Analysis of Financial Decision Making with Loss Aversion*

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Abstract

In this paper we consider a utility function that has a kink at the reference point and exhibits loss aversion. First we obtain an approximation of the expected utility of a loss averse decision maker. Then, in the spirit of Arrow and Pratt, we derive the expression for a risk premium. Finally we find an approximate solution to the optimal capital allocation problem and derive the expression for a portfolio performance measure. Our analysis generalizes the mean-variance utility of Tobin and Markowitz, the Arrow-Pratt measure of risk, and the Sharpe ratio. We show that a loss averse decision maker distinguishes between three sources of risk. Consequently, the characterization of the risk attitude of a loss averse decision maker involves three types of aversions, namely, aversion to loss, aversion to uncertainty in gains, and aversion to uncertainty in losses. We illustrate that if the decision maker exhibits a risk-seeking behavior in the domain for loss as in prospect theory, then neither the standard deviation nor the downside deviation can be used as a proper risk measure.

Key words: risk aversion, loss aversion, uncertainty aversion, risk measure, partial moments of distribution, optimal capital allocation, portfolio performance evaluation.

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1 Introduction

Expected utility theory of von Neumann and Morgenstern has long been the main workhorse of modern financial theory. A von Neumann and Morgenstern's utility function is defined over the decision maker's wealth. The properties of a von Neumann and Morgenstern's utility function have been studied in every detail. The concept of "risk aversion" was analyzed by Friedman and Savage (1948) and Markowitz (1952). They show that the realistic assumption of diminishing marginal utility of wealth explains why people are risk averse. Measurements of risk aversion was developed by Pratt (1964) and Arrow (1965). These authors analyze the risk premium in case risk is small and introduce a measure which is widely known now as the "Arrow-Pratt measure of risk aversion". The celebrated modern portfolio theory of Markowitz and the use of mean-variance utility function can be justified by approximating a von Neumann and Morgenstern's utility function by a function of mean and variance, see, for example Samuelson (1970) and Levy and Markowitz (1979). In this sense, the use of the Sharpe ratio (see Sharpe (1966)) as a measure of performance evaluation of risky portfolios is also well justified.

However, not very long ago after expected utility theory was formulated by von Neumann and Morgenstern (1944) questions were raised about its value as a descriptive model of choice under uncertainty. Allais (1953) and Ellsberg (1961) were among the first to challenge expected utility theory. Influential experimental studies have shown the inability of expected utility theory to explain many phenomena and reinforced the need to rethink much of the theory. Kahneman and Tversky (1979) propose an alternative descriptive model of choice under uncertainty that they call *prospect theory*. Prospect theory can predict correctly individual choices even in the cases in which expected utility theory is violated (for a brief description see, for example, Camerer (2000)). In prospect theory, utility function is defined over gains and losses relative to some reference point, as opposed to wealth in expected utility theory. The utility function has a kink at the origin, with the slope of the loss function steeper than the gain function. This is what is called *loss aversion* which is an important element of prospect theory. The marginal value of both gains and losses decreases with their size. All these properties give rise to an asymmetric S-shaped utility function, concave for gains and convex for losses. Moreover, in prospect theory the decision maker transforms the objective probability distribution into a subjective probability distribution.

The utility function of prospect theory was inspired by the results of numerous experimental studies where people were asked to make a choice among a few alternatives. In this sense, this utility is a behavioral utility which was obtained by "calibrating" the results of experiments to a utility function with a loose parametric specification. However, very little is known about general implications of this behavioral utility for financial decision making. What are the principal differences, if any, between the risk attitudes of a decision maker in prospect theory and a decision maker in expected utility theory? Up to now there are only some guesses. For examples, one usually believes that loss aversion in prospect theory plays the role of risk aversion in expected utility theory (see Bernartzi and Thaler (1995), pages 85-86) and investors are not averse to the variability of returns, only to losses (see Camerer (2000), page 290). If we approximate the expected utility in prospect theory, what will distinguish it with the mean-variance approximation of the von Neumann-Morgenstern expected utility? Moreover, to the best of the authors knowledge, no one has studied the implications of behavioral utility for the standard capital allocation problem and comparison of performances of different risky portfolios. The solutions of these problems involve not just the choice of the best alternative among several ones, but the search of the optimal decision in the continuum of possible choices. The goal of this paper is to provide some answers to these questions as well as some new insights on the decision maker's attitude toward risk and risk measurement.

In this paper we consider a generalized behavioral utility function. This utility has a kink at the reference point and different functions for losses and gains. We require only that the behavioral utility function is increasing and exhibits loss aversion. The first contribution of this paper is to obtain an approximation of the expected utility of a loss averse decision maker. Our approximation of the expected (behavioral) utility generalizes the Tobin-Markowitz mean-variance approximation of the von Neumann-Morgenstern expected utility. We show that in contrast to a decision maker with the mean-variance utility for whom the only source of risk is the variance, a loss averse decision maker distinguishes between three sources of risk: the expected loss, the uncertainty in losses, and the uncertainty in gains. A decision maker with loss aversion puts more weight to the uncertainty in losses than to the uncertainty in gains.

The second contribution of this paper is, in the spirit of Pratt (1964) and Arrow (1965), to derive an expression for a risk premium in case risk is small. We show that a utility function with loss aversion allows a much richer and detailed characterization of a risk premium. For a decision maker with a von Neumann-Morgenstern utility the risk premium is completely described by the Arrow-Pratt measure of risk aversion and the variance, which serves as a measure of risk. Since a decision maker with a von Neumann-Morgenstern utility does not exhibit loss aversion (that is, this decision maker treats equally infinitesimal losses and gains), we argue that the Arrow-Pratt measure of risk aversion should be properly denoted as the measure of "aversion to uncertainty". In contrast, our analysis shows that a decision maker with a behavioral utility exhibits three types of aversions: aversion to loss, aversion to uncertainty in gains, and aversion to uncertainty in losses (if the function for losses is concave, otherwise, if the function is convex as, for example, in prospect theory, a decision maker appreciates the uncertainty in losses). Uncertainties in gains and losses are measured by the second upper and lower partial moments of the probability distribution of risk. Besides the second partial moments, to compute the risk premium one needs to compute the first upper and lower partial moments which measure the expected gain and loss. The presence of loss aversion results in the fact that losses and gains have different weights in the computation of the risk premium.

The third contribution of this paper is to derive an approximate solution to the optimal capital allocation problem of an investor. In this setting the investor wants to allocate the wealth between a risk-free and a risky asset. It is well known that in the framework of expected utility theory the investor's utility function in such an analysis reduces to mean-variance utility function with a single measure of risk aversion. We show that a behavioral utility function reduces in this case to a "mean-partial moments" utility function with three sources of risk and, consequently, three measures of risk aversion. These measures of risk aversion are the same as in the expression for the risk premium of a decision maker with behavioral utility: one loss aversion and two uncertainty aversions. It is widely known that a mean-variance utility maximizer will always want to allocate some wealth to the risky asset if the risk premium is non-zero, no matter how small it might be. We discover here that a loss averse investor will want to allocate some wealth to the risky asset only when the perceived risk premium is sufficiently high (how high depends on the level of loss aversion). Otherwise, if the risk premium is small, a loss averse investor will invest only in the risk-free asset. This result has a potential to explain for why many investors do not invest in equities (see, for example, Agnew, Balduzzi, and Annika (2003) who report that about 48% of participants of retirement accounts do not invest in equities), this behavior clearly contradicts expected utility theory.

Our fourth contribution is to derive an expression for the performance measure of a loss averse investor. Instead of the mean and variance in the Sharpe ratio, in the performance measure of a loss averse investor one needs to use the (upper and lower) first and second partial moments of the return distribution of a risky asset. We show that in some specific cases the performance measure of an investor with behavioral utility reduces to the performance measure suggested by Sortino and Price (1994) and Ziemba (2005) who replace standard deviation in the Sharpe ratio by downside deviation. As compared with the Sharpe ratio where the investor's risk preferences completely disappear, to compute the performance measure of a loss averse investor one generally needs to take into account the investor's preferences. This means that this performance measure is not unique for all investors, but rather an individual performance measure. The explanation for this is the fact that a loss averse investor distinguishes between several sources of risk. Since each investor may exhibit different preferences to each source of risk, investors with different preferences might rank differently the same set of risky assets. In the paper we present a couple of examples that illustrate this.

The fifth contribution of this paper is to provide some new insights on risk measurement. In modern financial theory most often one uses the standard deviation as a risk measure. This is justified by the mean-variance analysis of the von Neumann-Morgenstern expected utility provided by Tobin and Markowitz. Our approximation results justify the use of downside deviation as a risk measure in some cases. It is worth noting that the idea of measuring the risk by downside deviation was expressed already by Markowitz (1959). In particular, Markowitz also proposes to use the (downside) semi-variance as an alternative measure for risk. Semi-variance is a deviation measure that differs from ordinary variance in one aspect, namely, it considers only returns below some target level. Technically, aggregating semi-variances from assets to portfolios is extremely difficult. That is probably why this idea was not pursued further. Afterwards the notion of a downside semi-variance was generalized by Fishburn (1977) and Bawa (1978) who introduce the notion of a lower partial moment as a risk measure. However, we discover that if the investor is equipped with the S-shaped utility function as in prospect theory, then neither the standard deviation nor the downside deviation can be used as a proper risk measure. This is the consequence of the fact that the investor with the S-shaped utility exhibits uncertainty-loving behavior with respect to losses (because the function for losses is convex). That is, such an investor appreciates uncertainty in losses. Usually, downside deviation serves as a risk measure meaning that the higher the downside deviation the greater the risk. But for an investor with a convex utility for losses the higher the downside deviation the lesser the risk. In this paper we present two examples that illuminate the point that a decision maker/investor with the S-shaped utility may prefer a more riskier (in a usual sense) lottery/asset to a less riskier lottery/asset.

The rest of the paper is organized as follows. In Section 2 we provide definitions and introduce notation. In Section 3 we perform the approximation of the decision maker's expected utility. In Section 4 we analyze the risk premium of a loss averse decision maker. In Section 5 we analyze the optimal capital allocation problem of a loss averse investor and derive the expression for a portfolio performance measure. Section 6 concludes the paper.

2 Preliminaries

Utility function. A von Neumann-Morgenstern utility function is defined over the decision maker's wealth, w, as a single function $U(w)$. In contrast, in prospect theory a utility always has a reference point, w_0, with respect to which one defines losses and gains, see, for example, Kahneman and Tversky (1979). That is, the utility function is defined as

$$U(w) = \begin{cases} U_+(w - w_0) & \text{if } w \geq w_0, \\ U_-(w - w_0) & \text{if } w < w_0, \end{cases}$$

where $U_-(\cdot)$ is the utility function for losses, and $U_+(\cdot)$ is the utility function for gains. In the framework of prospect theory, decision makers are more sensitive to losses than to gains. This implies that the utility function of a loss averse decision maker should be steeper for losses than for gains. Observe that $U(w)$ has a kink at w_0. Observe also that at the reference point w_0 the utility function is zero and, hence, by the continuity condition

$$U(w_0) = U_-(0) = U_+(0) = 0. \tag{1}$$

Moreover, in prospect theory the utility function $U(w)$ is concave for gains (that is, $U_+(w - w_0)$ is concave) to reflect risk aversion, but $U(w)$ is convex for losses (that is, $U_-(w - w_0)$ is convex) to reflect risk seeking. In our analysis we will also consider the case where $U(w)$ is concave for both losses and gains.

Reference point. The current level of the decision maker's wealth (the so-called "status quo") serves usually as the reference point w_0. However, as Khaneman and Tversky point out "gains and losses can be coded relative to an expectation or aspiration level that differs from the status quo" (see Kahneman and Tversky (1979) page 286).

Loss Aversion. Denote the left-sided derivative of $U(w)$ at point w_0 by[1] $U'_-(0)$. That is,

$$U'_-(0) = \lim_{w \to w_0 -} \frac{U_-(w) - U_-(w_0)}{w - w_0}.$$

Similarly, denote the right-sided derivative of $U(w)$ at point w_0 by $U'_+(0)$

$$U'_+(0) = \lim_{w \to w_0 +} \frac{U_+(w) - U_+(w_0)}{w - w_0}.$$

In our analysis we assume that both $U'_-(0)$ and $U'_+(0)$ exist and are positive and finite. Loss aversion does only hold when

$$U'_-(0) > U'_+(0).$$

The measure of loss aversion is given by

$$\lambda = \frac{U'_-(0)}{U'_+(0)}. \tag{2}$$

[1] To be more precise, we need to denote the left derivative as $U'_-(0-)$, but this would enlarge the notation.

This measure of loss aversion was proposed by Bernartzi and Thaler (1995) and formalized by Köbberling and Wakker (2005). Observe that loss aversion implies $\lambda > 1$.

Transformation of probability distribution. In prospect theory one uses not the objective probability distribution, but a transformation of the objective probability distribution. In prospect theory of Kahneman and Tversky (1979), which is applicable only to cases with a discrete probability distribution, the objective probability p of an outcome is replaced with a transformed probability $\varphi(p)$ also known as a decision weight. The weighting function $\varphi(\cdot)$ overweights low probabilities and underweights high probabilities. In cumulative prospect theory of Tversky and Kahneman (1992) the transformed probability distribution function $q(x)$ of a random variable x is obtained using the objective cumulative probability distribution function $F(x)$ in the following manner

$$q(x) = \frac{d}{dx}\varphi\left(F(x)\right).$$

In our analysis we denote by $Q(x)$ the cumulative probability distribution function of random variable x. This cumulative distribution function represents the beliefs of a decision maker. That is, it is generally a transformed distribution function. The distinction between the objective and the subjective beliefs does not influences the results presented in this paper. However, the reader should keep in mind this distinction and realize that the computation of the partial moments of the distribution of x might be done in the subjective world of a decision maker.

Lower and upper partial moments. Fishburn (1977) and Bawa (1978) introduced the notion of a *lower partial moment* as a measure of risk. Suppose that x is some random variable and $Q(x)$ is the cumulative distribution function of x. The definition of a lower partial moment of order n at some level l

$$LPM_n(x,l) = \int_{-\infty}^{l} (l-x)^n dQ(x).$$

Observe that the integral above is a Lebesque-Stieltjes integral. Similarly, we define an *upper partial moment* of order n at some level l

$$UPM_n(x,l) = \int_{l}^{\infty} (x-l)^n dQ(x).$$

Examples of utility functions with loss aversion. A possible general form of a utility function with loss aversion is proposed by Köbberling and Wakker (2005)

$$U(w) = \begin{cases} u(w-w_0) & \text{if } w \geq w_0, \\ -\lambda u(w_0 - w) & \text{if } w < w_0. \end{cases}$$

Here there is a single function u for both losses and gains. Parameter $\lambda > 1$ makes the utility function U be steeper for losses than for gains.

Kahneman and Tversky (1979) propose the following utility function

$$U(w) = \begin{cases} (w - w_0)^\alpha & \text{if } w \geq w_0, \\ -\lambda(w_0 - w)^\beta & \text{if } w < w_0. \end{cases} \tag{3}$$

This function has extreme (0 or ∞) derivatives at w_0 whenever the powers α and β are not 1. This complicates the definition of loss aversion. Khaneman and Tversky estimate the parameters to be: $\alpha = \beta = 0.88$ and $\lambda = 2.25$. In this exceptional case, the value of the parameter λ can still be interpreted as a measure of loss aversion.

The utility function of Khaneman and Tversky is motivated by power utility. Köbberling and Wakker (2005) propose the following utility function which is motivated by negative exponential utility

$$U(w) = \begin{cases} \frac{1-e^{-\gamma(w-w_0)}}{\gamma} & \text{if } w \geq w_0, \\ \lambda\left(\frac{e^{\eta(w-w_0)}-1}{\eta}\right) & \text{if } w < w_0. \end{cases}$$

This function does not encounter the problem with the existence of derivatives at w_0. In this function $\gamma > 0$ controls the concavity of utility for gains, η controls the convexity (if $\eta > 0$) of utility for losses, and λ is loss aversion.

3 Approximation of the Decision Maker's Expected Utility

In this paper we perform the approximation analysis of financial decision making with a generalized behavioral utility function. This utility has a kink at the reference point and different functions for losses and gains. We require only that the behavioral utility function is increasing in wealth and exhibits loss aversion. Our approximation analysis is based on the following technique. Suppose that w is the decision maker's random wealth, $Q(w)$ is the cumulative distribution function of w, and w_0 is the reference point. Then the decision maker's expected utility is given by

$$E[U(w)] = \int_{-\infty}^{w_0} U_-(w - w_0)dQ(w) + \int_{w_0}^{\infty} U_+(w - w_0)dQ(w).$$

We apply Taylor series expansions for $U_-(w-w_0)$ and $U_+(w-w_0)$ around 0 which yields

$$
\begin{aligned}
E[U(w)] &= \int_{-\infty}^{w_0} \left(\sum_{n=1}^{\infty} \frac{1}{n!} U_-^{(n)}(0)(w-w_0)^n \right) dQ(w) \\
&\quad + \int_{w_0}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{n!} U_+^{(n)}(0)(w-w_0)^n \right) dQ(w) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} U_-^{(n)}(0) \int_{-\infty}^{w_0} (-1)^n (w_0-w)^n dQ(w) \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{n!} U_+^{(n)}(0) \int_{w_0}^{\infty} (w-w_0)^n dQ(w) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} U_-^{(n)}(0)(-1)^n LPM_n(w,w_0) + \sum_{n=1}^{\infty} \frac{1}{n!} U_+^{(n)}(0) UPM_n(w,w_0),
\end{aligned}
\tag{4}
$$

where $U^{(n)}$ denotes the nth derivative of U and supposing that the Taylor series converge and the integrals exist. Observe that the summations in the equation above start with $n=1$ due to the fact that the utility is zero at the reference point (see property (1)). To simplify expression (4) we assume that

$$w = w_0 + x,$$

where x is a random variable whose probability distribution belongs to the family of "compact" or "small risk" distributions (for the definition, see, for example, Samuelson (1970)). This allows us to assume that all the terms in (4) with $LPM_3(w,w_0)$, $UPM_3(w,w_0)$, and other higher partial moments are of smaller order than the second partial moments. If we neglect all the lower and upper partial moments of higher order than 2, then

$$E[U(w)] \approx \sum_{n=1}^{2} \frac{1}{n!} U_-^{(n)}(0)(-1)^n LPM_n(w,w_0) + \sum_{n=1}^{2} \frac{1}{n!} U_+^{(n)}(0) UPM_n(w,w_0). \tag{5}$$

Since utility functions are equivalent up to a positive linear transformation, we find an equivalent expected utility by dividing the left- and right-hand sides of (5) by $U'_+(0) > 0$. This yields

$$
\begin{aligned}
E[\widehat{U}(w)] \approx &-\frac{U'_-(0)}{U'_+(0)} LPM_1(w,w_0) + \frac{1}{2} \frac{U''_-(0)}{U'_-(0)} \frac{U'_-(0)}{U'_+(0)} LPM_2(w,w_0) \\
&+ UPM_1(w,w_0) + \frac{1}{2} \frac{U''_+(0)}{U'_+(0)} UPM_2(w,w_0),
\end{aligned}
\tag{6}
$$

where $\widehat{U}(\cdot)$ is the equivalent utility function. Note that $\frac{U'_-(0)}{U'_+(0)} = \lambda$ by the definition of loss aversion (2). Denote in addition

$$
\begin{aligned}
\gamma_- &= -\frac{U''_-(0)}{U'_-(0)}, \\
\gamma_+ &= -\frac{U''_+(0)}{U'_+(0)}.
\end{aligned} \tag{7}
$$

Using this notation, the equivalent expected utility can be written as

$$
\begin{aligned}
E[\widehat{U}(w)] \approx\ & UPM_1(w, w_0) - \frac{1}{2}\gamma_+ UPM_2(w, w_0) \\
& - \lambda \left(LPM_1(w, w_0) + \frac{1}{2}\gamma_- LPM_2(w, w_0) \right).
\end{aligned} \tag{8}
$$

The question now is how to interpret correctly the values of γ_- and γ_+. It is tempting to interpret these values, in the spirit of Arrow and Pratt, as measures of risk aversions to losses and gains respectively. However, it is hard to make economic sense of the expression "risk aversion to gains". We argue that γ_- should be interpreted as the measure of *aversion to uncertainty in losses*. Similarly, γ_+ should be interpreted as the measure of *aversion to uncertainty in gains*.

Indeed, if the decision maker has a von Neumann-Morgenstern utility, then $\lambda = 1$ and $\gamma_- = \gamma_+ = \gamma = -\frac{U''(w_0)}{U'(w_0)}$. In this case the equivalent utility (8) reduces largely to the mean-variance utility (the Tobin-Markowitz approximation of the expected utility)

$$
E[\widehat{U}(w)] \approx E[w - w_0] - \frac{1}{2}\gamma \mathrm{Var}[w - w_0], \tag{9}
$$

if, for example, either $E[x] = 0$ or x is Wiener's Brownian motion where the mean and variance grow linearly with time interval Δt, and Δt is small (the illustrations will be provided in the subsequent sections).

As compared with a von Neumann-Morgenstern utility where the decision maker's risk attitude is completely described by only one parameter, the aversion to uncertainty, with a loss averse utility the decision maker's attitude toward risk needs to be described by three parameters. To illuminate the difference between (8) and (9), observe that

$$
E[w - w_0] = UPM_1(w, w_0) - LPM_1(w, w_0).
$$

Thus, the equivalent utility (8) can be rewritten as

$$
\begin{aligned}
E[\widehat{U}(w)] \approx\ & E[w - w_0] - (1 - \lambda) LPM_1(w, w_0) \\
& - \frac{1}{2}\gamma_+ UPM_2(w, w_0) - \frac{1}{2}\lambda\gamma_- LPM_2(w, w_0).
\end{aligned} \tag{10}
$$

In contrast to the decision maker with the mean-variance utility for whom the only source of risk is the variance, the decision maker with loss aversion distinguishes between three sources of risk: the lower partial moment of order 1 which is related to the expected loss, the lower partial moment of order 2 which is related to the uncertainty in losses, and the upper partial moment of order 2 which is related to the uncertainty in gains. Observe that a decision maker with loss aversion puts more weight to the uncertainty in losses than to the uncertainty in gains.

Observe that the results of our approximation of the decision maker's expected utility (either (8) or (10)) are exact irrespective of the distribution of x when both the loss and gain functions are quadratic. To illustrate our results and highlight the differences in decision making for different shapes of the decision maker's utility function, we will employ the following utility

$$U(w) = \begin{cases} (w - w_0) - \frac{1}{2}\gamma_+(w - w_0)^2 & \text{if } w \geq w_0, \\ \lambda\left((w - w_0) - \frac{1}{2}\gamma_-(w - w_0)^2\right) & \text{if } w < w_0. \end{cases} \tag{11}$$

In this function $\gamma_+ \geq 0$ controls the concavity of utility for gains, whereas γ_- controls either the concavity (if $\gamma_- > 0$) or the convexity (if $\gamma_- < 0$) of utility for losses. Some possible shapes of this utility function are presented in Figure 1.

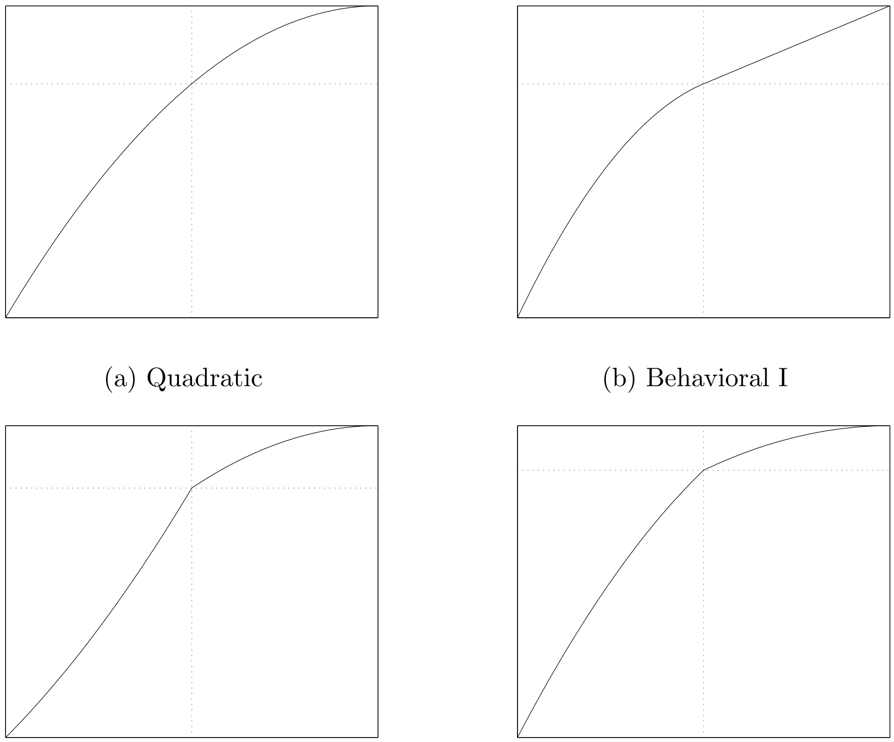
Quadratic: The shape of this utility is given by $\lambda = 1$ and $\gamma_- = \gamma_+ = \gamma > 0$. This utility largely corresponds to the quadratic utility in the expected utility theory framework. Note that this function is increasing in the interval

$$w - w_0 < \frac{1}{\gamma}.$$

Behavioral I: The shape of this utility is given by $\lambda = 1$, $\gamma_- > 0$, and $\gamma_+ = 0$. This is the utility function of Fishburn (1977) and Bawa (1978) where one uses the lower partial moment of order 2. The decision maker with this utility exhibits no loss aversion, risk neutrality in the domain for gains, and risk aversion in the domain for losses. Note that this function is increasing for all values of $w - w_0$.

Behavioral II: The shape of this utility is given by $\lambda > 1$, $\gamma_- < 0$, and $\gamma_+ > 0$. This utility largely corresponds to the utility function in prospect theory. The decision maker with this utility exhibits loss aversion, risk aversion in the domain for gains, and risk seeking in the domain for losses. Note that this function is increasing in the interval

$$\frac{1}{\gamma_-} < w - w_0 < \frac{1}{\gamma_+}.$$



(a) Quadratic (b) Behavioral I

(c) Behavioral II (d) Behavioral III

Figure 1: Alternative shapes of the utility function.

Behavioral III: The shape of this utility is given by $\lambda > 1$, $\gamma_- > 0$, and $\gamma_+ > 0$. The decision maker with this utility exhibits loss aversion, and risk aversion in the domains for losses and gains. Observe that the difference in the shapes of the Behavioral II and Behavioral III utilities results from the different signs of γ_-. Note that this function is increasing in the interval

$$w - w_0 < \frac{1}{\gamma_+}.$$

4 Risk Premium with Loss Aversion

In this section we consider a risk averse decision maker with (deterministic) wealth w and a utility function U which exhibits loss aversion. Let x be some uncertain amount. Since the decision maker is risk averse, then

$$U(w + E[x]) > E[U(w + x)].$$

This says that for a risk averse decision maker the expected utility of $w+x$ is less than the utility of $w+E[x]$. Observe that this is the most general definition of risk aversion. It is a standard in economics to characterize the risk aversion in terms of the risk premium, π, which is defined by the following indifference condition

$$U(w+E[x]-\pi) = E[U(w+x)]. \tag{12}$$

This says that the decision maker is indifferent between receiving x and receiving a non-random amount of $E[x]-\pi$. In the spirit of Pratt (1964) we will derive an expression for the risk premium π. A Taylor series expansion for $U(w+E[x]-\pi)$ around the reference point w_0 gives us

$$U(w+E[x]-\pi) = \sum_{n=0}^{\infty} \frac{1}{n!} U^{(n)}(w_0)(w+E[x]-\pi-w_0)^n. \tag{13}$$

Observe that $U^{(n)}(w_0)$ does not exist in a strict sense since the utility function U is not differentiable at the kink point. It is either $U_-^{(n)}(0)$ or $U_+^{(n)}(0)$ depending on the sign of $w+E[x]-\pi-w_0$. Now consider

$$\begin{aligned} E[U(w+x)] &= \int_{-\infty}^{\infty} U(w+x)dQ(x) \\ &= \int_{-\infty}^{w_0-w} U_-(w+x-w_0)dQ(x) + \int_{w_0-w}^{\infty} U_+(w+x-w_0)dQ(x), \end{aligned}$$

where $Q(x)$ is the cumulative probability distribution function of x. Taylor series expansions for $U_-(w+x-w_0)$ and $U_+(w+x-w_0)$ around 0 give

$$\begin{aligned} E[U(w+x)] &= \int_{-\infty}^{w_0-w} \left(\sum_{n=0}^{\infty} \frac{1}{n!} U_-^{(n)}(0)(w+x-w_0)^n \right) dQ(x) \\ &\qquad + \int_{w_0-w}^{\infty} \left(\sum_{n=0}^{\infty} \frac{1}{n!} U_+^{(n)}(0)(w+x-w_0)^n \right) dQ(x) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} U_-^{(n)}(0) \int_{-\infty}^{w_0-w} (-1(w_0-w-x))^n dQ(x) \\ &\qquad + \sum_{n=0}^{\infty} \frac{1}{n!} U_+^{(n)}(0) \int_{w_0-w}^{\infty} (x-(w_0-w))^n dQ(x) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} U_-^{(n)}(0)(-1)^n LPM_n(x, w_0-w) \\ &\qquad + \sum_{n=0}^{\infty} \frac{1}{n!} U_+^{(n)}(0) UPM_n(x, w_0-w). \end{aligned} \tag{14}$$

Now suppose that $E[x] = 0$, that is, x is a pure risk, and the reference point is $w_0 = w$, that is, the "status quo". Since the decision maker dislikes risk, the risk premium is positive, $\pi > 0$, and, thus, $U^{(n)}(w_0) = U_-^{(n)}(0)$ in equation (13). As in the seminal paper of Pratt (1964) we assume that the risk is small that allows us to neglect all the terms in (13) with π^2 and higher powers of π. Moreover, since the risk is small, the probability distribution of x belongs to the family of "compact" or "small risk" distributions. This allows us to neglect all the lower and upper partial moments of higher order than 2. More formally, we suppose that

$$U(w + E[x] - \pi) \approx -U'_-(0)\pi,$$

and

$$E[U(w+x)] \approx \sum_{n=1}^{2} \frac{1}{n!} U_-^{(n)}(0)(-1)^n LPM_n(x,0) + \sum_{n=1}^{2} \frac{1}{n!} U_+^{(n)}(0) UPM_n(x,0).$$

Given these assumptions, the indifference equation (12) becomes

$$-U'_-(0)\pi = \sum_{n=1}^{2} \frac{1}{n!} U_-^{(n)}(0)(-1)^n LPM_n(x,0) + \sum_{n=1}^{2} \frac{1}{n!} U_+^{(n)}(0) UPM_n(x,0). \tag{15}$$

To shorten the subsequent notation, denote

$$\begin{aligned} \mu_- &= LPM_1(x,0), \\ \mu_+ &= UPM_1(x,0), \\ \sigma_-^2 &= LPM_2(x,0), \\ \sigma_+^2 &= UPM_2(x,0). \end{aligned}$$

Then the solution for the risk premium, π, can be written as follows

$$\pi = \mu_- - \frac{1}{2}\frac{U''_-(0)}{U'_-(0)}\sigma_-^2 - \frac{U'_+(0)}{U'_-(0)}\mu_+ - \frac{1}{2}\frac{U''_+(0)}{U'_+(0)}\frac{U'_+(0)}{U'_-(0)}\sigma_+^2.$$

Note that $\frac{U'_+(0)}{U'_-(0)} = \frac{1}{\lambda}$ by the definition of loss aversion (2). Using the same notation as in (7), the expression for the risk premium can be written

$$\pi = \left(\mu_- + \frac{1}{2}\gamma_-\sigma_-^2\right) - \frac{1}{\lambda}\left(\mu_+ - \frac{1}{2}\gamma_+\sigma_+^2\right). \tag{16}$$

Observe that if the decision maker has a von Neumann-Morgenstern utility, then $\lambda = 1$ and $\gamma_- = \gamma_+ = \gamma = -\frac{U''(w_0)}{U'(w_0)}$. In this case the equation for π reduces to the famous result of Arrow-Pratt

$$\pi = \frac{1}{2}\gamma\sigma_-^2 + \frac{1}{2}\gamma\sigma_+^2 = \frac{1}{2}\gamma\sigma^2, \tag{17}$$

since $\mu_- = \mu_+$ and $\sigma_-^2 + \sigma_+^2 = \sigma^2$ (due to $E[x] = 0$). Since a decision maker with a von Neumann-Morgenstern utility exhibits no loss aversion, both (infinitesimal) losses and gains are treated similarly as it is clearly seen from equation (17). Consequently, the Arrow-Pratt measure of risk aversion is a measure of aversion to variance, or a measure of *aversion to uncertainty*. In other words, when risk is small, a decision maker with a von Neumann-Morgenstern utility exhibits only aversion to uncertainty. In addition, when risk is small, the risk premium is fully characterized by a measure of uncertainty aversion and the variance, which is a measure of uncertainty itself.

A utility function with loss aversion allows a much richer and detailed characterization of risk aversion. According to equation (16) a decision maker exhibits three different types of aversion: aversion to loss, aversion to uncertainty in gains, and aversion to uncertainty in losses. Uncertainties in gains and losses are measured by the second upper partial moment of x and the second lower partial moment of x respectively. If $LPM_3(x, 0)$, $UPM_3(x, 0)$, and higher partial moments of x are not of smaller order than the second partial moments, we need all the partial moments of x to characterize the risk premium. If $\pi^2 \ll \pi$ and we keep all the partial moments of x in equation (14), then the expression for the risk premium becomes

$$\pi = \left(\mu_- + \sum_{n=2}^{\infty} \frac{1}{n!} \frac{U_-^{(n)}(0)}{U_-'(0)} (-1)^n LPM_n(x, 0)\right) \\ - \frac{1}{\lambda}\left(\mu_+ - \sum_{n=2}^{\infty} \frac{1}{n!} \frac{U_+^{(n)}(0)}{U_+'(0)} UPM_n(x, 0)\right).$$

In this case $\frac{U_-^{(n)}(0)}{U_-'(0)}$ must be interpreted as a measure of aversion to the nth lower partial moment, and $\frac{U_+^{(n)}(0)}{U_+'(0)}$ as a measure of aversion to the nth upper partial moment.

Finally we present a brief comparative static analysis of the expression for the risk premium (16). First observe that since for a loss averse decision maker $\frac{1}{\lambda} < 1$, losses and gains have different weights in the computation of the risk premium. In particular, losses are λ times more important than gains. In the limit as $\lambda \to \infty$

$$\lim_{\lambda \to \infty} \pi = \mu_- + \frac{1}{2}\gamma_- \sigma_-^2,$$

which means that the upper partial moments become irrelevant for the computation of the risk premium. Then note that generally $\gamma_- \neq \gamma_+$, that is, the decision maker has different degrees of uncertainty aversions to losses and gains. As we suppose that the utility function U is always concave for gains, thus $\gamma_+ > 0$ which means that the greater the aversion to uncertainty in gains, the higher the risk premium. If the utility function is also concave for losses, then $\gamma_- > 0$ which means that the greater the aversion to uncertainty in losses, the higher the risk premium. In prospect theory, however, the utility function U is convex for losses, thus $\gamma_- < 0$. In this case the decision maker actually appreciates the uncertainty in losses and the higher the absolute value of γ_-, the lesser the risk premium. If the decision maker is completely neutral to uncertainties, then $\gamma_- = \gamma_+ = 0$ and the risk premium is computed in accordance

$$\pi = \mu_- - \frac{1}{\lambda}\mu_+.$$

Note that we have conducted the analysis assuming that $E[x] = 0$ and $w_0 = w$. This is a particular case of a more general assumption $w_0 = w + E[x]$. Observe that our result (16) remains valid for $w_0 = w + E[x]$ when the computation of the lower and upper moments is done with respect to the level $l = E[x]$. Recall that w does not need to be the reference point. The choice of $w_0 = w + E[x]$ is sensible in the meaning that for any $a > 0$ the lower and upper partial moments of the uncertain amount $y = ax$ have a very nice homogeneity property

$$LPM_n(y, E[y]) = a^n LPM_n(x, E[x]),$$
$$UPM_n(y, E[y]) = a^n UPM_n(x, E[x]).$$

If, for example, the risk is measured by a lower partial moment (as, for example, in Behavioral I utility function), it is natural to require that the risk measure satisfies the homogeneity property, see, for example[2], Artzner et al. (1999) or Rockafellar et al. (2006). Observe that if a decision maker has a von Neumann-Morgenstern utility and a Taylor series expansion is done around $w_0 = w + E[x]$, then the homogeneity property is also satisfied for all moments of the distribution of y

$$E[(y - E[y])^n] = a^n E[(x - E[x])^n].$$

Finally we provide an example that is constructed to illustrate a possible counter-intuitive decision making in prospect theory. Recall that in prospect theory one assumes that the decision maker's utility function is convex for losses, thus, the decision maker appreciates the uncertainty in losses. The data for the example are provided in Table 1. In short, we would like to find out which lottery, A or B, is considered to be more risky. Observe that the probabilities of the states are alike so that the presented results do not

[2]The lower partial moment is not a "risk measure" in the sense proposed by Artzner, Delbaen, Eber, and Heath (1999) in their landmark paper. In the sense of Artzner et al. (1999) the coherent risk measure of x based on a lower partial moment of order n is $\rho(x) = -E[x] + a\sqrt[n]{LPM_n(x, E[x])}$ for $a > 0$. This was proved by Fisher (2003) and later by Rockafellar, Uryasev, and Zabarankin (2006). We thank Tom Fisher for this remark.

State	1	2	3	4
Probability	0.25	0.25	0.25	0.25
Payoff lottery A	-10	-5	5	10
Payoff lottery B	-12	-3	5	10

Table 1: Probability distributions of the two lotteries.

Parameter	Lottery A	Lottery B
Expected payoff	0	0
Standard deviation, σ	7.91	8.34
μ_+	3.75	3.75
μ_-	3.75	3.75
σ_+	5.59	5.59
σ_-	5.59	6.18

Table 2: Descriptive parameters of the distributions of the two lotteries.

depend on the probability transformation that is used in prospect theory. Table 2 presents the descriptive parameters of the distributions of the two lotteries. Note that the two lotteries differ in their probability distributions of payoffs below zero so that their measures σ_+ are equal and since the expected payoffs are zero, so $\mu_+ = \mu_-$ for both the lotteries.

Example 1 (A more risky (in a usual sense) lottery can have a smaller risk premium than a less risky lottery). In this example we use the same general utility function (11) which can result in different shapes and preferences depending on the set of parameters $(\lambda, \gamma_-, \gamma_+)$. We consider the risk premiums of four different decision makers where each of them has a distinct shape of the utility function. Observe that the higher the risk premium of a lottery, the more riskier the lottery for a particular decision maker (given that the lotteries under question have the same expected payoff).

- For the decision maker with *Quadratic* utility with $\lambda = 1$ and $\gamma_- = \gamma_+ = 0.04$
$$\pi_A^{\text{Quadratic}} = 1.25 < \pi_B^{\text{Quadratic}} = 1.39.$$

- For the decision maker with *Behavioral I* utility for which $\lambda = 1$, $\gamma_- = 0.04$, and $\gamma_+ = 0$
$$\pi_A^{\text{Behavioral I}} = 0.63 < \pi_B^{\text{Behavioral I}} = 0.77.$$

- For the decision maker with *Behavioral II* utility for which $\lambda = 2$, $\gamma_- = -0.04$, and $\gamma_+ = 0.04$
$$\pi_A^{\text{Behavioral II}} = 1.56 > \pi_B^{\text{Behavioral II}} = 1.42.$$

- For the decision maker with *Behavioral III* utility for which $\lambda = 2$, $\gamma_- = 0.04$, and $\gamma_+ = 0.04$

$$\pi_A^{\text{Behavioral III}} = 2.81 < \pi_B^{\text{Behavioral III}} = 2.95.$$

In the computation of the risk premiums we make sure that the values of γ_- and γ_+ are chosen to satisfy the condition of positive marginal utility of a decision maker in all states. Observe that the decision maker with convex loss function considers lottery B to be less risky than lottery A, whereas for all the other decision makers lottery A is less risky than lottery B. This result seems to be counter-intuitive because we are used to think that the higher the standard deviation or the downside deviation, the more risky the lottery. That is, the standard deviation or the downside deviation are considered to be the measures of risk. However, if the investor has a utility function as in prospect theory, neither the standard deviation or the downside deviation is a proper risk measure! $\diamond$

5 Optimal Capital Allocation with Loss Aversion

5.1 Set Up

In this section we consider an investor who wants to allocate the wealth between a risk-free and a risky asset. The returns of the risky asset over a small time interval Δt are

$$x = \mu \Delta t + \sigma \sqrt{\Delta t} \varepsilon,$$

where μ and σ are, respectively, the mean and volatility of the risky asset return per unit of time, and ε is some (normalized) stochastic variable such that $E[\varepsilon] = 0$ and $\text{Var}[\varepsilon] = 1$. We assume that $Q(x)$ is the cumulative probability distribution function of x. The returns on the risk-free asset over the same time interval equal

$$r_f = r\Delta t,$$

where r is the risk-free interest rate per unit of time.

We further suppose that the investor has a wealth of w and invests a into the risky asset and, consequently, $w - a$ into the risk-free asset. Thus, the investor's wealth after Δt is

$$\widetilde{w} = a(x - r_f) + w(1 + r_f).$$

The investor's expected utility

$$E[U(\widetilde{w})] = E[U(a(x - r_f) + w(1 + r_f))]. \tag{18}$$

The investor's objective is to choose a to maximize the expected utility

$$E[U^*(\widetilde{w})] = \max_a E[U(\widetilde{w})]. \tag{19}$$

Before turning to the solution of the optimal capital allocation problem, we would like to derive a detailed general expression for the investor's expected utility as a function of a. It is important to observe here that, since the utility function is defined over losses and gains, the resulting expression for the investor's expected utility depends on whether the value of a is positive or negative. First we consider the case where $a > 0$, that is, the investor buys the risky asset. In this case

$$\begin{aligned} E[U(\widetilde{w})|a > 0] = &\int_{-\infty}^{r_f - \frac{\delta w}{a}} U_-(a(x - r_f) + \delta w) dQ(x) \\ &+ \int_{r_f - \frac{\delta w}{a}}^{\infty} U_+(a(x - r_f) + \delta w) dQ(x), \end{aligned}$$

where

$$\delta w = w(1 + r_f) - w_0.$$

The next step is to apply Taylor series expansions for $U_-(a(x - r_f) + \delta w)$ and $U_+(a(x - r_f) + \delta w)$ around 0

$$\begin{aligned} E[U(\widetilde{w})|a > 0] &= \int_{-\infty}^{r_f - \frac{\delta w}{a}} \left(\sum_{n=1}^{\infty} \frac{1}{n!} U_-^{(n)}(0)(a(x - r_f) + \delta w)^n \right) dQ(x) \\ &\quad + \int_{r_f - \frac{\delta w}{a}}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{n!} U_+^{(n)}(0)(a(x - r_f) + \delta w)^n \right) dQ(x) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} U_-^{(n)}(0) \int_{-\infty}^{r_f - \frac{\delta w}{a}} (-a)^n \left(r_f - \frac{\delta w}{a} - x \right)^n dQ(x) \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{n!} U_+^{(n)}(0) \int_{r_f - \frac{\delta w}{a}}^{\infty} a^n \left(x - \left(r_f - \frac{\delta w}{a} \right) \right)^n dQ(x) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} U_-^{(n)}(0)(-1)^n a^n LPM_n \left(x, r_f - \frac{\delta w}{a} \right) \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{n!} U_+^{(n)}(0) a^n UPM_n \left(x, r_f - \frac{\delta w}{a} \right). \end{aligned} \tag{20}$$

Similarly, for the case where $a < 0$ (that is, the investor sells short the risky asset) we obtain

$$
\begin{aligned}
E[U(\widetilde{w})|a<0] &= \int_{-\infty}^{r_f-\frac{\delta w}{a}} U_+(a(x-r_f)+\delta w)dQ(x) \\
&\quad + \int_{r_f-\frac{\delta w}{a}}^{\infty} U_-(a(x-r_f)-\delta w)dQ(x) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} U_+^{(n)}(0)(-1)^n a^n LPM_n\left(x, r_f - \frac{\delta w}{a}\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{n!} U_-^{(n)}(0) a^n UPM_n\left(x, r_f - \frac{\delta w}{a}\right).
\end{aligned}
\tag{21}
$$

5.2 Solutions of the Optimal Capital Allocation Problem

Prior to proceeding to the analysis of the investor's optimal capital allocation decision, we need to choose the investor's reference point w_0 to which gains and losses are compared. One possible reference point is the "status quo", that is, the investor's initial wealth w. Unfortunately, with this choice it is possible to arrive to a closed-form approximate solution of the optimal capital allocation problem only when $w = 0$. However, according to Khaneman and Tversky, the investor's initial wealth does not need to be the reference point. Following Barberis, Huang, and Santos (2001) we assume that the reference point is $w_0 = w(1 + r_f)$. This is the investor's initial wealth scaled up by the risk-free rate. This level of wealth serves as a "benchmark" wealth. Consequently, gains and losses for the investor are coded relative to the benchmark. The idea here is that the investor is likely to be disappointed if the risky asset provides a return below the risk-free rate of return. Moreover, the choice of $w_0 = w(1 + r_f)$ is also justified if we require that all partial moments should exhibit the homogeneity property in a (recall the discussion in the previous section). For example, if the investor is equipped with Behavioral I utility function, the risk of investing the amount of $a > 0$, as measured by downside deviation, equals to $a\sqrt{LMP_2(x, r_f)}$, which seems to be very natural.

Now we proceed to the analysis of the optimal capital allocation problem. First we consider the case where the optimal a is positive. In this case, given the choice $w_0 = w(1+r_f)$ which results in $\delta w = 0$, we can rewrite the equation (20) as

$$
\begin{aligned}
E[U(\widetilde{w})|a>0] &= \sum_{n=1}^{\infty} \frac{1}{n!} U_-^{(n)}(0) a^n (-1)^n LPM_n(x, r_f) \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{n!} U_+^{(n)}(0) a^n UPM_n(x, r_f).
\end{aligned}
\tag{22}
$$

Note that the first and the second (either lower or upper) partial moments are of order Δt, the third partial moments are of order $\Delta t^{\frac{3}{2}}$, the fourth partial moments are or order Δt^2, etc. To proceed further, we assume that the time interval Δt is rather small such that in equation (22) we can disregard all the terms but those which are of order Δt. Further denote

$$
\begin{aligned}
p_- &= LPM_1(x, r_f) = \int_{-\infty}^{r_f} (r_f - x) dQ(x), \\
p_+ &= UPM_1(x, r_f) = \int_{r_f}^{\infty} (x - r_f) dQ(x), \\
\sigma_-^2 &= LPM_2(x, r_f) = \int_{-\infty}^{r_f} (r_f - x)^2 dQ(x), \\
\sigma_+^2 &= UPM_2(x, r_f) = \int_{r_f}^{\infty} (x - r_f)^2 dQ(x).
\end{aligned}
$$

That is, p_- is the lower partial moment of order 1 at level r_f and σ_-^2 is the lower partial moment of order 2 at level r_f. Similarly, p_+ is the upper partial moment of order 1 at level r_f and σ_+^2 is the upper partial moment of order 2 at level r_f. With this assumption and notation we can rewrite equation (22) as

$$
E[U(\widetilde{w})|a > 0] \approx a\left(-U_-'(0)p_- + U_+'(0)p_+\right) + \frac{1}{2}a^2\left(U_-''(0)\sigma_-^2 + U_+''(0)\sigma_+^2\right). \tag{23}
$$

Note that if we divide the left- and right-hand sides of the equation above by $U_+'(0) > 0$, we arrive to the equivalent expected utility

$$
E[\widehat{U}(\widetilde{w})|a > 0] \approx a(p_+ - \lambda p_-) - \frac{1}{2}a^2(\gamma_+\sigma_+^2 + \lambda\gamma_-\sigma_-^2), \tag{24}
$$

where λ is loss aversion and γ_- and γ_+ are aversions to uncertainties in losses and gains as in (7). Observe that the investor's equivalent expected utility (24) is a quadratic function in a. To guarantee the existence of a maximum, the investor's utility should be concave in a, which means that the following conditions should be satisfied:

$$
\gamma_+\sigma_+^2 + \lambda\gamma_-\sigma_-^2 > 0 \quad \text{if } a > 0, \tag{25}
$$

The first-order condition of optimality of a in equation (24) gives us

$$
(p_+ - \lambda p_-) - a(\gamma_+\sigma_+^2 + \lambda\gamma_-\sigma_-^2) = 0.
$$

The solution with respect to a yields

$$
a = \frac{p_+ - \lambda p_-}{\lambda\gamma_-\sigma_-^2 + \gamma_+\sigma_+^2} \quad \text{if } a > 0. \tag{26}
$$

Inserting expression (26) for the optimal value of a into (24), we obtain the solution for the investor's maximum expected utility

$$
\begin{aligned}
E[\widehat{U}^*(\widetilde{w})|a>0] &\approx \left(\frac{p_+ - \lambda p_-}{\lambda\gamma_-\sigma_-^2 + \gamma_+\sigma_+^2}\right)(p_+ - \lambda p_-) \\
&\quad - \frac{1}{2}\left(\frac{p_+ - \lambda p_-}{\lambda\gamma_-\sigma_-^2 + \gamma_+\sigma_+^2}\right)^2 \left(\lambda\gamma_-\sigma_-^2 + \gamma_+\sigma_+^2\right) \qquad (27) \\
&= \frac{1}{2}\frac{(p_+ - \lambda p_-)^2}{\lambda\gamma_-\sigma_-^2 + \gamma_+\sigma_+^2}.
\end{aligned}
$$

We postpone the discussion of the investor's optimal capital allocation. Next we consider the case where the optimal a is negative. In this case we can rewrite the equation (21) as (disregarding again all the terms but those which are of order Δt)

$$E[U(\widetilde{w})|a<0] \approx a\left(-U'_+(0)p_- + U'_-(0)p_+\right) + \frac{1}{2}a^2\left(U''_+(0)\sigma_-^2 + U''_-(0)\sigma_+^2\right). \qquad (28)$$

Note that if we again divide the left- and right-hand sides of the equation above by $U'_+(0) > 0$, we arrive to the equivalent expected utility

$$E[\widehat{U}(\widetilde{w})|a<0] \approx -a(p_- - \lambda p_+) - \frac{1}{2}a^2(\gamma_+\sigma_-^2 + \lambda\gamma_-\sigma_+^2). \qquad (29)$$

Again observe that the investor's equivalent expected utility (29) is a quadratic function in a. To guarantee the existence of a maximum, the investor's utility should be concave in a, which means that the following conditions should be satisfied:

$$\gamma_+\sigma_-^2 + \lambda\gamma_-\sigma_+^2 > 0 \quad \text{if } a<0. \qquad (30)$$

Using the first-order condition of optimality of a and solving the resulting equation with respect to a we obtain

$$a = \frac{\lambda p_+ - p_-}{\gamma_+\sigma_-^2 + \lambda\gamma_-\sigma_+^2} \quad \text{if } a<0. \qquad (31)$$

Afterwards we insert expression (31) for the optimal value of a into (28) and obtain the solution for the investor's maximum expected utility

$$E[\widehat{U}^*(\widetilde{w})|a<0] \approx \frac{1}{2}\frac{(p_- - \lambda p_+)^2}{\gamma_+\sigma_-^2 + \lambda\gamma_-\sigma_+^2}. \qquad (32)$$

Finally observe that not for all possible sets of parameters $(\lambda, \gamma_-, \gamma_+)$ the optimal capital allocation problem has a solution. Since $\sigma_+^2 > 0$ and $\sigma_-^2 > 0$, the conditions (25) and (30) are always satisfied if the investor dislikes either both uncertainties or at least one uncertainty and is neutral to the other. For example, these conditions are violated if the investor appreciates the uncertainty in losses, $\gamma_- < 0$, but neutral to the uncertainty in gains, $\gamma_+ = 0$. The other example when the conditions are violated is if $\gamma_- = \gamma_+ = 0$, that is, the investor is neutral to both uncertainties. In this case the investor's utility can be represented by so-called "bilinear" utility. If, for example, $a > 0$ and $p_+ - \lambda p_- > 0$, then the investor's expected utility (24) is strictly increasing in a which means that the investor is willing to borrow an infinite amount at the risk-free rate and invest all in the risky asset. For bilinear utility the solution of the optimal capital allocation problem generally does not exist, this was noted by Sharpe (1998).

5.3 Discussion of Economic Implications

Observe that for a von Neumann-Morgenstern utility the solution for a reduces to the well-known result

$$a = \frac{\mu - r}{\gamma \sigma^2}, \tag{33}$$

since with no transformation of probability $p_+ - p_- = (\mu - r)\Delta t$ and $\sigma_-^2 + \sigma_+^2 = \sigma^2 \Delta t$ up to the leading terms of order Δt. Observe from (33) that for the investor with a von Neumann-Morgenstern utility it is always optimal to undertake a risky investment when

$$\mu \neq r.$$

If $\mu > r$, it is optimal for the investor to buy some amount of the risky asset, whereas if $\mu < r$, it is optimal for the investor to sell short some amount of the risky asset. However, if the investor has a utility function with loss aversion then it is optimal for the investor to buy some amount of the risky asset only in case

$$p_+ > \lambda p_-. \tag{34}$$

Similarly, it is optimal for the investor sell short some amount of the risky asset only under condition that

$$p_+ < \frac{1}{\lambda} p_-. \tag{35}$$

Both conditions, (34) and (35), say that the risk premium provided by the risky asset must be rather high to induce the investor to undertake a risky investment. With loss aversion, combining conditions (34) and (35), we conclude that the investor does not invests if

$$\frac{1}{\lambda} p_- < p_+ < \lambda p_-.$$

In other words, the absolute value of the risk premium[3], $|p_+ - p_-|$, must exceed some threshold amount to induce the investor to undertake a risky

[3] Observe that without a probability transformation it means that the absolute value of $\mu - r$ should be rather high.

investment. If the perceived risk premium provided by the risky asset is below some threshold, the investor avoids the risky asset and invests all in the risk-free asset only. Observe that loss aversion provides a possible explanation for why many investors do not invest in equities. For example, Agnew et al. (2003) study nearly 7,000 retirement accounts and report that about 48% of participants do not invest in equities. Note that according to expected utility theory this behavior is not rational.

Finally we present a brief comparative static analysis of the expressions (26) and (31) for the optimal amount invested in the risky asset, a. If the utility function is linear in the domain for gains, thus $\gamma_+ = 0$ which means that uncertainty in gains plays no role in investment decisions. As we suppose that the utility function is always concave for gains, thus $\gamma_+ > 0$ which means that the greater the aversion to uncertainty in gains, the lesser the amount invested in the risky asset. If the utility function is also concave for losses, then $\gamma_- > 0$ which means that the greater the aversion to uncertainty in losses, the lesser the amount invested in the risky asset. Observe that in prospect theory the utility function is convex for losses, thus $\gamma_- < 0$. In this case the investor appreciates the uncertainty in losses and the higher the absolute value of γ_-, the larger the amount invested in the risky asset. Observe that when $\gamma_- < 0$ and $a > 0$ the greater the value of σ_- the larger might be the amount invested in the risky asset. This seems to be counter-intuitive. That is, if it is optimal for the investor to buy the risky asset, then the value of σ_-, which is the downside deviation below the risk-free alternative, is often used as a risk measure. However, if the investor has a utility function as in prospect theory, either the standard deviation or the downside deviation is not a proper risk measure! In the next subsection we provide an example that illustrates this point.

When it comes to the dependence of the optimal amount invested in the risky asset on the loss aversion parameter λ, the higher the value of λ, the lesser the amount invested in the risky asset. To illustrate this, we consider only the case when $a > 0$. We find the derivative of a with respect to λ

$$\frac{da}{d\lambda} = -\frac{p_-\gamma_+\sigma_+^2 + p_+\gamma_-\sigma_-^2}{(\lambda\gamma_-\sigma_-^2 + \gamma_+\sigma_+^2)^2}.$$

Let us consider the numerator in the fraction above. Using condition (34) we conclude

$$p_-\gamma_+\sigma_+^2 + p_+\gamma_-\sigma_-^2 > p_-(\gamma_+\sigma_+^2 + \lambda\gamma_-\sigma_-^2) > 0,$$

where the last inequality is due to condition (25) and the positiveness of p_-. Consequently, we conclude that

$$\frac{da}{d\lambda} < 0.$$

The illustration for the case $a < 0$ can be done similarly.

5.4 Performance Evaluation

Suppose the investor wants to invest the wealth into a risk-free asset and a single risky asset that should be chosen among a universe of different risky assets (meaning that the risky assets are mutually exclusive investment alternatives), or to construct the optimal risky portfolio consisting of these risky assets. How to do this? The standard approach in financial theory and practice is to employ some portfolio performance measure to rank the various risky investments. Each portfolio performance measure calculates a score for each asset using its probability distribution of returns. The best asset to invest to is the asset with the highest score. For example, it is well known that if the investor exhibits quadratic preferences then the appropriate performance measure is the Sharpe ratio. But what about a loss averse investor?

Observe that in case the investor has a utility with loss aversion and $\gamma_+ > 0$, the maximum (equivalent) expected utilities can be written as (see equations (27) and (32))

$$E[\widehat{U}^*(\widetilde{w})|a > 0] \approx \frac{1}{2\gamma_+}\frac{(p_+ - \lambda p_-)^2}{\lambda\theta\sigma_-^2 + \sigma_+^2}, \tag{36}$$

$$E[\widehat{U}^*(\widetilde{w})|a < 0] \approx \frac{1}{2\gamma_+}\frac{(p_- - \lambda p_+)^2}{\sigma_-^2 + \lambda\theta\sigma_+^2}, \tag{37}$$

where

$$\theta = \frac{\gamma_-}{\gamma_+}.$$

Note that for any investor the higher the value of

$$\max\left[\frac{(p_+ - \lambda p_-)^2}{\lambda\theta\sigma_-^2 + \sigma_+^2}, \frac{(p_- - \lambda p_+)^2}{\sigma_-^2 + \lambda\theta\sigma_+^2}\right],$$

the higher the maximum expected utility. By analogy with the Sharpe ratio, the investor's (individual) performance measure of the risky asset can be written as

$$PM = \max\left[\frac{p_+ - \lambda p_-}{\sqrt{\lambda\theta\sigma_-^2 + \sigma_+^2}}, \frac{p_- - \lambda p_+}{\sqrt{\sigma_-^2 + \lambda\theta\sigma_+^2}}\right]. \tag{38}$$

Observe that the conditions for the existence of the solution of the optimal allocation problem (25) and (30) guarantee that the denominators in (38) are non-complex numbers. Easy to check that if the investor has a von Neumann-Morgenstern utility (for which $\lambda = \theta = 1$) then the performance measure PM (38) reduces to the Sharpe ratio

$$SR = \frac{E[x] - r_f}{\sqrt{E[(x - r_f)^2]}} = \frac{E[x] - r_f}{\sqrt{\mathrm{Var}[x]}}, \tag{39}$$

since $E[(x - r_f)^2] = \text{Var}[x]$ with an accuracy up to the leading terms of order Δt.

If $\gamma_+ = 0$, that is, the investor is neutral to uncertainty in gains, the conditions for the existence of the solution of the capital allocation problem give us $\gamma_- > 0$. In this case the maximum (equivalent) expected utilities can be written as

$$E[\widehat{U}^*(\widetilde{w})|a > 0] \approx \frac{1}{2\lambda\gamma_-}\frac{(p_+ - \lambda p_-)^2}{\sigma_-^2}, \tag{40}$$

$$E[\widehat{U}^*(\widetilde{w})|a < 0] \approx \frac{1}{2\lambda\gamma_-}\frac{(p_- - \lambda p_+)^2}{\sigma_+^2}, \tag{41}$$

and, consequently, the performance measure might be given by

$$PM = \max\left[\frac{p_+ - \lambda p_-}{\sigma_-}, \frac{p_- - \lambda p_+}{\sigma_+}\right]. \tag{42}$$

Observe that if $\lambda = 1$ (this is the case for the Behavioral I utility) then this measure reduces (if the risk premium is positive) to the performance measure suggested by Sortino and Price (1994) and Ziemba (2005) who replace standard deviation in the Sharpe ratio by downside deviation

$$DSR = \frac{E[x] - r_f}{\sigma_-}. \tag{43}$$

There are some cases when the performance measure (42) produces the same ranking of risky assets as that of (43) irrespective the value of the loss aversion parameter λ. To illustrate this, suppose, for example, that the investor wants to rank two investment funds. Suppose in addition that both funds have positive risk premiums, and when $\lambda = 1$ the first fund is better than the second fund meaning that

$$\frac{p_{1+} - \lambda p_{1-}}{\sigma_{1-}} > \frac{p_{2+} - \lambda p_{2-}}{\sigma_{2-}} \quad \text{for } \lambda = 1. \tag{44}$$

It is easy to show that the ranking of the funds remains the same for any $\lambda > 1$ under condition that

$$\frac{p_{1-}}{\sigma_{1-}} \leq \frac{p_{2-}}{\sigma_{2-}}.$$

To show this, rewrite inequality (44) as

$$\frac{p_{1+}}{\sigma_{1-}} > \frac{p_{2+}}{\sigma_{2-}} - \lambda\left(\frac{p_{2-}}{\sigma_{2-}} - \frac{p_{1-}}{\sigma_{1-}}\right). \tag{45}$$

As the quantity in the brackets on the right-hand side of inequality (45) is nonnegative, it is clear that the inequality will not be violated for any $\lambda > 1$.

State	1	2	3	4
Probability	0.25	0.25	0.25	0.25
Return asset A	-10%	0%	20%	30%
Return asset B	-15%	4%	20%	30%
Return asset C	-5%	4%	10%	20%
Risk-free return	4%	4%	4%	4%

Table 3: Probability distributions of three risky assets.

Parameter	**Asset A**	**Asset B**	**Asset C**
Expected return, $E[r]$	10%	9.75%	7.25%
Standard deviation, σ	16.91%	17.98%	9.66%
p_+	10.5%	10.5%	5.5%
p_-	4.5%	4.75%	2.25%
σ_+	15.26%	15.26%	8.54%
σ_-	7.28%	9.5%	4.5%

Table 4: Descriptive parameters of the return distributions of the three risky assets. Note that σ is computed with respect to r_f so that $\sigma^2 = \sigma_-^2 + \sigma_+^2$.

Recall that a loss averse investor invests only when the perceived risk premium is rather high. When neither of conditions (34) and (35) are satisfied, the investor avoids the risky asset and invests all into the risk-free asset. This means that when

$$PM < 0,$$

the investor avoids the risky asset.

As compared with the Sharpe ratio where the investor's risk preferences completely disappear, to compute the performance measure PM one generally needs to define the values of λ and θ (and, possible, the rule for the transformation of the objective probability distribution). This means that *the performance measure PM is not unique for all investors, but rather an individual performance measure.* That is, investors with different preferences might rank differently the same set of risky assets.

Finally we provide two illustrating examples. The data for both examples are provided in Table 3 which presents the probability distribution of three risky assets. Observe that the probabilities of the states are alike so that the presented results do not depend on the probability transformation as in prospect theory. Table 4 presents the descriptive parameters of the return distributions of the risky assets.

Example 2 (A more risky (in a usual sense) asset can be more attractive than a less risky asset). In this example we consider the investor's choice between assets A and B. Note that the two risky assets differ in their probability distributions of returns below the risk-free rate of return so that their measures p_+ and σ_+ are equal. In this example we use the same general utility function (11) which can result in different shapes and preferences depending on the set of parameters $(\lambda, \gamma_-, \gamma_+)$. We consider the performance measures of four different investors where each of them has a distinct shape of the utility function.

- For the investor with *Quadratic* utility for whom the performance measure is computed according to (39)

$$PM_A^{\text{Quadratic}} = 0.3548 > PM_B^{\text{Quadratic}} = 0.3198.$$

- For the investor with *Behavioral I* utility for whom the performance measure is computed according to (43)

$$PM_A^{\text{Behavioral I}} = 0.8242 > PM_B^{\text{Behavioral I}} = 0.6053.$$

- For the investor with *Behavioral II* utility for whom the performance measure is computed according to (38) with $\lambda = 2$ and $\theta = -1$

$$PM_A^{\text{Behavioral II}} = 0.1331 < PM_B^{\text{Behavioral II}} = 0.1380.$$

- For the investor with *Behavioral III* utility for whom the performance measure is computed according to (38) with $\lambda = 2$ and $\theta = 1$

$$PM_A^{\text{Behavioral III}} = 0.0815 > PM_B^{\text{Behavioral III}} = 0.0492.$$

In the computation of the performance measures we make sure that in the optimal allocation the investor's marginal utility is positive in all states. This is essential because otherwise it is very easy to arrive to some spurious results. Observe that the investor with convex loss function considers asset B to be more attractive than asset A, whereas for all the other investors asset A is more attractive than asset B. This result seems to be counter-intuitive because asset B provides lesser expected return and has greater risk (in a usual sense) as compared with asset A ($E[r_B] < E[r_A]$, $\sigma_B > \sigma_A$, and $\sigma_{B-} > \sigma_{A-}$). Standard intuition says that one should prefer asset A to asset B. $\diamond$

Example 3 (Ranking of risky assets depends on the investor's individual preferences). In the previous example we illustrated that the choice of the best risky asset might depend on whether the investor exhibits aversion to the uncertainty in losses or appreciates the uncertainty in losses. That is, the ranking of assets might be different depending on the sign of the preference parameter θ. The purpose of this example is to illustrate that the ranking of assets might be different even if the sign of the preference parameter θ is the same. In particular, in this example we consider the investor's choice between assets A and C. Observe that, as compared with asset C, asset A is more risky, but more rewarding too. First we assume that the investor is equipped with Behavioral III utility and the value of the preference parameter $\theta = 1$. We consider the ranking of these assets for two different levels of loss aversion. The computation of the performance measures for assets A and C yields

$$PM_A^{\text{Behavioral III}} = 0.2396 > PM_C^{\text{Behavioral III}} = 0.2334 \qquad \text{when } \lambda = 1.4,$$

$$PM_A^{\text{Behavioral III}} = 0.1324 < PM_B^{\text{Behavioral III}} = 0.1386 \qquad \text{when } \lambda = 1.8.$$

Observe that when the loss aversion is relatively low, the investor prefers more riskier asset A to less riskier asset C. However, when the loss aversion is relatively high, the investor prefers less riskier asset C to more riskier asset A. In contrast, if the investor has either Quadratic or Behavioral I utility function, the investor prefers asset A to C:

$$PM_A^{\text{Quadratic}} = 0.3548 > PM_C^{\text{Quadratic}} = 0.3366.$$

$$PM_A^{\text{Behavioral I}} = 0.8242 > PM_C^{\text{Behavioral I}} = 0.7222.$$

For the purpose of comparison, the investor with Behavioral II utility and $\theta = -1$ prefers asset C to A for any level of loss aversion for which the investment in any risky asset is sensible (that is, the performance measures of any asset is greater than zero)

$$PM_A^{\text{Behavioral II}} = 0.3333 < PM_C^{\text{Behavioral II}} = 0.3517 \qquad \text{when } \lambda = 1.4,$$

$$PM_A^{\text{Behavioral II}} = 0.2046 < PM_C^{\text{Behavioral II}} = 0.2398 \qquad \text{when } \lambda = 1.8.$$

$\diamond$

Let us elaborate more on the result illustrated in the last example because it is important to realize that investors with different degrees of loss aversion might rank differently risky assets. Similarly, if an investor is supposed to construct an optimal portfolio of several risky assets, then the composition of the optimal risky portfolio might depend on the investor's loss aversion. To illustrate this, we suppose that all risky assets have positive risk premiums so that the investor's performance measure is

$$\text{either } \frac{p_+ - \lambda p_-}{\sqrt{\lambda\theta\sigma_-^2 + \sigma_+^2}} \text{ or } \frac{p_+ - \lambda p_-}{\sigma_-}$$

depending on whether the investor exhibits aversion or neutrality to the uncertainty in gains. Observe, however, that when the investor is averse to the uncertainty in gains and λ is rather large, then σ_- is more important than σ_+ and

$$\frac{p_+ - \lambda p_-}{\sqrt{\lambda\theta\sigma_-^2 + \sigma_+^2}} \approx \frac{p_+ - \lambda p_-}{\sigma_- \sqrt{\lambda\theta}} \equiv \frac{p_+ - \lambda p_-}{\sigma_-},$$

since a performance measure is invariant to a positive affine transformation[4]. The problem of choosing the best risky asset can be formulated either as the maximization of the investor's expected utility or, alternatively, as the maximization of the investor's performance measure. In the latter case the investor's objective

$$\max \frac{p_+ - \lambda p_-}{\sigma_-} = \max\left(\frac{p_+}{\sigma_-} - \lambda\frac{p_-}{\sigma_-}\right),$$

which can be interpreted as a double objective: (1) maximization of $\frac{p_+}{\sigma_-}$ and (2) minimization of $\frac{p_-}{\sigma_-}$. Besides, the second objective is λ times more important than the first one. That is, when the investor's loss aversion increases, the minimization of $\frac{p_-}{\sigma_-}$ becomes more and more important[5] than the maximization of $\frac{p_+}{\sigma_-}$. Consequently, the investors with rather high loss aversion prefer risky assets with low $\frac{p_-}{\sigma_-}$. This explains why in Example 3 asset C becomes more attractive than asset A when we increase the level of loss aversion. Indeed

$$\frac{p_{C-}}{\sigma_{C-}} = 0.5 < \frac{p_{A-}}{\sigma_{A-}} = 0.62.$$

6 Summary and Conclusions

In this paper we considered a generalized behavioral utility function that has a kink at the reference point and different functions for losses and gains. We obtained an approximation of the expected utility of a loss averse decision maker. In the spirit of Arrow and Pratt we derived the expression for a risk premium. We showed that in contrast to a decision maker with the mean-variance utility for whom the only source of risk is the variance, a loss averse decision maker distinguishes between three sources of risk: the expected loss, the uncertainty in losses, and the uncertainty in gains. Consequently, a decision maker with a behavioral utility exhibits three types of aversions: aversion to loss, aversion to uncertainty in gains, and aversion to uncertainty in losses.

[4]That is, if PM is some performance measure then the affine transformation $PM' = c + d\,PM$, $d > 0$, is an equivalent performance measure to PM in the sense that they both produce equal ranking of alternative investments.

[5]Note that to minimize $\frac{p_-}{\sigma_-}$ one needs mainly to minimize the expected loss p_- because increasing σ_- is in conflict with the first objective since it decreases $\frac{p_+}{\sigma_-}$.

We have also found an approximate solution to the optimal capital allocation problem and derived the expression for a portfolio performance measure. We discovered here that a loss averse investor will want to allocate some wealth to the risky asset only when the perceived risk premium is sufficiently high. Otherwise, if the risk premium is small, a loss averse investor avoids the risky asset and invests only in the risk-free asset. As compared with the Sharpe ratio where the investor's risk preferences completely disappear, we showed that the performance measure of a loss averse investor is not unique, but rather an individual performance measure. The explanation for this is the fact that a loss averse investor distinguishes between several sources of risk. Since each investor may exhibit different preferences to each source of risk, investors with different preferences might rank differently the same set of risky assets. Moreover, we have shown that if the investor exhibits a risk-seeking behavior in the domain for loss as in prospect theory, then neither the standard deviation nor the downside deviation can be used as a proper risk measure.

References

Agnew, J., Balduzzi, P., and Annika, S. (2003). "Portfolio Choice and Trading in a Large 401(k) Plan", *American Economic Review*, *93*(1), 193–215.

Allais, M. (1953). "Le Comportement de l'Homme Rationnel devant le Risque: Critique des postulats et axiomes de l'Ecole Americaine", *Econometrica*, *21*, 503–546.

Arrow, K. J. (1965). *Aspects of the Theory of Risk-Bearing.* Helsinki: Yrjö Hahnsson Foundation.

Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). "Coherent Measures of Risk", *Mathematical Finance*, *9*(3), 203–228.

Barberis, N., Huang, M., and Santos, T. (2001). "Prospect Theory and Asset Prices", *Quarterly Journal of Economics*, *116*(1), 1–53.

Bawa, V. S. (1978). "Safety-First, Stochastic Dominance, and Optimal Portfolio Choice", *Journal of Financial and Quantitative Analysis*, *13*(2), 255–271.

Bernartzi, S. and Thaler, R. (1995). "Myopic Loss Aversion and the Equity Premium Puzzle", *Quarterly Journal of Economics*, *110*(1), 73–92.

Camerer, C. F. (2000). "Prospect theory in the wild: Evidence from the field. in D. Kahneman and A. Tversky, eds. *Choices, Values and Frames*, Cambridge University Press, New York",, 288–300.

Ellsberg, D. (1961). "Risk, Ambiguity, and the Savage Axioms", *Quarterly Journal of Economics*, *75*(4), 643–669.

Fishburn, P. C. (1977). "Mean-Risk Analysis with Risk Associated with Below-Target Returns", *American Economic Review*, *67*(2), 116–126.

Fisher, T. (2003). "Risk Capital Allocation by Coherent Risk Measures Based on One-Sided Moments", *Insurance: Mathematics and Economics*, *32*(1), 135–146.

Friedman, M. and Savage, L. J. (1948). "The Utility Analysis of Choices Involving Risk", *Journal of Political Economy*, *56*(4), 279–304.

Kahneman, D. and Tversky, A. (1979). "Prospect Theory: An Analysis of Decision Under Risk", *Econometrica*, *47*(12), 263–291.

Köbberling, V. and Wakker, P. P. (2005). "An Index of Loss Aversion", *Journal of Economic Theory*, *122*, 119–131.

Levy, H. and Markowitz, H. (1979). “Approximating Expected Utility by a Function of Mean and Variance”, *American Economic Review*, *69*(3), 308–317.

Markowitz, H. (1959). *Portfolio Selection. Efficient Diversification of Investments*. John Wiley, New York.

Markowitz, H. (1952). “The Utility of Wealth”, *Journal of Political Economy*, *60*(2), 151–158.

Pratt, J. W. (1964). “Risk Aversion in the Small and in the Large”, *Econometrica*, *32*(1-2), 122–136.

Rockafellar, R. T., Uryasev, S., and Zabarankin, M. (2006). “Generalized Deviations in Risk Analysis”, *Finance and Stochastics*, *10*(1), 51–74.

Samuelson, P. A. (1970). “The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variances, and Higher Moments”, *Review of Economic Studies*, *37*(4), 537–542.

Sharpe, W. F. (1966). “Mutual Fund Performance”, *Journal of Business*, *31*(1), 119–138.

Sharpe, W. F. (1998). “Morningstar’s Risk-Adjusted Ratings”, *Financial Analysts Journal*, *54*(4), 21 – 33.

Sortino, F. A. and Price, L. N. (1994). “Performance Measurement in a Downside Risk Framework”, *Journal of Investing*, *3*(3), 59 – 65.

Tversky, A. and Kahneman, D. (1992). “Advances in Prospect Theory: Cumulative Representation of Uncertainty”, *Journal of Risk and Uncertainty*, *5*(4), 297 – 323.

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press.

Ziemba, W. T. (2005). “The Symmetric Downside-Risk Sharpe Ratio”, *Journal of Portfolio Management*, *32*(1), 108–122.