Robust Investment Decisions and The Value of Waiting to Invest

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Abstract. We solve a firm's investment problem when there is uncertainty about the growth rate of the project value and about the investment cost. This uncertainty makes the firm ambiguity averse. We use a robust method to take this into account. In this setting, we provide explicit solutions when the value of the project as well as the investment cost are stochastic. Ambiguity aversion decreases the investment threshold and, in contrast to standard models, volatility can decrease the investment threshold. In fact, volatility increases the impact of ambiguity aversion. We also show that the effect of volatility is highly dependent on the correlation between the value of the project and the investment cost. Hence, ambiguity aversion is an important aspect to take into account when the firm considers its investment strategy.

Keywords: Real options, Ambiguity, Robustness, Correlation effects

JEL classification: G31, D81

1 Introduction

A fundamental question in corporate finance is whether a firm should exploit an investment opportunity. Furthermore, as firms often have discrepancy in the timing of the investment this element must be taken into account. The issue of finding the optimal time for undertaking the investment has been addressed by several papers since the seminal paper by McDonald and Siegel (1986). They base their analysis on the option like approach—leading to the so-called real options analysis—and a key assumption is that all parameters in their model are known to investors. However, it can be difficult to provide precise estimates of the parameters in practice. Our paper contributes to the optimal investment literature by considering the investment problem when a firm takes parameter uncertainty into account. We demonstrate that this significantly impacts the firm's investment decision. For example, the investment threshold can be decreasing in volatility.

A vast amount of literature has considered variations of the investment problem with McDonald and Siegel (1986) being the basic reference. Expansions have analyzed e.g. competition, asymmetric information, incomplete information, or risk aversion in combination with unspanned risk. Generally, the value of the option to invest is lowered by the cause of the friction introduced in the different models. Our paper supplements this literature by analyzing how ambiguity aversion against uncertainty in project value and the investment cost, respectively, impacts the value of the investment opportunity and, thus, when investment takes place.

As noted, one strand of real options literature relaxes the assumption of complete information by considering incomplete information models with updating of beliefs, see e.g. Decamps, Mariotti, and Villeneuve (2005) who add the friction that the investor does not have complete information about the parameters in the model. Instead, the investor has as prior probability measure over the states of the nature and as more observations occur over time, the investor uses data to update the probability distribution of the parameters. Thus, by Bayesian updating the investor has a perception of the drift and uses this to find his optimal investment threshold, see e.g. Liptser and Shiriaev (2001). However, a problem with this approach is that in principle one needs an infinite amount of data to reduce the variance of the parameters sufficiently. Consequently, the optimal investment decision depends on time as well as the underlying state variable. This feature makes it very hard to derive explicit solutions. That is, one needs to rely solely on numerical methods. Yang, Song, and Yang (2011) uses the method with an extensive numerical study and they find that the value of the project decreases in volatility. In contrast, e.g. Henderson (2007) analyzes a model with an incomplete market and risk aversion. She models a risk averse investor, who has access to a market portfolio and has the option to undertake an irreversible investment project, where the pay-off is not spanned by the market portfolio. This has the effect that the investor values the project less than in the complete market, invests for an lower profit, and the value is not increasing convex in volatility. This is also possible in the present paper albeit we consider a different setting.

A different approach takes as a starting point that the economic agent does not trust the reference model he employs in his analysis. Importantly, the agent is averse against this kind of uncertainty. An early example of this is the Ellsberg paradox by Ellsberg (1961) with the famous urn experiments. This setting is known as Knightian uncertainty in which ambiguity aversion is present. That is, the investor does not trust the probability measure employed in the model and is averse from this lack of knowledge. Similar to risk aversion, the investor can be more or less ambiguity averse. Ambiguity aversion has been modeled in three different ways in the literature: Smooth preferences, the multiple prior approach, or the multiplier approach. Smooth preferences is a framework that considers the preferences of an investor and uses a concave function of all the models the investor considers possible. In this framework, ambiguity aversion is similar to risk aversion, since it is also a measure for the level of concavity in the function, see e.g. Klibanoff, Marinacci, and Mukerji (2005). Unfortunately, the smooth preference approach is difficult to apply in continuous-time models as explained in e.g. Hansen and Sargent (2009).

In both the multiple prior approach and the multiplier approach the economic agent has a reference parameter as a starting point. For example, in a real options setting one can think of an estimate of the expected growth rate of the project's value as the reference parameter. However, the agent (i.e. the firm) worries that this estimator is not correct (or has a low precision). Hence, the firm fears that the project's value can evolve very differently than what is predicted. In the multiple prior setting, the worst outcome is chosen, and the model is completed as without ambiguity aversion with the important adjustment that the reference parameter is substituted with the worst outcome as a fixed parameter. Hence, the employed reference parameter only depends on the space of possible outcomes. In a real options setting this method is used by Nishimura and Ozaki (2007) and Trojanowska and Kort (2010). Both papers use cash flow as the underlying variable and they consider the growth rate as the parameter estimated with high uncertainty. In particular, the growth rate is assumed to lie within an interval where the boundaries cannot change over time. Thus, the worst possible outcome is constant (the lowest possible growth rate) and the model can be solved with dynamic programming as in (Dixit and Pindyck, 1994, Chapter 6). In the standard real options setting, the value of the option to invest has characteristics similar to a call option. in particular, the value is an increasing convex function of volatility, which induces investors to choose more uncertain projects, see e.g. Dixit and Pindyck (1994). However, in the multiple priors setting, Nishimura and Ozaki (2007) and Trojanowska and Kort (2010) show that the value of the option is no longer a monotonic increasing function of volatility. Related to the lack of information, the investor will invest for a higher level of cash flow, if the interval is wider, i.e. if the worst outcome gets worse.

In the multiplier approach—also known as the robust decision making approach—the worst possible outcome is also chosen, but there is an opposite working penalty for choosing a parameter. Hence, the parameter is chosen endogenously, see e.g. Hansen and Sargent (2008) for an introduction to robust methods. In a continuous-time framework Anderson, Hansen, and Sargent (2003) show how to derive a robust Hamilton-Jacobi-Bellman (HJB) equation. The robust HJB-equation is similar to the standard HJB-equation except for extra terms taking the above measuring penalty into account. The penalty is measured as the relative entropy between the reference measure and other probability measures considered. This robustness framework has been used in financial economic to address problems in asset allocation and asset pricing, see e.g. Epstein and Schneider (2008), Chen and Epstein (2002), and Maenhout (2004).

The present paper is, as far as the authors know, the first to use the multiplier approach in a real options setting. To derive the optimal time to undertake an investment our model employs a set-up similar to the one in McDonald and Siegel (1986). For a start, we focus on a setting in which the underlying variable is the value of the project. That is, the project value is uncertain in the future and, in particular, it has an expected growth rate which we consider to be estimated with low precision so that the firm wants to make a robust investment decision taking this into account. Subsequently, we also address the investment problem when the investment cost is a state variable. Since we use the robust HJB-equation, we end up with a partial differential equation that is significantly different than the Euler-differential equation from the standard problem without ambiguity aversion and from the literature using the multiple priors approach. In technical terms, the present paper has two main results. First, we are able to derive the explicit solution to the robust differential equation. Second, using this solution we can derive the explicit value of the option to invest together with the optimal investment threshold. In economic terms, we find that the threshold value of the project—at which the investment is undertaken—has a functional form similar to the one in the non-ambiguity aversion problem as well as the multiple prior approach. However, our results reveal that ambiguity aversion enters in a more complicated manner and, therefore, it has multiple effects. In particular, the threshold value of the project is not a monotonic increasing function of volatility, since an increase in volatility can decrease the expected growth in value. Furthermore, when ambiguity aversion increases, we show that the threshold value converges to a limit. If the reference growth rate of the project value is low enough compared to volatility, the firm employs the simple NPV rule. In contrast, if the growth rate is high enough the firm will not invest until the project's NPV is at a level strictly higher than 0, thus violating the simple NPV rule.

Finally, McDonald and Siegel (1986) also consider the setting in which the investment cost is uncertain. They show by homogeneity that the problem can be reduced in dimension, where the underlying variable is the ratio between project value and the investment cost. It is not evident that we have the same feature in our model due to the penalty function. However, we show that homogeneity prevails, and the value of the project can be found by only considering the pay-off ratio. Thus, we are able to study correlation effects. We show that different degrees of correlation imply important differences in the investment decision. For example, if the value of the project has higher volatility, this may decrease the investment threshold, when the correlation is positive. In contrast, with negative correlation the investment threshold can increase in the project value's volatility.

The remainder of the paper is organized as follows. We set up the general model in Section 2 in which we derive explicit formulas for the value of the option to investment and the investment threshold. In Section 3 we restrict the analysis to have a fixed investment cost. This allows us to make easier interpretations and to compare with results using the multiple prior method. We return to the general setting in Section 4 in which we elaborate on correlation effects. Finally, Section 5 concludes. A number of technical results and proofs are postponed to the appendix.

2 The robust decision to invest

In this section we extend the fundamental model by McDonald and Siegel (1986) to capture model uncertainty. Thus, we consider a firm facing the standard real options problem. That is, the firm has a perpetual option to pay the irreversible investment cost I. In return it receives a project with value V. What complicates matters for the firm is the fact that it is uncertain about the probability law driving the evolution of the value of the project and the investment cost. If the firm takes this uncertainty into account it is ambiguity averse. We now consider how to introduce ambiguity aversion into the basic irreversible real investment problem

Let $(\Omega, \mathcal{F})$ be a measure space and assume that the firm uses $\mathbb{P}$ as a reference probability measure. That is, the firm considers this probability measure as the most likely. This is the measure an ambiguity neutral firm would apply. Let $\boldsymbol{B} = (B^1, B^2)^\top$ be a Brownian motion with respect to $\mathbb{P}$ and the filtration $\mathbb{F}$. At time t the process for the project's value and the investment cost have the dynamics:

$$dV_t = V_t \left(\mu_V dt + \sigma_V dB_t^1\right), \tag{1}$$

$$dI_t = I_t \left[\mu_I dt + \sigma_I \left(\rho dB_t^1 + \sqrt{1-\rho^2} dB_t^2\right)\right], \tag{2}$$

where μ_V is the expected growth rate and σ_V is the drift regarding the value of the project. Similarly, μ_I is the expected growth rate and σ_I is the drift of the investment cost. ρ measures the correlation between the project's value and the investment cost. All these parameters are constants and, hence, V and I are geometric Brownian motions under the reference measure.

As mentioned above, the firm is aware that it cannot be certain that the reference measure provides a correct specification of the true model. Thus, the firm only knows some approximation of the true model, for example due to parameter uncertainty. We follow Anderson et al. (2003), Hansen and Sargent (2009) and others in modeling this uncertainty. Thus, in addition to the references measure, the firm has to consider a larger set of probability measures. We assume that the firm has knowledge about all nulsets in the model, and that is only has to consider measures equivalent to $\mathbb{P}$.[1] For this reason, we can apply Girsanov's theorem in which any measure $\tilde{\mathbb{P}} \sim \mathbb{P}$ fulfil the relation

$$\tilde{\mathbb{P}}_t(A) = \mathbb{E}_t\left[1_A \mathcal{E}(\boldsymbol{u})_t\right], \quad A \in \mathcal{F}_t \tag{3}$$

where

$$\mathcal{E}(\boldsymbol{u})_t = \exp\left[\int_0^t \boldsymbol{u}_s d\boldsymbol{B}_s - \frac{1}{2}\int_0^t |\boldsymbol{u}_s|^2 ds\right]. \tag{4}$$

Here the process $\boldsymbol{u} = (u_1, u_2)^\top$ is denoted the distortion process. It is a progressively measurable process satisfying

$$\int_0^\infty |\boldsymbol{u}_s|^2 ds < \infty \quad a.s. \tag{5}$$

Thus, we have the Radon-Nikodym derivative $\frac{d\tilde{\mathbb{P}}_t}{d\mathbb{P}_t} = \mathcal{E}(\boldsymbol{u})_t$ where $\mathbb{P}_t$ is the restriction to $\mathcal{F}_t$ and similar for $\tilde{\mathbb{P}}_t$. Under the new measure we have the new Brownian motions

$$dB_t = d\tilde{B}_t^1 + u_{1t}dt, \quad dB_t^2 = d\tilde{B}_t^2 + u_{2t}dt. \tag{6}$$

Denote $\mathcal{U}$ as the set of all distortion processes $\boldsymbol{u}$ such that $\tilde{\mathbb{P}} \sim \mathbb{P}$. For a $\boldsymbol{u} \in \mathcal{U}$ the expected growth rates become

$$\mu_V - \sigma_V u_{1t}, \tag{7}$$

$$\mu_I - \sigma_I\left(\rho u_{1t} + \sqrt{1-\rho^2}u_{2t}\right). \tag{8}$$

Inserting these into the evolution of the value of the project and the investment cost, (1) and (2), respectively, we obtain

$$dV_t = V_t\left((\mu - \sigma_V u_{1t})dt + \sigma_V d\tilde{B}_t^1\right), \tag{9}$$

$$dI_t = I_t\left((\mu_I - \sigma_I(\rho u_{1t} + \sqrt{1-\rho^2}u_{2t}))dt + \sigma_I\left(\rho d\tilde{B}_t^1 + \sqrt{1-\rho^2}d\tilde{B}_{2t}\right)\right). \tag{10}$$

Since the firm is ambiguity averse it worries that an alternative measure yields a less beneficial option to invest. Specifically, the reference drift for the project value could

[1] We use this methods with equivalent measures and Girsanov's theorem such that the drift of the processes can change in both directions; some model use Choquet Brownian, but there the drift can only be decreased and the volatility is also decreased, which is an unfortunate result. See e.g. Kast, Lapied, and Roubaud (2010)

be too high compared to the true drift, while the reference drift for the investment cost could be too low compared to the true drift. To deal with this, the firm considers this as a max-min problem in which an imaginary malevolent counter player chooses the drift that minimizes the value of the option to invest. However, since the firm has a reference measure—which it considers to be the most likely measure—it has less faith in measures that lie far away from it. Therefore, we assume that there is a penalty to the counter player for choosing an alternative probability measure away from the reference measure. Thus, the firm faces the problem of maximizing the value of the option to invest, while the counter player wants to minimize it subject to the penalty from choosing an alternative measure.

Formally, let denote $\tilde{F}(V, I)$ the alternative value of the option to invest in which both the expected profit and the penalty from the counter player choosing a different probability measure are included:

$$\tilde{F}(V,I) = \sup_{\tau \in \mathcal{T}} \inf_{\boldsymbol{u} \in \mathcal{U}} \left[\mathbb{E}^{\tilde{\mathbb{P}}} \left[e^{-\delta(\tau - t)} \left(V_\tau - I_\tau \right) \right] + \underline{\underline{\Psi}}^{-1} \int_0^\tau \int_\Omega \log \left(\frac{d\tilde{\mathbb{P}}_s}{d\mathbb{P}_s} \right) d\tilde{\mathbb{P}}_s ds \right]. \tag{11}$$

The second term in (11) is the penalty for choosing the alternative measure $\mathbb{P}$ and not the reference measure $\tilde{\mathbb{P}}$. The penalty is measured by the relative entropy of the two measures, $\mathcal{R}(\tilde{\mathbb{P}})$, where

$$\mathcal{R}(\tilde{\mathbb{P}}) = \int_0^\tau \int_\Omega \log \left(\frac{d\tilde{\mathbb{P}}_s}{d\mathbb{P}_s} \right) d\tilde{\mathbb{P}}_s ds = \int_0^\tau \int_\Omega \left(\int_0^s \boldsymbol{u}_r d\boldsymbol{B}_r - \frac{1}{2} \int_0^s |\boldsymbol{u}_r|^2 dr \right) d\tilde{\mathbb{P}} ds. \tag{12}$$

Note that choosing the alternative measure $\mathbb{P}$ is basically equivalent to choosing the function $\boldsymbol{u}$.

To handle how much the firm fears alternatives to the reference measure, i.e. its degree of ambiguity aversion, the entropy penalty in (11) is scaled by $\underline{\underline{\Psi}}^{-1}$, where

$$\underline{\underline{\Psi}} = \begin{pmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{pmatrix}.$$

Thus, Ψ_1 is the firm's subjective ambiguity parameter regarding the value of the project, while Ψ_2 is the firm's subjective ambiguity parameter regarding the investment cost.[2] If the firm has less faith in the reference model it considers models with a larger relative

[2]Since $\mathcal{R}(\tilde{\mathbb{P}}) = \infty$, if $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are not equivalent we are restricted to having a fixed maturity, see Karatzas and Shreve (1998) for the maturity problem and Hansen et al. (2006) about the relative entropy.

entropy by reducing the penalty to the counter player choosing a measure further away from the reference measure. In the special case $\underline{\underline{\Psi}} = \underline{\underline{0}}$, the counter player chooses $\boldsymbol{u} = 0$ to minimize the payoff. Thus, this case simply corresponds to an expected payoff maximizer, i.e. the standard setting without ambiguity (ambiguity neutrality). Note that we allow the penalty weighting to be different for the two state variables. We consider this to be reasonable as the firm's information quality regarding the state variables can differ. For instance, the firm can have better knowledge regarding changes in the investment cost than in the project value. We can model this by letting the firm have a lower Ψ_2 than Ψ_1.

To solve the optimal stopping problem we use robust dynamic programming developed in Anderson et al. (2003) and applied in e.g. Maenhout (2006). That is, we solve the robust Hamilton-Jacobi-Bellman (HJB) equation which is an adjusted usual (non-robust) HJB. To see how the HJB is affected by the demand for robustness we first state the HJB for the case in which the firm is ambiguity neutral. The usual HJB states that the infinitesimal generator must be equal to the demanded rate of return, $\delta\tilde{F}$, in the continuation (waiting) region. Loosely speaking the infinitesimal generator is interpreted as the infinitesimal expected change in $\tilde{F}$, i.e. $\mathbb{E}[d\tilde{F}]/dt$. Hence, from Itô'a Lemma and the ambiguity neutral expected growth rates in (1)–(2) we have

$$0 = \tilde{F}_V V\mu_V + \tilde{F}_I I\mu_I + \frac{1}{2}\tilde{F}_{VV}V^2\sigma_V^2 + \frac{1}{2}\tilde{F}_{II}I^2\sigma_I^2 + VI\sigma_V\sigma_I\rho\tilde{F}_{VI} - \delta\tilde{F}. \qquad (13)$$

To extend (13) to the robust HJB two changes are needed. First, the evolution of the value of the project and the investment cost must be specified under the alternative measure as in (9)–(10). Second, we must add the weighted derivative of the relative entropy. Using (12) the derivative of the relative entropy is

$$\mathcal{R}'(\tilde{\mathbb{P}}) = \frac{\partial\mathcal{R}(\tilde{\mathbb{P}})}{\partial t} = \frac{1}{2}\boldsymbol{u}^\top\boldsymbol{u}. \qquad (14)$$

With the above ingredients we can now set up the robust HJB:

$$\begin{aligned}0 = \inf_{\boldsymbol{u}\in\mathcal{U}} \Big[&\tilde{F}_V V(\mu_V - \sigma_V u_1) + \tilde{F}_I I(\mu_I - \sigma_I(\rho u_1 + \sqrt{1-\rho^2}u_2)) \\ &+\frac{1}{2}\tilde{F}_{VV}V^2\sigma_V^2 + \frac{1}{2}\tilde{F}_{II}I^2\sigma_I^2 + VI\sigma_V\sigma_I\rho\tilde{F}_{VI} - \delta\tilde{F} + \frac{1}{2}tr(\underline{\underline{\Psi}}^{-1}\boldsymbol{u}^\top\boldsymbol{u})\Big], \qquad (15)\end{aligned}$$

where tr is the trace of a matrix, i.e. $tr(\underline{\underline{\Psi}}^{-1}\boldsymbol{u}^\top\boldsymbol{u}) = \frac{1}{\Psi_1}u_1^2 + \frac{1}{\Psi_2}u_2^2$. Hence, we obtain the robust HJB

$$0 = \inf_{\boldsymbol{u}\in\mathcal{U}} \Big[\tilde{F}_V V(\mu_V - \sigma_V u_1) + \tilde{F}_I I(\mu_I - \sigma_I(\rho u_1 + \sqrt{1-\rho^2}u_2))$$

$$+\frac{1}{2}\tilde{F}_{VV}V^2\sigma_V^2+\frac{1}{2}\tilde{F}_{II}I^2\sigma_I^2+VI\sigma_V\sigma_I\rho\tilde{F}_{VI}-\delta\tilde{F}+\frac{1}{2\Psi_1}u_1^2+\frac{1}{2\Psi_2}u_2^2\Bigg]. \tag{16}$$

From the two first order conditions with respect of the distortion process u we get that

$$u_1^*=\Psi_1\left(\tilde{F}_V V\sigma_V+\rho\sigma_I\tilde{F}_I I\right), \tag{17}$$

$$u_2^*=\Psi_2\tilde{F}_I I\sqrt{1-\rho^2}\sigma_I. \tag{18}$$

Inserting the solution into the HJB equation we get the partial differential equation (PDE)

$$\begin{aligned}0=&\tilde{F}_V V\left(\mu_V-\sigma_V\Psi_1\left(\tilde{F}_V V\sigma_V+\rho\sigma_I\tilde{F}_I I\right)\right)+\tilde{F}_I I\left(\mu_I-\sigma_I\left(\rho\Psi_1\left(\tilde{F}_V V\sigma_V+\rho\sigma_I\tilde{F}_I I\right)\right.\right.\\&\left.\left.+(1-\rho^2)\Psi_2\tilde{F}_I I\sigma_I\right)\right)+\frac{1}{2}\tilde{F}_{VV}V^2\sigma_V^2+\frac{1}{2}\tilde{F}_{II}I^2\sigma_I+VI\rho\sigma_V\sigma_I\tilde{F}_{VI}-\delta\tilde{F}\\&+\frac{1}{2}\Psi_1\left(\tilde{F}_V V\sigma_V+\rho\sigma_I\tilde{F}_I I\right)^2+\frac{1}{2}\Psi_2\tilde{F}_I^2 I^2(1-\rho^2)\sigma_I^2.\end{aligned} \tag{19}$$

The PDE (19) characterizes the value of the option to invest together with the following boundary boundary conditions[3]

$$\lim_{V\to 0}\tilde{F}(V,I)=0, \tag{20}$$

$$\lim_{I\to\infty}\tilde{F}(V,I)=0, \tag{21}$$

$$\tilde{F}(V^*,I^*)=V^*-I^*, \tag{22}$$

$$\left.\frac{\partial\tilde{F}(V,I)}{\partial V}\right|_{V=V^*}=1, \tag{23}$$

$$\left.\frac{\partial\tilde{F}(V,I)}{\partial I}\right|_{I=I^*}=-1. \tag{24}$$

Condition (20) ensures that the option becomes worthless as the value of the project tends to zero. Similarly, (21) states than the value of the option to invest is zero when the investment cost is infinitely high. Condition (22) is the value matching condition saying that the value of the option to invest equals the value of the project minus the investment cost. Finally, (23)–(24) are the smooth pasting conditions needed to derive the optimal investment threshold. These conditions are standard in the real options literature, see e.g. Dixit and Pindyck (1994), Schwartz and Trigeorgis (2004), and Alili and Kyprianou (2005).

[3]It is not evident that these requirements are enough for the optimal stopping time. However, Riedel (2010) analyzes this problem in a general form and has this solution, when the state variables are Markov processes.

To obtain a closed form solution we impose more structure on the subjective ambiguity parameter. Consider the distortion process from the first order conditions, i.e. (17) and (18). From this we see a dependence on the value of the option to invest and the two underlying processes V and I. If we assume that the structure of the functional form of the value is the same as with ambiguity neutrality—and if Ψ_1 and Ψ_2 are constants—we obtain some less desired properties of the distortion process. Maenhout (2004, 2006) faces a similar problem in a portfolio choice problem. To circumvent this, he assumes that Ψ is state dependent. We follow his approach and define the penalty parameter as

$$\Psi_i = \Psi_i(V, I) = \frac{\theta_i}{\tilde{F}(V, I)}. \tag{25}$$

Intuitively, the option value decreases, when the project value decreases or the investment cost increases. Therefore, the above specification increases the weight Ψ_i, i.e. the firm takes larger distortions away from the reference measure into account. This seems plausible, as the firm should fear a mis-specification of the growth of the project value more when it is low. Similarly, it should fear a mis-specification of the growth of the investment cost more when the investment cost is high.

In the following we will consider the profit ratio $v \triangleq V/I$. In a standard real options problem it is well known that the profit ratio is sufficient to describe the problem, see e.g. Dixit and Pindyck (1994, Chapter 6). In this case, the two-dimensional problem reduces to a one-dimensional one. It turns out that this is also possible when ambiguity aversion is present.

An application of Itô's Lemma shows that the dynamics of the profit ratio is

$$dv_t = v_t(\mu_V - \mu_I + \sigma_I^2 - \rho\sigma_V\sigma_I)dt + v_t \left(\sigma_V - \rho\sigma_I, \sigma_I\sqrt{1-\rho^2}\right)^\top d\boldsymbol{B_t}. \tag{26}$$

When we consider the investment problem with the profit ration as the state variable, we denote the value of the option to invest as $f(v)$. Using this together with the penalty specification, we obtain the robust HJB

$$\begin{aligned} 0 =& \frac{1}{2} f'' \left(\sigma_V^2 + \sigma_I^2 - 2\rho\sigma_V\sigma_I\right) v^2 + f' \left(\mu_V - \mu_I + \rho^2\sigma_I^2\theta_1 - \rho\sigma_V\sigma_I\theta_1 + \sigma_I^2(1-\rho^2)\theta_2\right) v \\ &+ f \left(-\delta + \mu_I - \frac{1}{2}\rho^2\sigma_I^2\theta_1 - \frac{1}{2}\sigma_I^2(1-\rho^2)\theta_2\right) \\ &- \frac{1}{2} f'^2 f^{-1} \left(\sigma_V^2\theta_1 + \sigma_I^2(1-\rho^2)\theta_2 + \rho^2\sigma_I^2\theta_1 - 2\rho\sigma_V\sigma_I\theta_1\right) v^2. \end{aligned} \tag{27}$$

We note that (27) is significantly different from the ODE in the original problem by McDonald and Siegel (1986). Indeed, the last term has the non-linear factor $(f'(v))^2/f(v)$. However, the following lemma which shows that—as in McDonald and Siegel (1986)—our problem with ambiguity aversion exhibits homogeneity.

Lemma 2.1. *Let the investor have ambiguity aversion parameters θ_1 and θ_2. Then the value of the option to invest can be written*

$$\tilde{F}(V, I) = I f(v) \tag{28}$$

where $f(v)$ is a function depending on $v = V/I$.

Hence, we reduce the problem into solving the corresponding differential equation that depends only on v. For future reference we set up some notation. From (26) the volatility is on the vector form

$$\boldsymbol{\sigma} = \left(\sigma_V - \rho\sigma_I, \sqrt{1-\rho^2}\sigma_I\right)^\top, \tag{29}$$

and we also specify the ambiguity aversion matrix and another volatility vector as

$$\underline{\underline{\theta}} = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}, \tag{30}$$

$$\boldsymbol{\sigma_I} = (\rho\sigma_I, \sqrt{1-\rho^2}\sigma_I)^\top. \tag{31}$$

We can now state the general solution to (27).

Lemma 2.2. *1. Assume that $\underline{\underline{\theta}} \neq \underline{\underline{I}}$. The solution of equation (27) is on the form*

$$f(v) = (y_1(v) + y_2(v))^{Q(\underline{\underline{\theta}})}, \tag{32}$$

where

$$Q(\underline{\underline{\theta}}) = \frac{\boldsymbol{\sigma}^\top\boldsymbol{\sigma}}{\boldsymbol{\sigma}^\top\left(\underline{\underline{I}} - \underline{\underline{\theta}}\right)\boldsymbol{\sigma}},$$

and y_1 and y_2 are two linear independent functions.

2. If $\underline{\underline{\theta}} = \underline{\underline{I}}$, then

$$f(v) = C_2 v^{\frac{2\left(\delta - \mu_I + \frac{1}{2}\sigma_I^2\right)}{2\left(\mu_V - \mu_I - \rho\sigma_V\sigma_I + \sigma_I^2\right) - \boldsymbol{\sigma}^\top\boldsymbol{\sigma}}} \times \exp\left[\frac{\boldsymbol{\sigma}^\top\boldsymbol{\sigma}}{\boldsymbol{\sigma}^\top\boldsymbol{\sigma} - 2\left(\mu_V - \mu_i - \rho\sigma_V\sigma_I + \sigma_I^2\right)}\right]. \tag{33}$$

Given the solution to the HJB equation (27), the value of the option to invest and the optimal investment threshold is then found as in the standard problem without ambiguity.

Theorem 2.1. *The optimal investment threshold is*

$$V^*/I^* = v^* = \frac{\beta}{\beta - 1}, \tag{34}$$

where

$$\beta = \frac{\boldsymbol{\sigma}^\top \boldsymbol{\sigma}}{\boldsymbol{\sigma}^\top (\underline{\underline{I}} - \underline{\underline{\theta}})\boldsymbol{\sigma}} \times \frac{1}{2}\left[-\left(\frac{2\left(\mu_V - \mu_I - \theta_1 \rho \sigma_V \sigma_I + \boldsymbol{\sigma_I}^\top \underline{\underline{\theta}} \boldsymbol{\sigma_I}\right)}{\boldsymbol{\sigma}^\top \boldsymbol{\sigma}} - 1 \right) \right.$$
$$\left. + \left[\left(\frac{2\left(\mu_V - \mu_I - \theta_1 \rho \sigma_V \sigma_I + \boldsymbol{\sigma_I}^\top \underline{\underline{\theta}} \boldsymbol{\sigma_I}\right)}{\boldsymbol{\sigma}^\top \boldsymbol{\sigma}} - 1 \right)^2 + 8 \frac{\boldsymbol{\sigma}^\top \left(\underline{\underline{I}} - \underline{\underline{\theta}}\right) \boldsymbol{\sigma} \left(\delta + \frac{1}{2}\boldsymbol{\sigma_I}^\top \underline{\underline{\theta}} \boldsymbol{\sigma_I} - \mu_I\right)}{\left(\boldsymbol{\sigma}^\top \boldsymbol{\sigma}\right)^2} \right]^{1/2} \right].$$

The value of the project is given by

$$\tilde{F}(V, I) = I\left(v^* - 1\right)\left(\frac{V}{v^* I}\right)^\beta. \tag{35}$$

2.1 The investment problem without ambiguity

To compare the effect of ambiguity, we want to use the solution from the investment problem without ambiguity as a benchmark. Since this is the in standard problem already studied e.g. McDonald and Siegel (1986) and Dixit and Pindyck (1994) we omit the details of the derivation.

Thus, the firm has to find the optimal time to invest and, hence, receive the profit $V - I$. Therefore, the firm's problem is

$$F(V, I) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\delta\tau}\left(V_\tau - I_\tau\right)\right] = \mathbb{E}\left[e^{-\delta\tau^*}\left(V_{\tau^*} - I_{\tau^*}\right)\right].$$

This can be reduced to only depend on the profit ratio $v = V/I$, i.e.

$$F(V, I) = I f(v),$$

where $f(v)$ satisfies the standard HJB (27) with $\theta_1 = \theta_2 = 0$. Using the value matching and smooth-pasting conditions the value of the option is

$$F(V, I) = I\left(v^* - 1\right)\left(\frac{v}{v^*}\right)^\alpha,$$

where

$$\alpha = \frac{\left(\hat{\mu} - \frac{1}{2}\boldsymbol{\sigma}^\top\boldsymbol{\sigma}\right) + \left[\left(\hat{\mu} - \frac{1}{2}\boldsymbol{\sigma}^\top\boldsymbol{\sigma}\right)^2 + 2\delta\boldsymbol{\sigma}^\top\boldsymbol{\sigma}\right]^{1/2}}{\boldsymbol{\sigma}^\top\boldsymbol{\sigma}},$$

where we use $\boldsymbol{\sigma} = (\sigma_v, \sigma_I - \rho\sigma_V)^\top$ and $\hat{\mu} = \mu_V - \mu_I$. The optimal investment threshold is given by

$$v^* = \frac{\alpha}{\alpha - 1}.$$

We have that $\alpha > 1$ and, thus, the firm waits until the project value is sufficiently higher than the investment cost. We also see that there exists a ray through the origin separating waiting and investment in the (V, I) space. Here, the slope of the threshold line has the standard option value interpretation. That is, the slope is increased when either σ_V or σ_I is increased, since α is decreasing in both.

To simplify the problem, it is often assumed that the investment cost is constant and the comparative statics of the problem is easier to interpret. In the present setting this implies no loss of generality. We obtain

$$F(V) = AV^{\hat{\alpha}},$$

where

$$\hat{\alpha} = \frac{-\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\delta}}{\sigma^2} > 1, \tag{36}$$

and $A = (V^* - I)(V^*)^{-\hat{\alpha}}$.

We now turn to analyze how ambiguity aversion impacts the firm's investment decision in more detail.

3 Solution with fixed investment cost

In this section we assume that the investment cost is fixed. This allows us to consider a simpler problem and to provide easier interpretations of the results. Also, we can compare our framework to the multiple prior framework. In Section 4 we consider the case of stochastic investment cost.

Recall that the firm faces the investment problem

$$\tilde{F}(V,t) = \sup_{\tau\in\mathcal{T}} \inf_{u\in\mathcal{U}} \left[\mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{\delta(\tau-t)}\left(V_\tau - I\right)|\tilde{\mathcal{F}}_t\right] + \frac{1}{\Psi}\int_0^\infty \int \log\left(\frac{d\mathbb{P}_s}{d\tilde{\mathbb{P}}_s}\right) d\mathbb{P}_s ds\right].$$

As in Section 2 we set $\Psi(V) = \theta/\tilde{F}(V)$. The corresponding robust HJB equation yields the ODE[4]

$$0 = \frac{1}{2}\sigma^2 V^2 \tilde{F}_{VV} + \mu V \tilde{F}_V - \delta \tilde{F} - \frac{1}{2}\sigma^2 V^2 \theta \tilde{F}(V)^{-1} \tilde{F}_V^2. \tag{37}$$

With this version of $\Psi(V)$, the ODE in (37) differs from the standard one commonly seen in the literature—e.g. Dixit and Pindyck (1994) and Nishimura and Ozaki (2007). In particular, the last term makes the ODE nonlinear and different from the Euler equation. The boundary conditions for the problem are

$$\lim_{V \to 0} \tilde{F}(V) = 0, \tag{38}$$

$$\tilde{F}(V^*) = V^* - I, \tag{39}$$

$$\tilde{F}'(V^*) = 1, \tag{40}$$

where we, of course, assume that the option is not exercised if $V < I$. With the above boundary conditions, the solution collapses to the same structure seen in the standard non-ambiguous framework. Therefore, we highlight this solution below.

Lemma 3.1. *The solution depends on θ.*

- *Assume $\theta \neq 1$. Then the general solution to (37) can be written on the form*

$$\tilde{F}(V) = (y_1(V) + y_2(V))^{\frac{1}{1-\theta}} \tag{41}$$

  where y_1 and y_2 are two linear independent functions of V.

- *Assume $\theta = 1$. Then there are two cases for the solution to equation (37):*

  1. *If $2\mu - \sigma^2 \neq 0$. Then the solution can be written on the form:*

$$\tilde{F}(V) = C_1 V^{\frac{2\delta}{2\mu - \sigma^2}} \exp\left[C_2 V^{1-\frac{2\mu}{\sigma^2}}\right] \tag{42}$$

  2. *$2\mu - \sigma^2 = 0$. The solution is written*

$$\tilde{F}(V) = C_2 V^{C_1} \exp\left[\frac{\delta}{\sigma^2}\log(V)^2\right]$$

  for a arbitrary constant C_1.

[4] To simplify notation we let $\theta_1 = \theta$, $\sigma_V = \sigma$ and $\mu_V = \mu$.

From the proof of Lemma 3.1 it is seen that the functions y_1 and y_2 depend on the value of θ related to the value of $K = 1 + \left(\frac{2\mu}{\sigma^2} - 1\right)^2 \frac{\sigma^2}{8\delta}$. As we will see in Theorem 3.1, we have that $\theta \leq K$. To derive the value of the option to invest, we need to consider conditions (38)–(40), for all three cases of θ. We also make the following assumption:

Assumption 3.1. *The optimal investment threshold V^* is a continuous function of θ.*

The assumption is not restrictive, since it would not make sense that the investment threshold should have jumps as a function of the ambiguity parameter. Therefore, we use the limit of V^*, as θ converges to 1. We also note that the threshold will depend on the sign of $\mu - \frac{1}{2}\sigma^2$. If the sign is negative, then the value of the option to wait will be zero for a ambiguity parameter lower than one. That is, for a θ high enough, but less than one, the firm is willing to invest, if just the net present value is zero. We collect our results below.

Theorem 3.1. *Let the investor have ambiguity aversion θ. Then the value of the project can be written as*

$$F(V) = A_1 V^{\beta_1}, \tag{43}$$

where

$$A_1 = (V^* - I)(V^*)^{-\beta_1},$$

and

- *if $\theta \neq 1$ and $\theta \leq 1 + \left(\frac{2\mu}{\sigma^2} - 1\right)^2 \frac{\sigma^2}{8\delta}$, then*

$$\beta_1 = \frac{-\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2(1-\theta)\sigma^2\delta}}{\sigma^2(1-\theta)}, \tag{44}$$

- *if $\theta = 1$, then*

$$\beta_1 = \frac{\delta}{\mu - \frac{1}{2}\sigma^2},$$

- *if $\theta > 1 + \left(\frac{2\mu}{\sigma^2} - 1\right)^2 \frac{\sigma^2}{8\delta}$, then the value of the option to wait is zero.*

The optimal investment threshold occurs as the project value

$$V^* = \frac{\beta_1}{\beta_1 - 1} I. \tag{45}$$

From the proof of Theorem 3.1 it follows that the firm is concerned with the degree of ambiguity aversion and the sign of $2\mu - \sigma^2$. If $2\mu - \sigma^2 < 0$, then the value of the option to wait is zero for a $\hat{\theta} < 1$. If $2\mu - \sigma^2 = 0$, then the value of the option is zero for $\theta \geq 1$, since the optimal threshold is I. In addition, for a small enough σ we have that $V^* > I$ for all θ, i.e. the value of the option is positive. This may seem odd, but even in the case of no volatility and $\theta = 0$, which makes it a deterministic problem, the optimal threshold is also higher than I, $V^* = \frac{\delta}{\delta - \mu}$, see Dixit and Pindyck (1994, Section 5.1).

3.1 Loss in value by not using the robust rule

We now want to measure the firm's loss by not taking ambiguity into account. This is inspired by the certainty equivalent from asset pricing and portfolio choice. Denote the value of the option to invest—conditional on using the optimal robust investment rule—as $\tilde{F}(V; V^*) = \tilde{F}(V)$. Let $\hat{F}$ be the option value for an arbitrary investment rule, i.e. $\hat{F}(V) = \tilde{F}(V; \hat{V})$. $\hat{F}$ satisfies the absorbing as well as the value matching condition, so we know that

$$\hat{V} = V^*(0), \tag{46}$$

where V^* is considered as a function of θ.

Definition 3.1. *We measure the loss as L, where L denotes how large a fraction of the option value the firm is willing to give up in order to know the optimal robust investment strategy. Thus, L satisfies*

$$(1 - L)\tilde{F}(V) = \hat{F}(V).$$

Rewriting the two option expressions we see that the loss is

$$L = 1 - \frac{\beta_1(\theta) - 1}{\beta_1(0) - 1} \left(\frac{\beta_1(0)}{\beta_1(0) - 1} \right)^{-\beta_1(\theta)} \left(\frac{\beta_1(\theta)}{\beta_1(\theta) - 1} \right)^{\beta_1(\theta)}. \tag{47}$$

Note, that the loss vanishes, if there is no ambiguity aversion.

3.2 The multiple prior method

Our solution in Section 2 is derived using the multiplier approach. That is, we rely on the robust HJB and the specification of the subjective ambiguity parameter Ψ. Another framework to address ambiguity is the so-called multiple prior or maximin approach. In this subsection we use a different functional form for Ψ than previously to illustrate how our results relate to the multiple prior model from e.g. Nishimura and Ozaki (2007).

To see the differences between the two approaches we first briefly describe the multiple prior model by Nishimura and Ozaki (2007) and Trojanowska and Kort (2010). In these papers it is assumed that the distortion process is in a fixed, compact interval called κ-ignorance; i.e. $u \in [-\kappa, \kappa]$. Moreover, the dynamics of the value process is given by

$$dV_t = V_t\left((\mu - \kappa\sigma)dt + \sigma d\tilde{B}_t\right),$$

while the investment cost is kept constant. The next result follows from the standard analysis in e.g. Dixit and Pindyck (1994, Chapter 5)[5]

Proposition 3.1. *Let the investor have κ-ignorance and let the investment cost be fixed to I. The optimal threshold is given by*

$$V^* = \frac{\tilde{\beta}_1}{\tilde{\beta}_1 - 1} I, \tag{48}$$

where

$$\tilde{\beta}_1 = \frac{-\left((\mu - \kappa\sigma) - \frac{1}{2}\sigma^2\right) + \sqrt{((\mu - \kappa\sigma) - \frac{1}{2}\sigma^2)^2 + 2\delta\sigma^2}}{\sigma^2}. \tag{49}$$

It is immediate from (48) that the value of the option to invest is of the same functional form as in Section 2 and Section 3. However, there is a substantial difference due to the fact that the ambiguity aversion only shows up as a decrease in drift. Thus $\tilde{\beta}_1$ in (49) differs from e.g. β_1 in (44). Furthermore, from Girsanov's theorem we know that u is multiplied with σ. Therefore, it is not clear whether the investment threshold behaves similar in the two frameworks for different degrees of volatility.

It is interesting to consider whether the multiplier approach can be more directly related to the multiple prior approach. Suppose we set $\Psi(V) = \theta/(V\tilde{F}'(V))$. Then it follows by the method in Section 2 that $u^* = \theta\sigma$. Inserting this into the robust HJB we get

$$0 = \tilde{F}'(V)V(\mu - \frac{1}{2}\sigma^2\theta) + \frac{1}{2}\sigma^2V^2\tilde{F}''(V) - \delta\tilde{F}(V).$$

[5] Nishimura and Ozaki (2007) and Trojanowska and Kort (2010) consider the problem where the firm receives a cash flow from the point of investment. Both papers are able to solve the problem with their assumption of $\kappa-$ignorance, since that enables them to employ dynamic programming.

Thus, if we set $\kappa = \frac{1}{2}\sigma\theta$, we obtain exactly the same solution as in the multiple prior approach.

Since the two different specifications of Ψ appear to be closely related, one could be tempted to conclude that the approaches provide the same qualitative results. However, as we demonstrate below, there are important differences. More importantly, in Section 2 we solved the firm's investment problem in the general case with stochastic project value as well as stochastic investment cost. This allows us to study correlation effects as will be done in Section 4. In contrast, correlation effects seem to be harder to analyze in the multiple prior approach, since the results in Nishimura and Ozaki (2007) and Trojanowska and Kort (2010) cannot immediately be extended to include the case in which the investment cost is random and correlated with the value of the project.

3.3 Comparative statics

We now turn to an numerical analysis to show how the option value and the investment decision changes with ambiguity aversion. To restrict the analysis, we focus our attention to effects ambiguity aversion as well as volatility. To make comparisons we consider a base case with parameters given in Table 1. These parameters are inspired by McDonald and Siegel (1986). Note that $\mu - \frac{1}{2}\sigma^2 = -0.01 < 0$, cf. Theorem 3.1.

[Table 1 about here.]

[Figure 1 about here.]

Figure 1 depicts the value of the option to invest as a function of the project value V. The green (blue) curve depicts the option value with (without) ambiguity. We see that introducing ambiguity aversion lowers the value of the option, since the firm is unsure about the dynamics of the project value. Consequently, the value of waiting is lower with ambiguity aversion.

[Figure 2 about here.]

The impact of ambiguity aversion is further studied in Figure 2. In Figure 2(a) we depict the investment threshold as a function of the investor's ambiguity aversion. The threshold is clearly decreasing, i.e. the investor invests for a lower V in order to eliminate the uncertainty about the drift μ, and the more ambiguity averse the earlier does the investor want to eliminate this uncertainty. As we have seen we have that the investment threshold converges to $\hat{V} < I$ as $\theta \to \hat{\theta}$, and for $\hat{\theta} > 1$ the value of the project is zero.[6] This result is different from the results by both Nishimura and Ozaki (2007) and Trojanowska and Kort (2010), where the value of the project is always positive. The intuition is that as θ is increased above our threshold for positive investment the penalty from changing the probability measure in form of the relative entropy is so large, that the investor will choose an investment threshold that is actually negative.

Figure 2(b) illustrates the loss of neglecting ambiguity aversion. We use the loss to measure whether the importance of taking ambiguity into account. Intuitively, the higher the ambiguity aversion is, the higher the loss will be. From Figure 2(a) we know that the investment threshold decreases in ambiguity aversion. Even if the firm is only slightly ambiguity avers, the drop in the investment threshold seems to be quit large. However, Figure 2(b) shows that the loss is small. On the other hand, the loss increases a lot for larger degrees of ambiguity aversion. We also consider the effect of volatility. Interestingly, a higher volatility yields a higher loss. This is due to the fact that the robust adjustment of the drift is scaled directly by volatility. Thus, volatility emphasizes the effect of ambiguity aversion. For example, when $\theta = 0.2$, the loss is about 3% in the base case, but it increases to 10% when the volatility is increased to 0.15.

We now turn to a comparison between the multiplier approach and the multiple prior approach. Specifically, we consider how the investment threshold behaves as volatility is changed. This is illustrated in panel (c) and (d) in Figure 2. Figure 2(c) employs the model of the present paper, while Figure 2(d) uses the multiple prior approach in the sense of κ-ignorance as in Nishimura and Ozaki (2007). Recall, that the latter method implies that the drift is adjusted downward so that the expected growth rate is reduced from μ to $\mu - \kappa\sigma$. To make the comparisons as equal as possible, we have used $\kappa = 0.0556$ when $\theta = 0.3$, and when $\kappa = 0.5$ we set $\theta = 0.868$. These specifications ensure that the investment threshold is equal in the two specifications when $\sigma = 0.20$. As seen in Figure 2, the volatility effect on the investment threshold is somewhat different in the two implementations of ambiguity aversion. The multiplier approach seems to provide

[6]This is due to $\mu - \sigma^2/2 < 0$, but close to zero in the base case. Otherwise we have a maximal admissible θ, cf. Theorem 3.1, at which $V^* > I$.

a monotone effect, while the multiple prior approach is clearly non-monotonic. As in standard real options theory, the investment threshold is increasing in volatility when ambiguity aversion is absent (i.e. $\theta = 0$), cf. the blue curve in Figure 2(c) and Figure 2(d). However, as the importance of ambiguity aversion increases with volatility, the value of waiting to invest is dampened when volatility increases. When ambiguity aversion is high enough, we see from Figure 2(c) that the investment threshold can be decreasing in volatility. This decreasing effect is also present in Figure 2(d), but the usual increasing effect due to the value of flexibility enters when the volatility of the project value is (very) small compared to the volatility.

4 Comparative statics with stochastic investment cost

In this section we extend the analysis to include stochastic investment cost. Intuitively, the sensitivity analysis from Section 3 regarding the parameters for the project value basically carries over to a partial sensitivity analysis regarding the parameters for the investment cost in the opposite direction. Also, as previously noted, the studies in Nishimura and Ozaki (2007) and Trojanowska and Kort (2010) cannot be readily extended to include stochastic project value together with stochastic investment cost. For these reasons we focus on how correlation plays a role when ambiguity aversion is at play.

[Figure 3 about here.]

Correlation between the the value of the project and the investment cost can have a significant effect based on two reasons. First, there is a spill over effect from the first coordinate of the distortion process, u_1, see (8). Second, correlation obviously plays a role even without ambiguity aversion. Figure 3(a) illustrates the value of the option to invest. As in Section 3.3 the general picture is that ambiguity aversion lowers the investment threshold. When there is no ambiguity aversion, the investment threshold is 2.32 (not shown in the figure). It decreases to 2.29 if there is ambiguity aversion regarding the investment cost ($\theta_1 = 0, \theta_2 = 0.3$), and further down to 1.79 if there is ambiguity aversion regarding the value of the project ($\theta_1 = 0.3, \theta_2 = 0$). Overall, the figure shows that the impact of ambiguity aversion regarding the project value is more influential than ambiguity aversion regarding the investment cost.

[Figure 4 about here.]

We next consider how correlation and volatility regarding the project value affects the investment threshold. This is done in Figure 3(b). In the figure, the blue (purple/gray) curve corresponds to a negative (zero/positive) correlation. When correlation is zero, we have the same picture as the green curve in Figure 2(c). That is, the investment threshold is a convex-concave function of volatility. When the correlation is negative, the investment threshold is increases in volatility. This relates to the usual result in real option analysis, i.e. the value of waiting increases in volatility. For positive correlation, the investment threshold is no longer monotone in volatility. When volatility increases there is a decreasing effect in the project value's expected growth due to ambiguity. This drift-effect is dominating when volatility is low. However, as the project value's volatility is further increased the usual volatility effect is dominates.

In Figure 4 we consider the investment threshold depending on the volatility parameters σ_V and σ_I for positive and negative correlation. Consider first the case when the volatility of the project value is high ($\sigma_V = 0.4$). When the volatility of the investment cost increases, we clearly see a different shape of v^* depending on the correlation. When the correlation is positive, an increase in σ_I lowers the drift of the profit ratio even further. Therefore, we have an even lower threshold. Thus, the investment threshold is non-monotonic in the investment cost's volatility. Hence, comparing to the results in Figure 3(b), we see that volatility impacts the investment threshold depending on the correlation. Turning to the case with negative correlation, an increase in σ_I increases the drift and the threshold is increased. This is due to that fact, that with negative correlation, a higher volatility in the investment cost increases the value of waiting.

Interestingly, if we consider a high level of volatility of the investment cost, Figure 4 demonstrates the project value's volatility impacts the investment threshold differently. With positive correlation, ambiguity aversion makes more volatility (in addition to the high σ_I) less desirable. Thus, the value of waiting decreases in σ_V. On the contrary, when correlation is negative, the value of waiting increases in σ_V as long as it is not too large. To conclude, it is clear that correlation significantly impacts the firm's investment behavior.

5 Conclusion

We examine a firm's investment decision when it is ambiguity averse. That is, the firm does not trust its reference model and it wants to take account of this uncertainty. Despite the fact that we need to solve a non-standard differential equation, we are able to derive closed form solutions for the value of the option to invest as well as the investment threshold. Overall, ambiguity aversion decreases the value of the option to invest. Thus, the firm invests for a lower level of project value compared to the case with no ambiguity aversion. Interestingly, in contrast to standard real option analysis, the value of waiting is no longer monotonically increasing in volatility. This is due to the fact that volatility emphasizes ambiguity aversion. Thus, we demonstrate that it is more important to take ambiguity into account, the larger the ambiguity aversion is and, in particular, the larger the volatility is.

Since we can let both the project value and the investment cost be uncertain, we are able to study correlation effects. We show different correlations imply important different effects regarding the investment decision. For example, if the value of the project has higher volatility, this may decrease the investment threshold, when the correlation is positive. In contrast, with negative correlation the investment threshold can increase in the project value's volatility. Hence, ambiguity aversion is an important aspect to take into account when the firm considers its investment strategy.

A Proofs

A.1 Proofs from section 2

Proof of Lemma 2.1. The pay-off function is naturally homogeneous of degree one in both variables, and thus we need to show that the robust HJB equation can be rewritten into a differential equation depending on only one variable. First we solve for the controls, and we have the partial differential equation. We guess that the value of the project can be written as a function of the pay off ratio times the investment cost, which is put into the robust HJB. The HJB equation 19

$$\begin{aligned}
0 =& \tilde{F}_V \left(\mu_V - \sigma_V \Psi_1 \left(\tilde{F}_V V \sigma_V + \rho\sigma_I \tilde{F}_I I\right)\right) \\
&+ \tilde{F}_I I \left(\mu_I - \sigma_I \left(\rho \Psi_1 \left(\tilde{F}_V V \sigma_V + \rho\sigma_I \tilde{F}_I I\right) + (1-\rho^2)\Psi_2 \tilde{F}_I I \sigma_I\right)\right) \\
&+ \frac{1}{2}\tilde{F}_{VV} V^2 \sigma_V^2 + \frac{1}{2}\tilde{F}_{II} I^2 \sigma_I - \delta \tilde{F} + V I \rho \sigma_V \sigma_I \tilde{F}_{VI} \\
&+ \frac{1}{2}\Psi_1 \left(\tilde{F}_V V \sigma_V + \rho\sigma_I \tilde{F}_I I\right)^2 + \frac{1}{2}\Psi_2 \tilde{F}_I^2 I^2 (1-\rho^2)\sigma_I^2
\end{aligned} \tag{50}$$

If we gues a solution on the form

$$\tilde{F}(V, I) = I f(v)$$

with $v = V/I$. Then the partial derivatives of $\tilde{F}$ are

$$\tilde{F}_V = f'(v),\ \tilde{F}_I = f(v) - v f'(v),\ \tilde{F}_{VV} = f''(v)/I,\ \tilde{F}_{II} = v^2 f''(v)/I,\ \tilde{F}_{VI} = -v f''(v)/I$$

These are inserted into equation (19) and for simplicity we leave out the variables in f

$$\begin{aligned}
0 =& \mu_V V f' + \mu_I I (f - v f') - \delta I f + \frac{1}{2}\sigma_V^2 V^2 f''/I + \frac{1}{2}\sigma_I^2 I^2 v^2 f''/I \\
&+ \rho\sigma_V\sigma_I V I (-f'' v/I) - \frac{1}{2}\sigma_V^2 V^2 f'^2 \Psi_1 - \frac{1}{2}\rho^2\sigma_I^2 I^2 (f - v f')^2 \Psi_1 \\
&- \rho\sigma_V\sigma_I V I f'(f - v f')\Psi_1 - \frac{1}{2}\sigma_I^2 (1-\rho^2) I^2 (f - v f')^2 \Psi_2
\end{aligned} \tag{51}$$

As in the simple model we set the penalties to be state dependent:

$$\Psi_1 = \frac{\theta_V}{\tilde{F}(V, I)} = \frac{\theta_V}{I f(v)}, \quad \Psi_2 = \frac{\theta_I}{\tilde{F}(V, I)} = \frac{\theta_I}{I f(v)}$$

and insert these into (51) and we can divide by I to get

$$0 = \frac{1}{2} f'' \left(\sigma_V^2 + \sigma_I^2 - 2\rho\sigma_V\sigma_I\right) v^2$$

$$+ f'\left(\mu_V - \mu_I + \rho^2\sigma_I^2\theta_1 - \rho\sigma_V\sigma_I\theta_1 + \sigma_I^2(1-\rho^2)\theta_2\right)v$$
$$+ f\left(-\delta + \mu_I - \frac{1}{2}\rho^2\sigma_I^2\theta_1 - \frac{1}{2}\sigma_I^2(1-\rho^2)\theta_2\right)$$
$$- \frac{1}{2}f'^2 f^{-1}\left(\sigma_V^2\theta_1 + \sigma_I^2(1-\rho^2)\theta_2 + \rho^2\sigma_I^2\theta_1 - 2\rho\sigma_V\sigma_I\theta_1\right)v^2 \tag{52}$$

Hence the problem can be reduced to a problem depending of $v = V/I$ similar to the the original problem. □

Proof of Lemma 2.2. Since we have already shown that the problem is homogeneous, we need to solve the ordinary differential equation. Equation (27) can be rewritten as

$$\begin{aligned}
0 =& \frac{1}{2}f''v^2\boldsymbol{\sigma}^\top\boldsymbol{\sigma} - \frac{1}{2}v^2 f'^2 f^{-1}\boldsymbol{\sigma}^\top\underline{\underline{\theta}}\boldsymbol{\sigma} \\
&+ f'v\left(\mu_V - \mu_I - \theta_1\rho\sigma_V\sigma_I + \boldsymbol{\sigma_I}^\top\underline{\underline{\theta}}\boldsymbol{\sigma_I}\right) \\
&+ f\left(-\delta + \mu_I - \frac{1}{2}\boldsymbol{\sigma_I}^\top\underline{\underline{\theta}}\boldsymbol{\sigma_I}\right) \\
=& \frac{1}{2}v^2\boldsymbol{\sigma}^\top\left(f'' - f'^2 f^{-1}\underline{\underline{\theta}}\right)\boldsymbol{\sigma} \\
&+ f'v\left(\mu_V - \mu_I - \theta_1\rho\sigma_V\sigma_I + \boldsymbol{\sigma_I}^\top\underline{\underline{\theta}}\boldsymbol{\sigma_I}\right) \\
&+ f\left(-\delta + \mu_I - \frac{1}{2}\boldsymbol{\sigma_I}^\top\underline{\underline{\theta}}\boldsymbol{\sigma_I}\right)
\end{aligned}$$

With the assumption $\underline{\underline{\theta}} \neq \underline{\underline{I}}$ we can make the same tranformations as in lemma 3.1 and we have that

$$f(v) = u(v)^{Q(v)}$$

and the solution to $u(v)$ has the same form as before, but the solution to the quadratic equation has the form

$$\begin{aligned}
\hat{\beta} =& \frac{1}{2}\left[-\left(\frac{2\left(\mu_V - \mu_I - \theta_1\rho\sigma_V\sigma_I + \boldsymbol{\sigma_I}^\top\underline{\underline{\theta}}\boldsymbol{\sigma_I}\right)}{\boldsymbol{\sigma}^\top\boldsymbol{\sigma}} - 1\right)\right. \\
&\pm \left.\left[\left(\frac{2\left(\mu_V - \mu_I - \theta_1\rho\sigma_V\sigma_I + \boldsymbol{\sigma_I}^\top\underline{\underline{\theta}}\boldsymbol{\sigma_I}\right)}{\boldsymbol{\sigma}^\top\boldsymbol{\sigma}} - 1\right)^2 + 8\frac{\boldsymbol{\sigma}^\top\left(\underline{\underline{I}} - \underline{\underline{\theta}}\right)\boldsymbol{\sigma}\left(\delta + \frac{1}{2}\boldsymbol{\sigma_I}^\top\underline{\underline{\theta}}\boldsymbol{\sigma_I} - \mu_I\right)}{\left(\boldsymbol{\sigma}^\top\boldsymbol{\sigma}\right)^2}\right]^{1/2}\right]
\end{aligned}$$

and thus the solution to f depends on how many solutions there are to $\hat{\beta}$. □

Proof of Theorem 2.1. The proof very similar to the proof for Theorem 3.1 which is given below. Basically, the proof for Theorem 2.1 is only extended in dimension with minimum modifications. □

A.2 Proof from section 3

Proof of Lemma 3.1 . The structure of the proof is to transform the ODE in (37) to a differential equation with recognizable solution, and then transform this solution back to the initial problem. First we rewrite (37)

$$0 = \frac{1}{2}\sigma^2 V^2 \left(\tilde{F}_{VV} - \theta F(V)^{-1}\tilde{F}_V^2\right) + \mu V \tilde{F}_V - \delta \tilde{F}, \tag{53}$$

and abstracting from F=0 we get

$$0 = \frac{\sigma^2}{2} V^2 \left(\frac{\tilde{F}\tilde{F}_{VV} - \tilde{F}_V^2}{\tilde{F}^2(V)} + (1-\theta)\left(\frac{\tilde{F}_V}{F}\right)^2 \right) + \mu V \frac{\tilde{F}_V}{\tilde{F}} - \delta. \tag{54}$$

Let $g(V) = \frac{\tilde{F}_V}{\tilde{F}}$. Then $g' = \frac{\tilde{F}\tilde{F}_{VV} - \tilde{F}_V^2}{F^2}$ and (54) becomes

$$g'(V) = \underbrace{\frac{2\delta}{\sigma^2}\frac{1}{V^2}}_{q_0(V)} + \underbrace{\frac{-2\mu}{\sigma^2}\frac{1}{V}}_{q_1(V)} g(V) + \underbrace{(-(1-\theta))}_{q_2(V)} g^2(V), \tag{55}$$

which we recognize as a Ricatti equation. Therefore, consider the transformation $h(V) = q_2(V)g(V)$ yielding

$$h'(V) = \underbrace{q_0(V)q_2(V)}_{S(V)} + \underbrace{\left(q_1(V) + \frac{q_2'(V)}{q_2(V)}\right)}_{R(V)} h(V) + h^2(V). \tag{56}$$

Finally, let u satisfy

$$h(V) = \frac{u'(V)}{u(V)} \tag{57}$$

then

$$h'(V) = \frac{-u(V)u''(V) + (u'(V))^2}{u^2(V)}, \tag{58}$$

wherewithal

$$\frac{-u(V)u''(V) + (u'(V))^2}{u^2(V)} = S(V) + R(V)\frac{-u'(V)}{u(V)} + \left(\frac{u'(V)}{u(V)}\right)^2. \tag{59}$$

Thus,

$$0 = u''(V) - R(V)u'(V) + S(V)u(V) \tag{60}$$

and inserting for R and S we get

$$0 = u''(V) + \frac{2\mu}{\sigma^2}\frac{1}{V}u'(V) - (1-\theta)\frac{2\delta}{\sigma^2}\frac{1}{V^2}u(V), \tag{61}$$

hence

$$0 = V^2u''(V) + \frac{2\mu}{\sigma^2}Vu'(V) - (1-\theta)\frac{2\delta}{\sigma^2}u(V), \tag{62}$$

which we recognize as a second order Euler differential equation. By setting $V = e^t$ we get an linear second order differential equation.

$$u''(t) + \frac{2\mu}{\sigma^2}u'(t) - (1-\theta)\frac{2\delta}{\sigma^2}u(t) = 0$$

We have the characteristic function

$$\hat{\beta}^2 + \left(\frac{2\mu}{\sigma^2} - 1\right)\hat{\beta} - (1-\theta)\frac{2\delta}{\sigma^2} = 0. \tag{63}$$

where

$$\hat{\beta} = \frac{1}{2}\left[-\left(\frac{2\mu}{\sigma^2} - 1\right) \pm \sqrt{\left(\frac{2\mu}{\sigma^2} - 1\right)^2 + (1-\theta)\frac{8\delta}{\sigma^2}}\right] \tag{64}$$

If the solutions to the characteristic equation are complex, we write $\hat{\beta} = \psi \pm i\varphi$. The solution of u with respect to t then depends on the roots of the characteristic function, which depends mainly on θ. Denote $K = 1 + \left(\frac{2\mu}{\sigma^2} - 1\right)^2 \frac{\sigma^2}{8\delta}$. The solutions can be found in Spiegel and Liu (1999).

[Table 2 about here.]

Hence all solutions is a linear combination of two linear independent functions, y_1 and y_2. Now recall that

$$g(V) = \frac{1}{1-\theta}\frac{u'(V)}{u(V)}$$

but we also have

$$g(V) = \frac{F'(V)}{F(V)}$$

i.e.

$$\frac{\tilde{F}'(V)}{\tilde{F}(V)} = \frac{1}{1-\theta}\frac{u'(V)}{u(V)}$$

hence it follows that

$$\tilde{F}(V) = \hat{C}_3 u(V)^{\frac{1}{1-\theta}},$$

where $\hat{C}_3$ is an integration constant to be determined. Using the expression for u we obtain

$$\tilde{F}(V) = \hat{C}_3 \left(\hat{C}_1 y_1(V) + \hat{C}_2 y_2(V)\right)^{\frac{1}{1-\theta}}, \tag{65}$$

$$= \left(\hat{C}_3^{1-\theta}\hat{C}_1 y_1(V) + \hat{C}_3^{1-\theta}\hat{C}_2 y_2(V)\right)^{\frac{1}{1-\theta}}, \tag{66}$$

which we rewrite as

$$\tilde{F}(V) = (C_1 y_1(V) + C_2 y_2(V))^{\frac{1}{1-\theta}}, \tag{67}$$

as asserted in equation (41) □

Proof of Lemma 3.1 in the case $\theta = 1$. We only do the case with $\mu - \frac{1}{2}\sigma^2 \neq 0$. We assume that $\tilde{F}(V) \neq 0$. The differential equation is now written

$$\begin{aligned} 0 =& \frac{1}{2}\sigma^2 V^2 \left(tF_{VV} - \frac{F_V^2}{F}\right) + \mu V \tilde{F}_V - \delta \tilde{F} \\ =& \frac{1}{2}\sigma^2 V^2 \left(\frac{\tilde{F}_{VV}}{\tilde{F}} - \left(\frac{\tilde{F}_V}{\tilde{F}}\right)^2\right) + \mu V \frac{\tilde{F}_V}{\tilde{F}} - \delta \end{aligned}$$

We define

$$g(V) = \frac{\tilde{F}_V}{\tilde{F}} \quad g_V = \frac{\tilde{F}_{VV}}{\tilde{F}} - \left(\frac{\tilde{F}_V}{\tilde{F}}\right)^2$$

Thus the differential equation is written

$$g_V + \frac{2\mu}{\sigma^2} V^{-1} g = \frac{2\delta}{\sigma^2} V^{-2}$$

Set $G(V) = \frac{2\mu}{\sigma^2}\int V^{-1} dV = \frac{2\mu}{\sigma^2}\log(V)$ and the solution to g is

$$\begin{aligned} g(V) &= \exp\left[-\frac{2\mu}{\sigma^2}\log(V)\right]\left[\frac{2\delta}{\sigma^2}\int V^{-2}\exp\left[\frac{2\mu}{\sigma^2}\log(V)\right] dV + C_1\right] \\ &= \frac{2\delta}{\sigma^2} V^{-1} + C_1 V^{-\frac{2\mu}{\sigma^2}} \end{aligned}$$

To find $\tilde{F}$ we set

$$\begin{aligned} P(V) &= \int \left(-\frac{2\delta}{2\mu - \sigma^2} V^{-1} - C_1 V^{1-\frac{2\mu}{\sigma^2}}\right) dV \\ &= -\frac{2\delta}{2\mu - \sigma^2}\log(V) - \frac{C_1 \sigma^2}{\sigma^2 - 2\mu} V^{1-\frac{2\mu}{\sigma^2}} \end{aligned}$$

and the solution for $\tilde{F}$ is

$$\tilde{F}(V) = C_2 \exp\left[-P(V)\right] = C_2 \exp\left[-\left(\log(V)^{-\frac{2\delta}{2\mu-\sigma^2}} - \frac{C_1\sigma^2}{\sigma^2 - 2\mu} V^{1-\frac{2\mu}{\sigma^2}}\right)\right]$$
$$= C_2 V^{\frac{2\delta}{2\mu-\sigma^2}} \exp\left[C_1 V^{1-\frac{2\mu}{\sigma^2}}\right]$$

□

Proof of Theorem 3.1. Assume that $\theta < 1$. Then from Lemma 3.1 we have that $\hat{\beta}_1 > 0$ and $\hat{\beta}_2 < 0$ and $1/(1-\theta) > 0$. From the conditions (38)–(40) we can set $C_2 = 0$ and the $\tilde{F}$ can be written

$$\tilde{F}(V) = (C_1 V^{\hat{\beta}_1})^{1/(1-\theta)}$$
$$= A_1 V^{\beta_1}$$

where

$$\beta_1 = \frac{-\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2(1-\theta)\sigma^2\delta}}{\sigma^2(1-\theta)}$$

Since we are interested in $\hat{\beta}_1/(1-\theta)$ we study the associated quadratic equation

$$\hat{\beta}^2 + \frac{\left(\frac{2\mu}{\sigma^2} - 1\right)}{1-\theta}\hat{\beta} - \frac{(1-\theta)}{1-\theta}\frac{2\delta}{\sigma^2} = 0,$$

i.e.

$$\underbrace{(1-\theta)\frac{\sigma^2}{2}\hat{\beta}^2 + \left(\mu - \frac{\sigma^2}{2}\right)\hat{\beta} - \delta}_{Q(\beta)} = 0.$$

Since $(1-\theta) > 0$ we have a parabola with upward turning branches. Moreover, as $\delta > \mu$, we get $Q(0) = -\delta < 0$ and $Q(1) = -\theta\frac{\sigma^2}{2} - (\delta - \mu) < 0$. From this it follows that the positive root $\beta \triangleq \hat{\beta}_1/(1-\theta) > 1$. The value matching condition yields

$$F(V^*) = A_1(V^*)^{\beta_1} \triangleq V^* - I,$$

i.e.

$$A_1 = (V^* - I)\,(V^*)^{-\beta_1}.$$

The smooth pasting condition yields

$$F'(V^*) = A_1\beta_1(V^*)^{\beta_1-1} \triangleq 1,$$

i.e.

$$V^* = \frac{\beta_1}{\beta_1 - 1}I,$$

as asserted.

Denote the investment factor:

$$C = \frac{\beta_1}{\beta_1 - 1} = \frac{-\left(\frac{2\mu}{\sigma^2} - 1\right) + \left[\left(\frac{2\mu}{\sigma^2} - 1\right) + 4(1-\theta)\frac{2\delta}{\sigma^2}\right]^{1/2}}{-\left(\frac{2\mu}{\sigma^2} - 1\right) + \left[\left(\frac{2\mu}{\sigma^2} - 1\right) + 4(1-\theta)\frac{2\delta}{\sigma^2}\right]^{1/2} - 2(1-\theta)}$$

If we let $\theta \to 1$ we have that

$$\lim_{\theta \to 1^-} C(\theta) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We use l'Hopital's rule and we get

$$\lim_{\theta \to 1^-} C(\theta) = \frac{\delta}{\delta - (\mu - \frac{1}{2}\sigma^2)}$$

From the proof of Lemma 3.1 in the case of $\theta = 1$, we have that

$$\tilde{F}(V) = C_1 V^{\beta_1} \exp\left[C_2 V^{\beta_2}\right]$$

where $\beta_1 = \frac{2\delta}{2\mu - \sigma^2}$ and $\beta_2 = 1 - \frac{2\mu}{\sigma^2}$ and we have that

$$\frac{\beta_1}{\beta_1 - 1} = \frac{2\delta}{2\delta - 2\mu - \sigma^2}$$

and so for the case $\theta = 1$ we set $C_2 = 0$. With $1 < \theta \leq 1 + \left(\frac{2\mu}{\sigma^2} - 1\right)^2 \frac{\sigma^2}{8\delta}$ we can again set $C_2 = 0$ due to continuous $V^*(\theta)$. For the case $\theta = 1 + \left(\frac{2\mu}{\sigma^2} - 1\right)^2 \frac{\sigma^2}{8\delta}$, we also set $C_2 = 0$. When $\theta > 1 + \left(\frac{2\mu}{\sigma^2} - 1\right)^2 \frac{\sigma^2}{8\delta}$ we have to set both C_1 and C_2 equal to zero, otherwise the function $u(V) = y_1(V) + y_2(V)$ from the proof of Lemma 3.1 will oscillate and become negative. Recall that the actual penalty of changing the measure was Ψ and therefore that we can set $\tilde{F} \equiv 0$ for the case $\theta > 1 + \left(\frac{2\mu}{\sigma^2} - 1\right)^2 \frac{\sigma^2}{8\delta}$ □

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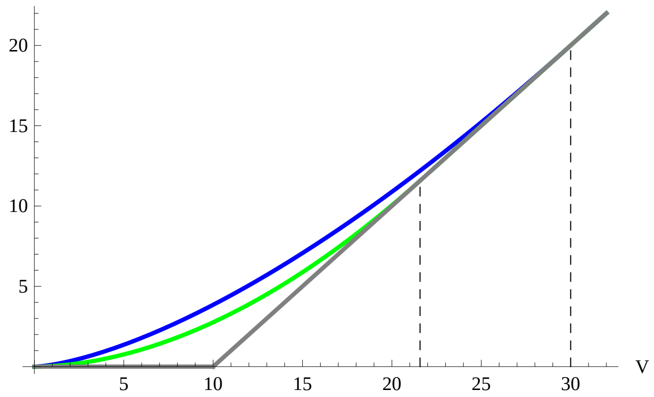
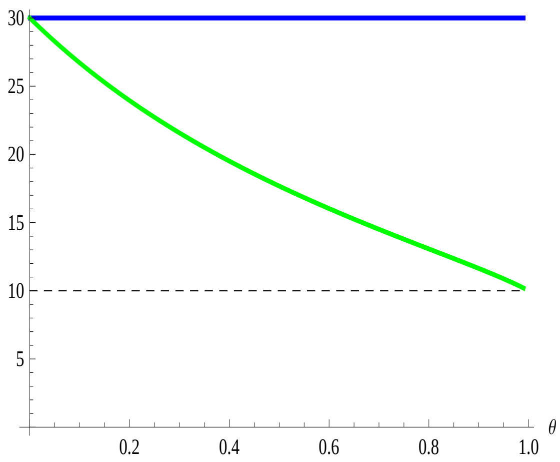
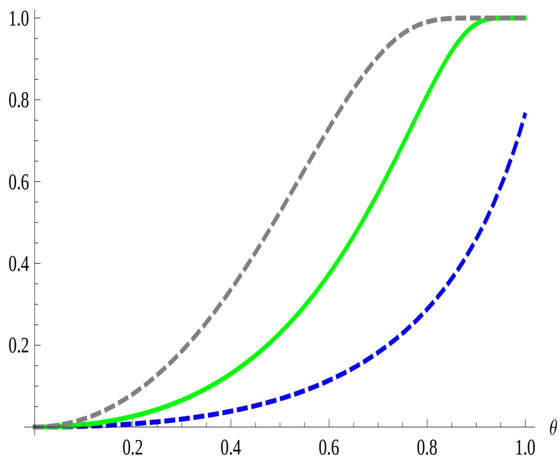


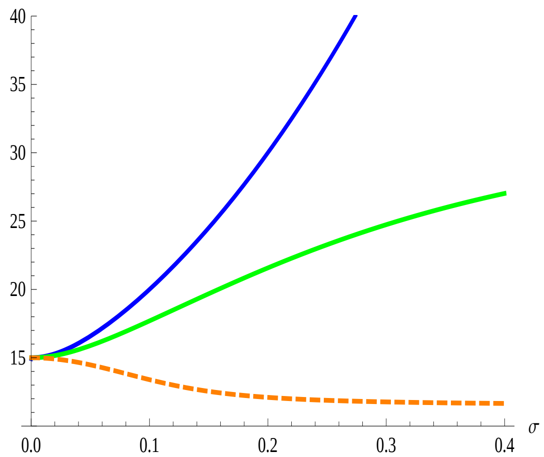
Figure 1: The value of the option to invest as a function of the value of V. The green curve is with ambiguity aversion. The vertical dashed lines indicate the investment threshold.



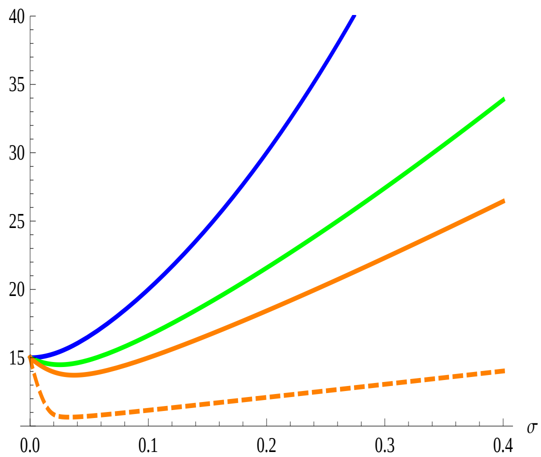
(a) The investment threshold, V^*, with (green curve) and without (blue curve) ambiguity aversion. The dashed line indicates the investment cost level



(b) Loss of ignoring ambiguity aversion for three cases of volatility: 0.15 (dashed, blue), 0.2 (solid, green), and 0.25 (dashed gray)

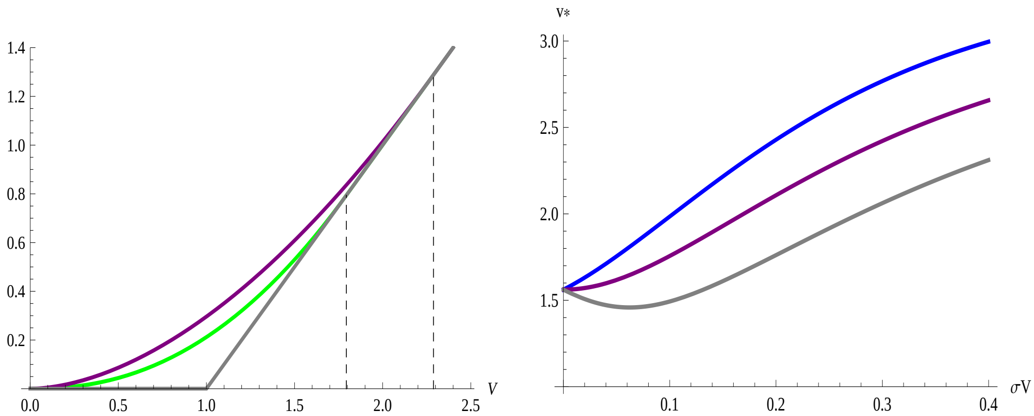


(c) The investment threshold without ambiguity aversion (blue curve) and with (green curve) using base case parameters and $\theta = 0.868$ (orange, dashed)



(d) The investment threshold without ambiguity aversion (blue curve) and with using base case parameters and κ being 0.056 (green curve), 0.1(orange line), and 0.5 (orange, dashed)

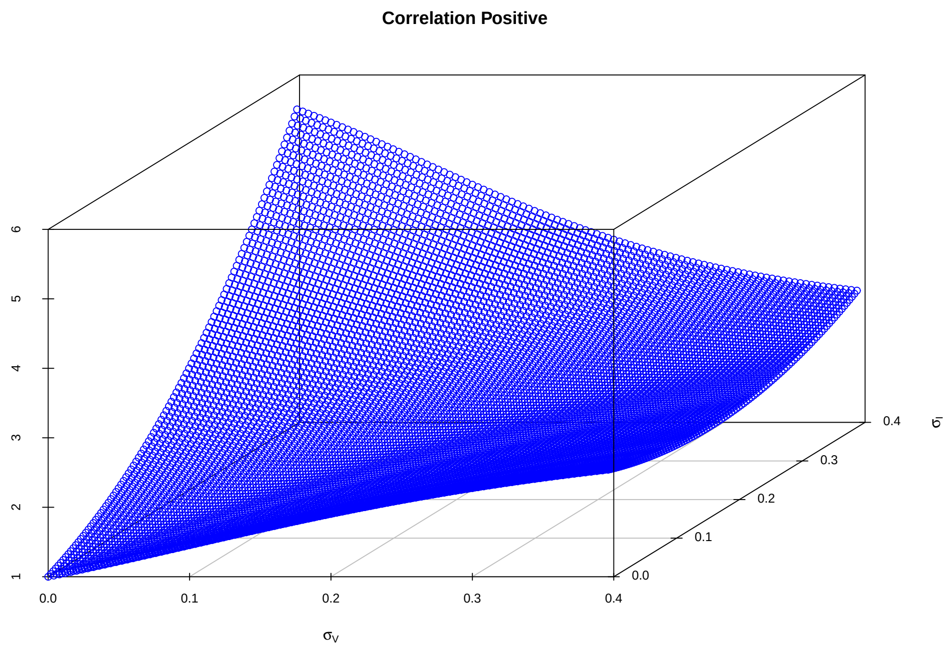
Figure 2: Effects of ambiguity aversion θ and volatility, σ.



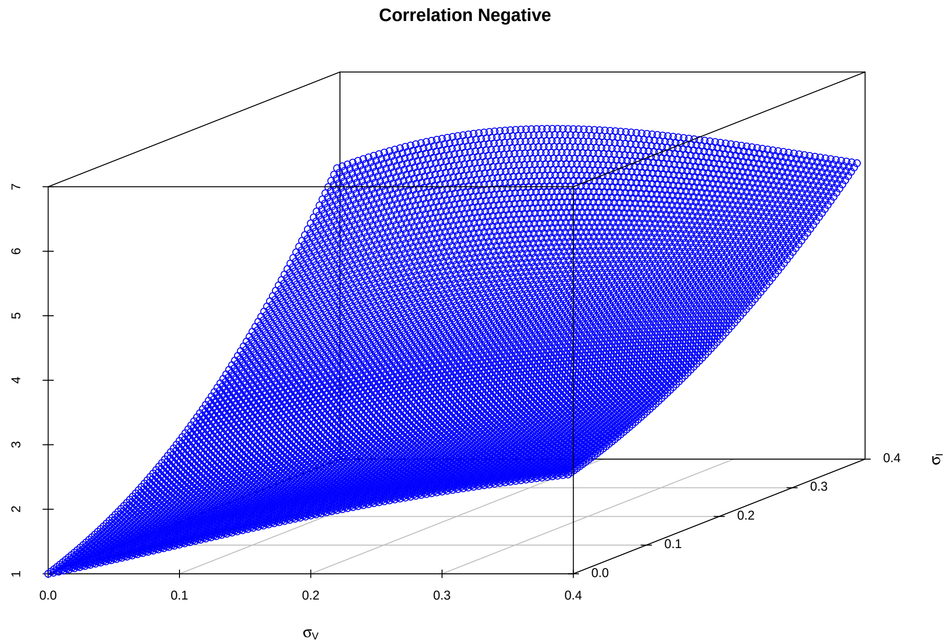
(a) The value of the option to invest as a function of V. The purple (green) curve is for the case $\theta_1 = 0$ ($\theta_2 = 0$).

(b) The investment threshold v^* as a function of the volatility in the value for three different correlations from above: Negative, zero and positive.

Figure 3: The value of the project and the investment threshold.



(a) Positive correlation, $\rho = 1/2$



(b) Negative correlation, $\rho = -1/2$

Figure 4: The investment threshold v^* as a function of the volatilities for two cases of correlation

δ	μ	σ	θ	I
3%	1 %	0.2	0.3	10

Table 1: Parameters for the base case.

Condition	$u(t)$	u(V)
$\theta < K$	$C_1 e^{\hat{\beta}_1 t} + C_2 e^{\hat{\beta}_2 t}$	$C_1 V^{\hat{\beta}_1} + C_2 V^{\hat{\beta}_2}$
$\theta = K$	$C_1 e^{\hat{\beta} t} + C_2 t e^{\hat{\beta} t}$	$C_1 V^{\beta} + C_2 \log(V) V^{\hat{\beta}}$
$\theta > K$	$e^{\psi t}\left(C_1 \cos(\varphi t) + C_2 \sin(\varphi t)\right)$	$V^{\psi}\left(C_1 \cos(\log V) + C_2 \sin(\log V)\right)$

Table 2: Solutions of the differential with respect to t and V.