. To avoid this problem, we rely on the so-called full truncation scheme as described in Lord *et al.* (2006).⁸

Parameter values are set very close to the ones obtained by Jensen (2001) who estimates the Longstaff-Schwartz model on the weekly US 3-month T-Bill over the 1954 – 1998 period. Specifically, we take $\alpha = 0.0003$, $\beta = 0.015$, $\gamma = 40$, $\delta = 0.25$, $\xi = 4$ and $\eta = 5$.

⁸Lord *et al.* (2006) and Andersen (2007) numerically find that among Euler discretizations that correct for negative variances, the full truncation scheme produces the smallest discretization bias.

Setting the initial values of $r = 0.05$ and $V = 0.0002$, we obtain an increasing term structure similar to the one generated by interest rate environment Ω_1 .

As shown in Table 5, results are very similar to the Vasicek case. In particular, the performance ranking among duration measures is the same, with the two traditional durations producing almost identical mean-variance results across all maturities considered. Our findings are therefore robust to the choice of the term structure model.

6 Concluding remarks

This paper has challenged several durations as coupon reinvestment horizon for immunization purposes. Both traditional and stochastic durations almost always lie below the minimum variance duration, inducing mean-variance inefficient strategies. As we consider a more sophisticated strategy with frequent rebalancing, the ranking between durations (in terms of mean-variance performance) remains the same, but differences across durations are narrowed. The analysis is conducted under a Gaussian interest rate model, which allows for analytical solutions for the mean and the variance of strategies and for an explicit characterization of the mean-variance efficient set. Yet, the same performance ranking across duration measures holds under an alternative two-factor term structure model with stochastic volatility. Our results explain why previous empirical studies on immunization (i) find disappointing performance for all duration-based strategies, and (ii) conclude that strategy sophistication matters more than the choice of duration.

A possible reason may stem from the ambivalent definition of duration. Most available durations, especially the stochastic one, are defined as the bond price elasticity with respect to the relevant risk factor(s). These measures are essentially *Hicks-type* durations built for hedging purposes. Alternatively, duration can be defined as the holding period over which the portfolio made of all reinvested coupons plus the present value of the bond is immune to interest rate changes. This is the *Macaulay-type* duration, which justifies its use in immunization strategies. Unfortunately, it is well known that the hedging and immunization definitions coincide only when the term structure is flat and subsequently

undergoes parallel shifts. Our *ex ante* mean-variance analysis and previous *ex post* empirical studies both emphasize the limitations in implementing an immunization strategy with a duration defined in a hedging sense.

Our paper therefore suggests that duration-based immunization strategies should not rely on available measures, but rather on adapted durations that are specifically designed for trading off coupon reinvestment risk with resale price risk.

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Appendix

Proof of Proposition 1

The mean of the value process is given by

$$E(\pi_\theta) = c \int_\theta^T E(P(\theta, k - \theta)) dk + E(P(\theta, T - \theta)) + c \int_0^\theta E\left(\frac{1}{P(s, \theta - s)}\right) ds.$$

For any given t and τ , we have that

$$E(P(t, \tau)) = a(\tau) E(\exp(-b(\tau) r_t)).$$

Since r_t is Gaussian with mean

$$m_t := E(r_t) = r_0 e^{-\alpha t} + \beta(1 - e^{-\alpha t})$$

and variance

$$v_t := \text{var}(r_t) = \frac{\eta^2}{2\alpha} (1 - e^{-2\alpha t}),$$

we obtain

$$E(P(t, \tau)) = a(\tau) \exp\left(-b(\tau) m_t + \frac{1}{2} b^2(\tau) v_t\right).$$

Similarly,

$$E\left(\frac{1}{P(t, \tau)}\right) = \frac{1}{a(\tau)} \exp\left(b(\tau) m_t + \frac{1}{2} b^2(\tau) v_t\right).$$

Hence

$$\begin{aligned} E(\pi_\theta) &= c \int_\theta^T a(k - \theta) \exp\left(-b(k - \theta) m_\theta + \frac{1}{2} b^2(k - \theta) v_\theta\right) dk \\ &\quad + a(T - \theta) \exp\left(-b(T - \theta) m_\theta + \frac{1}{2} b^2(T - \theta) v_\theta\right) \\ &\quad + c \int_0^\theta \frac{1}{a(\theta - s)} \exp\left(b(\theta - s) m_s + \frac{1}{2} b^2(\theta - s) v_s\right) ds. \end{aligned}$$

As for variance, we have that

$$\begin{aligned}
var(\pi_\theta) &= c^2 var\left(\int_\theta^T P(\theta, k - \theta) dk\right) + var(P(\theta, T - \theta)) \\
&+ 2c \cdot cov\left(\int_\theta^T P(\theta, k - \theta) dk, P(\theta, T - \theta)\right) \\
&+ c^2 var\left(\int_0^\theta \frac{1}{P(s, \theta - s)} ds\right) \\
&+ 2c^2 cov\left(\int_\theta^T P(\theta, k - \theta) dk, \int_0^\theta \frac{1}{P(s, \theta - s)} ds\right) \\
&+ 2c \cdot cov\left(P(\theta, T - \theta), \int_0^\theta \frac{1}{P(s, \theta - s)} ds\right).
\end{aligned}$$

Which yields, for $k \geq j$ and $u \geq s$

$$\begin{aligned}
var(\pi_\theta) &= 2c^2 \int_\theta^T dj \int_j^T cov(B(\theta, k - \theta), B(\theta, j - \theta)) dk \\
&+ var(B(\theta, T - \theta)) \\
&+ 2c \int_\theta^T cov(B(\theta, k - \theta), B(\theta, T - \theta)) dk \\
&+ 2c^2 \int_0^\theta ds \int_s^\theta cov\left(\frac{1}{B(u, \theta - u)}, \frac{1}{B(s, \theta - s)}\right) du \\
&+ 2c^2 \int_0^\theta ds \int_\theta^T cov\left(B(\theta, k - \theta), \frac{1}{B(s, \theta - s)}\right) dk \\
&+ 2c \int_0^\theta cov\left(B(\theta, T - \theta), \frac{1}{B(s, \theta - s)}\right) ds.
\end{aligned}$$

First, we compute $var(P(t, \tau))$ for any given t and τ . We have that

$$var(P(t, \tau)) = a^2(\tau) var(\exp(-b(\tau) r_t)).$$

Since $\exp(-b(\tau) r_t)$ is log-normal, we get

$$var(P(t, \tau)) = a^2(\tau) \exp(-2b(\tau) m_t + b^2(\tau) v_t) (\exp(b^2(\tau) v_t) - 1).$$

Similarly,

$$var\left(\frac{1}{P(t, \tau)}\right) = \frac{1}{a^2(\tau)} \exp(2b(\tau) m_t + b^2(\tau) v_t) (\exp(b^2(\tau) v_t) - 1).$$

Next, we compute $\text{cov}(P(t, \tau), P(t, \zeta))$ for any given t, τ and ζ . We have that

$$\text{cov}(P(t, \tau), P(t, \zeta)) = a(\tau) a(\zeta) \text{cov}(\exp(-b(\tau) r_t), \exp(-b(\zeta) r_t)).$$

Note that the last covariance can also be written as

$$E(\exp(-(b(\tau) + b(\zeta)) r_t)) - E(\exp(-b(\tau) r_t)) E(\exp(-b(\zeta) r_t)),$$

which yields

$$\begin{aligned} & \text{cov}(P(t, \tau), P(t, \zeta)) \\ &= a(\tau) a(\zeta) \left[\exp\left(- (b(\tau) + b(\zeta)) m_t + \frac{1}{2} ((b(\tau) + b(\zeta)))^2 v_t\right) \right] \\ & \quad - a(\tau) a(\zeta) \left[\exp\left(- (b(\tau) + b(\zeta)) m_t + \frac{1}{2} b^2(\tau) v_t + \frac{1}{2} b^2(\zeta) v_t\right) \right] \\ &= a(\tau) a(\zeta) \exp\left(- (b(\tau) + b(\zeta)) m_t + \frac{1}{2} b^2(\tau) v_t + \frac{1}{2} b^2(\zeta) v_t\right) \\ & \quad \times [\exp(b(\tau) b(\zeta) v_t) - 1]. \end{aligned}$$

Now we compute $\text{cov}\left(\frac{1}{P(s, \tau)}, \frac{1}{P(u, \zeta)}\right)$ for any given $s, u \geq s, \tau$ and ζ . We have that

$$\text{cov}\left(\frac{1}{P(s, \tau)}, \frac{1}{P(u, \zeta)}\right) = \frac{1}{a(\tau) a(\zeta)} \text{cov}(\exp(b(\tau) r_s), \exp(b(\zeta) r_u)).$$

Note that the last covariance can also be written as

$$E(\exp(b(\tau) r_s + b(\zeta) r_u)) - E(\exp(b(\tau) r_s)) E(\exp(b(\zeta) r_u)),$$

or, equivalently

$$\begin{aligned} & \exp\left(b(\tau) m_s + b(\zeta) m_u + \frac{1}{2} [b^2(\tau) v_s + b^2(\zeta) v_u + 2b(\tau) b(\zeta) g_{s,u}]\right) \\ & - \exp\left(b(\tau) m_s + \frac{1}{2} b^2(\tau) v_s\right) \exp\left(b(\zeta) m_u + \frac{1}{2} b^2(\zeta) v_u\right), \end{aligned}$$

with

$$g_{s,u} := \text{cov}(r_s, r_u) = \eta^2 \exp(-\alpha(s+u)) \frac{\exp(2\alpha s) - 1}{2\alpha}.$$

Hence

$$\begin{aligned}
& \text{cov} \left(\frac{1}{P(s, \tau)}, \frac{1}{P(u, \zeta)} \right) \\
&= \frac{1}{a(\tau) a(\zeta)} \exp \left(b(\tau) m_s + b(\zeta) m_u + \frac{1}{2} [b^2(\tau) v_s + b^2(\zeta) v_u + 2b(\tau) b(\zeta) g_{s,u}] \right) \\
&\quad - \frac{1}{a(\tau) a(\zeta)} \exp \left(b(\tau) m_s + \frac{1}{2} b^2(\tau) v_s \right) \exp \left(b(\zeta) m_u + \frac{1}{2} b^2(\zeta) v_u \right).
\end{aligned}$$

$$\begin{aligned}
& \text{cov} \left(\frac{1}{P(s, \tau)}, \frac{1}{P(u, \zeta)} \right) \\
&= \frac{1}{a(\tau) a(\zeta)} \exp \left(b(\tau) m_s + b(\zeta) m_u + \frac{1}{2} b^2(\tau) v_s + \frac{1}{2} b^2(\zeta) v_u \right) \\
&\quad [\exp(b(\tau) b(\zeta) g_{s,u}) - 1].
\end{aligned}$$

Finally, the computation of $\text{cov} \left(P(\theta, \tau), \frac{1}{P(u, \zeta)} \right)$ for any given θ , $\theta \geq u$, τ and ζ yields

$$\begin{aligned}
& \text{cov} \left(P(\theta, \tau), \frac{1}{P(u, \zeta)} \right) \\
&= \frac{a(\tau)}{a(\zeta)} \exp \left(-b(\tau) m_\theta + b(\zeta) m_u + \frac{1}{2} b^2(\tau) v_\theta + \frac{1}{2} b^2(\zeta) v_u \right) \\
&\quad [\exp(-b(\tau) b(\zeta) g_{u,\theta}) - 1].
\end{aligned}$$

Combining all these results completes the proof.

Proof of Proposition 2

Applying Itô's lemma,

$$dP_c(t, T-t) = \frac{\partial P_c(t, T-t)}{\partial t} dt + \frac{\partial P_c(t, T-t)}{\partial r} dr + \frac{1}{2} \frac{\partial^2 P_c(t, T-t)}{\partial r^2} (dr)^2. \quad (3)$$

Note that

$$\begin{aligned}
\frac{\partial P_c(t, T-t)}{\partial t} &= c \int_t^T \frac{\partial P(t, k-t)}{\partial t} dk - cP(t, 0) + \frac{\partial P(t, T-t)}{\partial t} \\
\frac{\partial P_c(t, T-t)}{\partial r} &= c \int_t^T \frac{\partial P(t, k-t)}{\partial r} dk + \frac{\partial P(t, T-t)}{\partial r} \\
\frac{\partial^2 P_c(t, T-t)}{\partial r^2} &= c \int_t^T \frac{\partial^2 P(t, k-t)}{\partial r^2} dk + \frac{\partial^2 P(t, T-t)}{\partial r^2}.
\end{aligned}$$

Hence, equation (3) becomes

$$\begin{aligned}
dP_c(t, T-t) &= \left(c \int_t^T \frac{\partial P(t, k-t)}{\partial t} dk - cP(t, 0) + \frac{\partial P(t, T-t)}{\partial t} \right. \\
&\quad + \alpha(\beta - r_t) c \int_t^T \frac{\partial P(t, k-t)}{\partial r} dk + \alpha(\beta - r_t) \frac{\partial P(t, T-t)}{\partial r} \\
&\quad \left. + \frac{\eta^2}{2} c \int_t^T \frac{\partial^2 P(t, k-t)}{\partial r^2} dk + \frac{\eta^2}{2} \frac{\partial^2 P(t, T-t)}{\partial r^2} \right) dt \\
&\quad + \eta \left(c \int_t^T \frac{\partial P(t, k-t)}{\partial r} dk + \frac{\partial P(t, T-t)}{\partial r} \right) dZ_t.
\end{aligned}$$

Terms in dt can be simplified as

$$\begin{aligned}
E(dP_c(t, T)) &= c \left(\int_t^T \left(\frac{\partial P(t, k)}{\partial t} + \alpha(\beta - r_t) \frac{\partial P(t, k)}{\partial r} + \frac{\eta^2}{2} \frac{\partial^2 P(t, k)}{\partial r^2} \right) dk \right) dt \\
&\quad - cdt + \left(\frac{\partial P(t, T)}{\partial t} + \alpha(\beta - r_t) \frac{\partial P(t, T)}{\partial r} + \frac{\eta^2}{2} \frac{\partial^2 P(t, T)}{\partial r^2} \right) dt.
\end{aligned}$$

In absence of arbitrage, we have, for any $u > t$

$$\frac{\partial P(t, u-t)}{\partial t} + \alpha(\beta - r_t) \frac{\partial P(t, u-t)}{\partial r} + \frac{\eta^2}{2} \frac{\partial^2 P(t, u-t)}{\partial r^2} = r_t P(t, u-t).$$

Hence

$$\begin{aligned}
E(dP_c(t, T-t) + cdt) &= r_t c \left(\int_t^T P(t, k-t) dk \right) dt + r_t P(t, T-t) dt, \\
E\left(\frac{dP_c(t, T-t) + cdt}{P_c(t, T-t)} \right) &= r_t dt.
\end{aligned}$$

As for standard deviation, we obtain from equation (3)

$$\sigma \left(\frac{dP_c(t, T-t) + cdt}{P_c(t, T-t)} \right) = \frac{\eta}{P_c(t, T-t)} \left(c \int_t^T \frac{\partial P(t, k-t)}{\partial r} dk + \frac{\partial P(t, T-t)}{\partial r} \right).$$

Hence, from the definition of θ^{sto}

$$\sigma \left(\frac{dP_c(t, T-t) + cdt}{P_c(t, T-t)} \right) = \eta b(\theta^{sto}).$$

Dynamic strategy description

For the minimum variance duration, the next reinvestment horizon $\theta_{(i)}^*$ is calculated as $\arg \min_{\theta} [\sigma(R_{i, \theta})]$ where the value process of the dynamic immunization strategy is given

by

$$\pi_{i,\theta} = P_c(\theta, T - \theta) + c \int_{i\Delta t}^{\theta} \frac{ds}{P(s, \theta - s)} + \frac{acc_i}{P(i\Delta t, \theta - i\Delta t)}.$$

Note that the last term is known at date $i\Delta t$ and therefore does not interfere in the computation of $var_i(\pi_{i,\theta})$.

Similarly, the next reinvestment horizon $\theta_{(i)}$ for all other durations is calculated as

$$\theta_{(i)}^{mac} = \frac{c \int_{i\Delta t}^T (k - i\Delta t) e^{-y(k-i\Delta t)} dk + (T - i\Delta t) e^{-y(T-i\Delta t)}}{c \int_{i\Delta t}^T e^{-y(k-i\Delta t)} dk + e^{-y(T-i\Delta t)}},$$

where y is the continuously compounded yield-to-maturity on the coupon bond

$$\theta_{(i)}^{fw} = \frac{c \int_{i\Delta t}^T (k - i\Delta t) P(i\Delta t, k - i\Delta t) dk + (T - i\Delta t) P(i\Delta t, T - i\Delta t)}{c \int_{i\Delta t}^T P(i\Delta t, k - i\Delta t) dk + P(i\Delta t, T - i\Delta t)},$$

$$\theta_{(i)}^{sto} = b^{-1} \left(\frac{c \int_{i\Delta t}^T b (k - i\Delta t) P(i\Delta t, k - i\Delta t) dk + b (T - i\Delta t) P(i\Delta t, T - i\Delta t)}{c \int_{i\Delta t}^T P(i\Delta t, k - i\Delta t) dk + P(i\Delta t, T - i\Delta t)} \right),$$

The annualized return of the dynamic immunization strategy is then given by

$$R_{\Theta} = \frac{1}{\Theta} \frac{P_c(\Theta, T) + acc_{\Theta} - P_c(0, T)}{P_c(0, T)}.$$

Figures

Figure 1: Sets of parameters and their implied zero-coupon yield curves.

The three sets of parameters are:

Set	Shape	r_0	α	β	η
Ω_1	Increasing	0.05	0.3	0.07	0.03
Ω_2	Decreasing	0.05	0.3	0.04	0.03
Ω_3	Humped	0.05	0.1	0.07	0.03

and $\lambda = 0$ across the three sets.

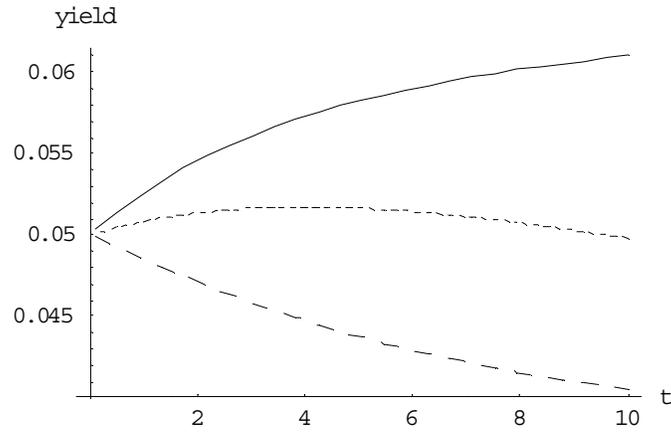


Figure 1 plots the zero-coupon yield curve corresponding to the set Ω_1 (straight line), Ω_2 (long-dashed line), and Ω_3 (short-dashed line).

**Figure 2: Variance decomposition
of the basic immunization strategy returns.**

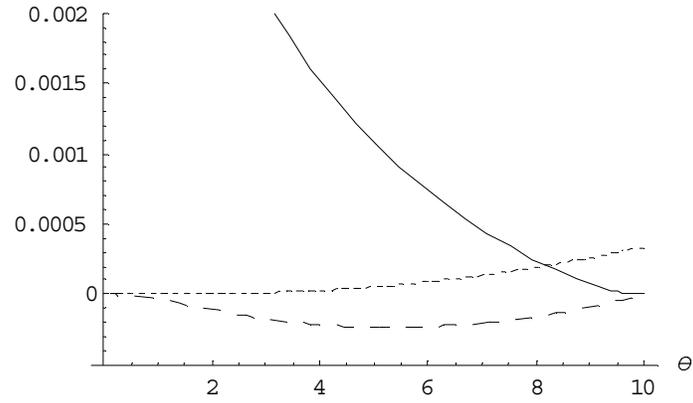


Figure 2 plots the total variance components of basic immunization strategy returns. The straight line plots the variance corresponding to resale price risk. The short-dashed line plots the variance corresponding to coupon reinvestment risk. The long-dashed line plots the covariance between the two risks. Interest rate environment is Ω_1 . Bond maturity is $T = 10$, coupon rate is $c = 0.1$.

**Figure 3: Basic immunization strategy returns
in the mean-standard deviation space.**

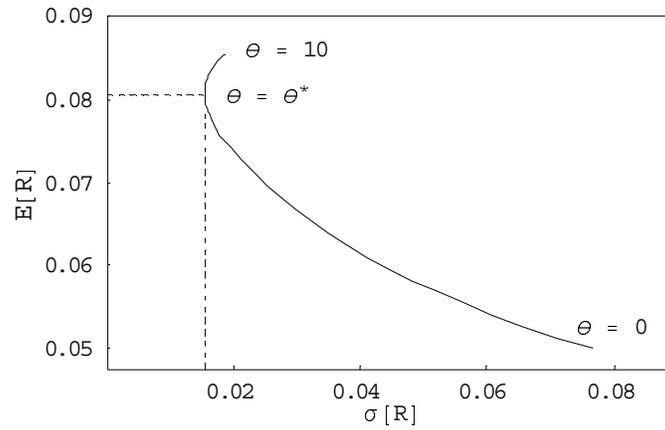


Figure 3 plots the returns of the basic immunization strategy in the mean-standard deviation space for every horizon θ . Interest rate environment is Ω_1 . Bond maturity is $T = 10$, coupon rate is $c = 0.1$.

**Figure 4: Basic immunization strategy returns
in the mean-standard deviation space
with different interest rate environments.**

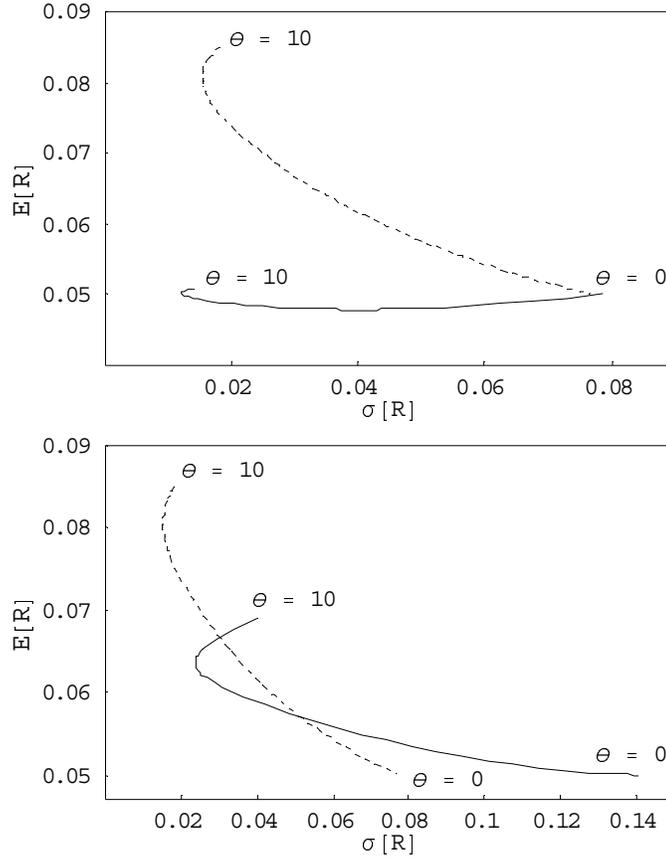


Figure 4 plots the returns of the basic immunization strategy in the mean-standard deviation space for every horizon θ . Bond maturity is $T = 10$, and coupon rate is $c = 0.1$. For the top figure, interest rate environment is Ω_1 (dashed line) and Ω_2 (straight line). For the bottom figure, interest rate environment is Ω_1 (dashed line) and Ω_3 (straight line).

**Figure 5: Basic immunization strategy returns
in the mean-standard deviation space
with different bond characteristics.**

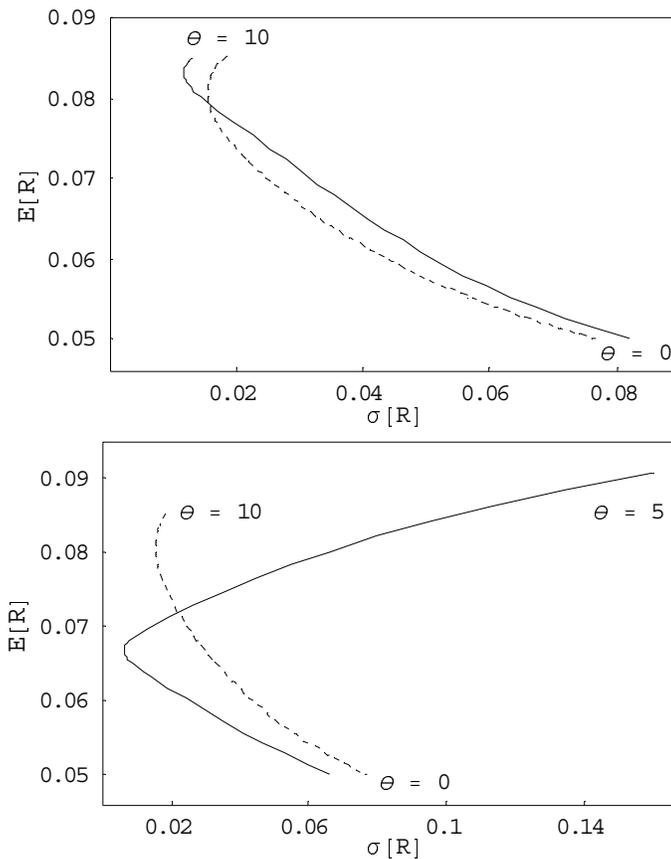


Figure 5 plots the returns of the basic immunization strategy in the mean-standard deviation space for every horizon θ . Interest rate parameters are Ω_1 . For the top figure, bond maturity is $T = 10$, and coupon rate is $c = 0.05$. For the bottom figure, bond maturity is $T = 5$, and coupon rate is $c = 0.1$.

**Figure 6: Duration-based strategies in the mean-standard deviation space:
The case for basic immunization strategy returns.**

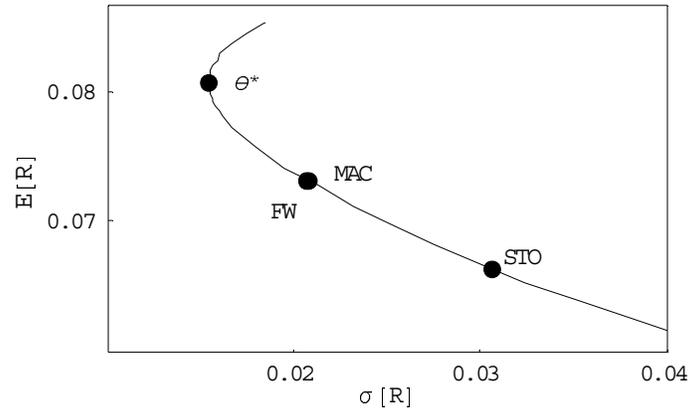


Figure 6 plots the returns of the basic immunization strategy in the mean-standard deviation space for every horizon θ . The respective positions of strategies based on traditional and stochastic durations are highlighted. Interest rate environment is Ω_1 . Bond maturity is $T = 10$, coupon rate is $c = 0.1$.

**Figure 7: Duration-based strategies in the mean-standard deviation space:
The case for dynamic immunization strategy returns.**

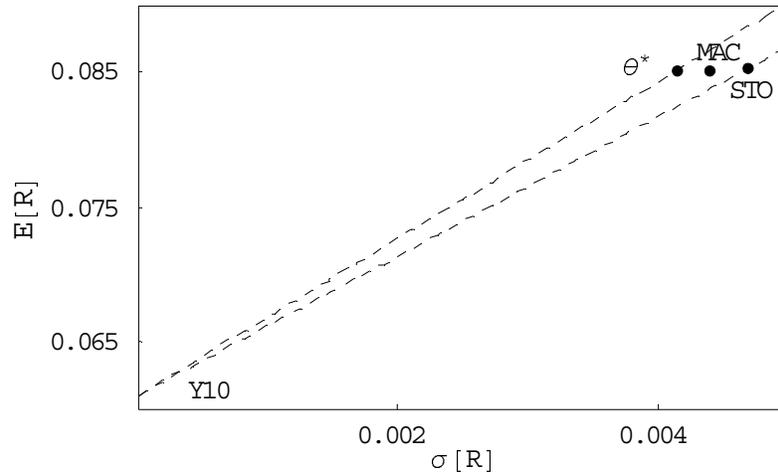


Figure 7 plots the returns of the dynamic immunization strategy in the mean-standard deviation space for strategies based on minimum variance, traditional and stochastic durations. The Sharpe ratio for each strategy is the slope of the dashed line with intercept equal to the ten-year yield. Interest rate environment is Ω_1 . Bond maturity is $T = 10$, coupon rate is $c = 0.05$.

Tables

Table 1: Mean-variance analysis of the basic immunization strategy:
Different interest rate environments.

	Min. var. duration θ^*	Traditional durations		Stochastic duration θ^{sto}
		Macaulay θ^{mac}	Fisher-Weil θ^{fw}	
Panel A: Interest rate parameters are Ω_1				
θ (years)	8.802	6.820	6.795	4.895
$E(R_\theta)$ (%)	8.062	7.326	7.316	6.642
$\sigma(R_\theta)$ (%)	1.538	2.055	2.066	3.059
Sharpe ratio	1.306	0.668	0.660	0.270
Panel B: Interest rate parameters are Ω_2				
θ (years)	9.102	7.043	7.070	5.167
$E(R_\theta)$ (%)	5.019	4.888	4.890	4.812
$\sigma(R_\theta)$ (%)	1.210	1.834	1.821	2.889
Sharpe ratio	0.765	0.367	0.371	0.156
Panel C: Interest rate parameters are Ω_3				
θ (years)	8.251	6.939	6.949	6.321
$E(R_\theta)$ (%)	6.366	6.011	6.014	5.859
$\sigma(R_\theta)$ (%)	2.375	3.314	3.302	4.131
Sharpe ratio	0.549	0.271	0.272	0.175

For various interest rate environments, Table 1 reports the duration (in years), the expected return (in percentage), the volatility (in percentage) and the Sharpe ratio of the associated basic immunization strategy. The Sharpe ratio is defined as $(E(R_\theta) - y(\theta)) / \sigma(R_\theta)$ where $y(\theta) = -(1/\theta) \ln P(0, \theta)$ is the zero coupon yield. Bond maturity is $T = 10$, and coupon rate is $c = 0.1$.

**Table 2: Mean-variance analysis of the basic immunization strategy:
Different maturities and 10% coupon.**

	Min. var.	Traditional durations		Stochastic
	duration	Macaulay	Fisher-Weil	duration
	θ^*	θ^{mac}	θ^{fw}	θ^{sto}
Bond maturity is $T = 1$				
θ (years)	0.984	0.953	0.953	0.948
$E(R_\theta)$ (%)	5.395	5.382	5.382	5.380
$\sigma(R_\theta)$ (%)	0.039	0.089	0.089	0.100
Sharpe ratio	3.560	1.498	1.497	1.324
Bond maturity is $T = 5$				
θ (years)	4.673	4.038	4.034	3.611
$E(R_\theta)$ (%)	6.674	6.433	6.431	6.273
$\sigma(R_\theta)$ (%)	0.614	1.085	1.090	1.595
Sharpe ratio	1.432	0.646	0.642	0.369
Bond maturity is $T = 10$				
θ (years)	8.802	6.820	6.795	4.895
$E(R_\theta)$ (%)	8.062	7.326	7.316	6.642
$\sigma(R_\theta)$ (%)	1.538	2.055	2.066	3.059
Sharpe ratio	1.306	0.668	0.660	0.270

For various bond maturities, Table 2 reports the duration (in years), the expected return (in percentage), the volatility (in percentage) and the Sharpe ratio of the associated basic immunization strategy. The Sharpe ratio is defined as $(E(R_\theta) - y(\theta)) / \sigma(R_\theta)$ where $y(\theta) = -(1/\theta) \ln P(0, \theta)$ is the zero coupon yield. Interest rate environment is Ω_1 . Coupon is $c = 0.1$.

**Table 3: Mean-variance analysis of the basic immunization strategy:
Different maturities and 5% coupon.**

	Min. var.	Traditional durations		Stochastic
	duration	Macaulay	Fisher-Weil	duration
	θ^*	θ^{mac}	θ^{fw}	θ^{sto}
Bond maturity is $T = 1$				
θ (years)	0.992	0.975	0.975	0.973
$E(R_\theta)$ (%)	5.398	5.391	5.391	5.390
$\sigma(R_\theta)$ (%)	0.021	0.049	0.049	0.055
Sharpe ratio	6.739	2.796	2.795	2.471
Bond maturity is $T = 5$				
θ (years)	4.847	4.411	4.408	4.083
$E(R_\theta)$ (%)	6.725	6.556	6.555	6.431
$\sigma(R_\theta)$ (%)	0.397	0.834	0.838	1.313
Sharpe ratio	2.305	0.942	0.937	0.529
Bond maturity is $T = 10$				
θ (years)	9.557	7.760	7.740	5.771
$E(R_\theta)$ (%)	8.315	7.624	7.617	6.909
$\sigma(R_\theta)$ (%)	1.172	2.080	2.094	3.331
Sharpe ratio	1.903	0.778	0.770	0.308

For various bond maturities, Table 2 reports the duration (in years), the expected return (in percentage), the volatility (in percentage) and the Sharpe ratio of the associated basic immunization strategy. The Sharpe ratio is defined as $(E(R_\theta) - y(\theta)) / \sigma(R_\theta)$ where $y(\theta) = -(1/\theta) \ln P(0, \theta)$ is the zero coupon yield. Interest rate environment is Ω_1 . Coupon is $c = 0.05$.

**Table 4: Mean-variance analysis of the dynamic immunization strategy:
Different maturities and 5% coupon.**

	Min. var. duration θ^*	Traditional durations		Stochastic duration θ^{sto}
		Macaulay θ^{mac}	Fisher-Weil θ^{fw}	
Bond maturity is $T = 1$				
$E(R_\theta)$ (%)	5.412	5.412	5.412	5.412
$\sigma(R_\theta)$ (%)	0.030	0.031	0.031	0.031
Sharpe ratio	4.982	4.942	4.942	4.938
Bond maturity is $T = 5$				
$E(R_\theta)$ (%)	6.811	6.812	6.812	6.813
$\sigma(R_\theta)$ (%)	0.233	0.241	0.241	0.245
Sharpe ratio	4.231	4.107	4.107	4.043
Bond maturity is $T = 10$				
$E(R_\theta)$ (%)	8.512	8.518	8.518	8.523
$\sigma(R_\theta)$ (%)	0.413	0.439	0.439	0.467
Sharpe ratio	5.845	5.506	5.509	5.188

For various bond maturities (in years), Table 4 reports the expected return (in percentage), the volatility (in percentage) and the Sharpe ratio of the associated dynamic immunization strategy. The Sharpe ratio is defined as $(E(R_\theta) - y(\theta)) / \sigma(R_\theta)$ where $y(\theta) = -(1/\theta) \ln P(0, \theta)$ is the zero coupon yield. Interest rate environment is Ω_1 . Coupon is $c = 0.05$.

**Table 5: Mean-variance analysis of the dynamic immunization strategy
under the Longstaff-Schwartz model:
Different maturities and 5% coupon.**

	Traditional durations		Stochastic
	Macaulay	Fisher-Weil	duration
	θ^{mac}	θ^{fw}	θ^{sto}
Bond maturity is $T = 1$			
$E(R_\theta)$ (%)	5.760	5.760	5.760
$\sigma(R_\theta)$ (%)	0.019	0.019	0.019
Sharpe ratio	9.290	9.290	9.201
Bond maturity is $T = 5$			
$E(R_\theta)$ (%)	7.166	7.166	7.169
$\sigma(R_\theta)$ (%)	0.151	0.151	0.153
Sharpe ratio	7.469	7.470	7.421
Bond maturity is $T = 10$			
$E(R_\theta)$ (%)	9.059	9.059	9.080
$\sigma(R_\theta)$ (%)	0.328	0.328	0.345
Sharpe ratio	8.517	8.527	8.162

For various bond maturities (in years), Table 5 reports the expected return (in percentage), the volatility (in percentage) and the Sharpe ratio of the associated dynamic immunization strategy. The Sharpe ratio is defined as $(E(R_\theta) - y(\theta)) / \sigma(R_\theta)$ where $y(\theta) = -(1/\theta) \ln P(0, \theta)$ is the zero coupon yield. Simulations are run under the Longstaff and Schwartz (1992) two-factor model. Coupon is $c = 0.05$.