High-low range in GARCH models of stock return volatility

Peter Molnár*

January 11, 2012

Abstract

GARCH volatility models should not be considered as data-generating processes for volatility but just as filters. Based on this insight we suggest a simple and general way to improve the GARCH volatility models using the difference between the highest and the lowest price of the day. We illustrate this idea on the GARCH(1,1) model, which we modify into the Range GARCH(1,1) model. An empirical analysis confirms that the RGARCH(1,1) model performs significantly better than the standard GARCH(1,1) model regarding both in-sample fit and out-of-sample forecasting ability.

Key words: volatility, GARCH, range, high, low

*Norwegian University of Science and Technology (NTNU), NO-7491 Trondheim, Norway, Tel.: +4773593576, E-mail address: Peter.Molnar@iot.ntnu.no. This paper is based on the chapter from my dissertation which was written while I was at Norwegian School of Economics (NHH), Helleveien 30, 5045 Bergen, Norway. I would like to thank Jonas Andersson, Jan Antell, Milan Bašta, Ray Chou, Stein-Erik Fleten, Ove Hetland, Svein-Olav Krakstad, Lukáš Lafférs, Gudbrand Lien, Per Mykland, Kjell Nyborg and participants in various seminars and conferences for helpful comments.

1 Introduction

The most fundamental variables of finance and economics are changes of asset prices (returns) and their variances. As was observed a long time ago, even though returns of most financial assets are to a large extent unpredictable, their variances display high temporal dependency and are predictable. Starting with the work of Engle (1982) and Bollerslev (1986), the ARCH and GARCH classes of models have become standard tools to for volatility modelling and forecasting. Some of the widely used extensions are EGARCH of Nelson (1991), GJR-GARCH of Glosten et al. (1993) and FIEGARCH of Bollerslev and Mikkelsen (1996). See e.g. Andersen et al. (2006) and Engle and Patton (2001) for surveys and further references.

Starting with the work of Taylor (1986), a new class of volatility models emerged – stochastic volatility models. For an overview see e.g. Ghysels et al. (1996) and Shephard (1996). The main difference between stochastic volatility and GARCH models is the following. Stochastic volatility models assume that volatility evolves over time as some stochastic process and returns are drawn from a distribution parametrized by this volatility. Past returns have no direct effect on the future volatility.[1] In GARCH models, returns are generated the same way as in the stochastic volatility models. The only stochastic element in the volatility equation are past returns and therefore past returns determine future volatility. In other words, returns are stochastic, but once the return is realized, future volatility is determined. A direct comparison of stochastic volatility models and GARCH models is inconclusive (Kim et al. (1998)). The main reason why stochastic volatility models did not become as popular as GARCH models is the practical one - stochastic

[1] Abstracting from leverage effect, which can be possibly incorporated into the stochastic volatility models.

volatility models are more difficult to estimate (see e.g. Broto and Ruiz (2004)).

If we consider GARCH models and stochastic volatility models as data generating processes, they are obviously mutually exclusive. However, important insights from Nelson (1990) and Nelson (1992) tell us that even when a GARCH model is misspecified (it is not a true data generating process for the volatility), it can provide consistent estimates of the volatility if the volatility changes "slowly" relative to the sampling interval (Nelson and Foster (1994)). Simply said, we can think of the GARCH model as a filter through which we pass the data to produce an estimate of the conditional volatility.

Our work is based on a similar intuition. The GARCH model can fit the data quite well even when the volatility itself is not generated by the process specified by the GARCH model. In GARCH type of models, demeaned[2] squared returns serve as a way to calculate innovations to the volatility. Rewriting GARCH models in terms of observed variables (returns) only shows that the GARCH model in fact calculates volatility as a weighted moving average of past squared returns. If volatility is changing gradually over time, the GARCH model will work simply because squared returns are daily volatility estimates and therefore the GARCH model essentially calculates volatility as a weighted moving average of the past volatilities.

This intuition has interesting implications. Most importantly, replacing the squared returns by more precise volatility estimates will produce better GARCH models, regarding

[2]For most of the assets, mean daily return is much smaller than its standard deviation and therefore can be considered equal to zero. From now on we assume that it is indeed zero. This assumption not only makes further analysis simpler, but it actually helps to estimate volatility more precisely. In the words of Poon and Granger (2003): "The statistical properties of sample mean make it a very inaccurate estimate of the true mean, especially for small samples, taking deviations around zero instead of the sample mean typically increases volatility forecast accuracy."

both in-sample fit and out-of-sample forecasting performance. Additionally, coefficients of GARCH models based on more precise volatility estimates than squared returns will be changed in such a way that they will put more weight on more recent observations. We examine both these implications.

To test our idea, we estimate a GARCH(1,1) model using both squared returns and a more precise volatility proxy, in particular the Parkinson (1980) volatility estimator based on range (the difference between high and low). The results confirm our expectations.

In this way our work becomes closely related to Alizadeh et al. (2002), Chou (2005) and Brandt and Jones (2006) who use range-based volatility measures to estimate volatility models. Alizadeh et al. (2002) estimate a stochastic volatility model. Brandt and Jones (2006) estimate EGARCH and FIEGARCH models based on log range. Chou (2005) uses range in standard deviation GARCH. These papers employ range-based volatility proxies in different volatility models. However, standard GARCH models are estimated to fit the conditional distribution of returns, whereas the previously mentioned models are estimated to fit the conditional distribution of range (log-range). This in turn means that only our model can be estimated using standard econometric software without any programming.

Our contribution is threefold. First, we construct a range-based GARCH model (RGARCH). This model is a simple modification of the standard widely used GARCH(1,1) model, but still outperforms it significantly. Second, our paper should be viewed as an illustration of how the existing GARCH models can be easily improved by using more precise volatility proxies. Even though this paper devotes most of the space to illustrate that the RGARCH models outperforms the standard GARCH(1,1) model, our main goal is not to convince the reader that our model is the best one. On the contrary, since leverage effect is a well-documented phenomenon, an asymmetric RGARCH model is very likely to outperform our model. However, we focus on the GARCH(1,1) model, as it is arguably the most popular volatility model and the incorporation of the range into this model illustrates the general idea well enough. Third, we confirm that GARCH models should indeed be considered just filtering devices, not models for data generating processes.

The rest of the paper is organized in the following way: Section 2 provides a basic introduction to volatility modelling and an overview of existing range-based volatility estimators. Section 3 describes the data, methodology and results. Finally, Section 4 concludes.

2 Theoretical background

2.1 GARCH models

Let P_t be the price of a speculative asset at the end of day t. Define return r_t as

$$r_t = \log(P_t) - \log(P_{t-1}). \quad (1)$$

Daily returns are known to be basically unpredictable and their expected value is very close to zero. On the other hand, variance of daily returns changes significantly over time. We assume that daily returns are drawn from a normal distribution with a zero mean and time-varying variance:

$$r_t \sim N\left(0, \sigma_t^2\right). \quad (2)$$

Both assumptions, zero mean and normal distribution, are not necessary and can be abandoned without any difficulty. For the sake of exposition, we maintain these assumptions throughout the whole paper. This allows us to focus on the modelling of conditional variance (volatility) only. The first model to capture the time variation of volatility is Engle's

(1982) Auto Regressive Conditional Heteroskedasticity (ARCH) model. The ARCH(p) has the form

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i r_{t-i}^2, \tag{3}$$

where r_t is a return in day t, σ_t^2 is an estimate of the volatility in day t and ω and α_i's are positive constants. The Generalized ARCH model was afterwards introduced by Bollerslev (1986). The GARCH(p,q) has the following form:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{j=1}^{q} \beta_i \sigma_{t-j}^2, \tag{4}$$

where the β_i's are positive constants. The GARCH model has become more popular, because with just a few parameters it can fit data better than a more parametrized ARCH model. Particularly popular is its simplest version, the GARCH(1,1) model[3]:

$$\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2. \tag{5}$$

Estimation of the GARCH(1,1) typically yields the following results. Parameter ω is very small (e.g. 0.0006), $\alpha + \beta$ is close to one, but smaller than one. Moreover, most of the weight is on the β coefficient, e.g. $\alpha = 0.04$, $\beta = 0.95$. In other words, the estimated GARCH(1,1) model is usually very close to its reduced form, the Exponential Weighted Moving Average (EMWA) model

$$\sigma_t^2 = \alpha r_{t-1}^2 + (1 - \alpha)\, \sigma_{t-1}^2. \tag{6}$$

The EMWA model is useful particularly for didactic purposes. In this model the new volatility estimate is estimated as a weighted average of the most recently observed volatility proxy (squared returns) and the last estimate of the volatility. Loosely speaking, we

[3]Even though the GARCH(1,1) is a very simple model, it still works surprisingly well in comparison with much more complex volatility models (see Hansen and Lunde (2005)).

gradually update our belief about the volatility as new information (noisy volatility proxy) becomes available. If the new information indicates that the volatility was larger than our previous belief about it, we update our belief upwards and vice versa. The coefficient α tells us how much weight we put on the new information. If we use a less noisy volatility proxy instead of squared returns, the optimal α should be larger and the performance of the model should be better.

The same intuition applies to GARCH models too. This naturally leads to the proposal of the modified GARCH(1,1)

$$\sigma_t^2 = \omega + \alpha \widehat{\sigma^2_{proxy,t-1}} + \beta \sigma_{t-1}^2 \tag{7}$$

where $\widehat{\sigma^2_{proxy,t-1}}$ is the less noisy volatility proxy.

Next we need to decide upon what should be used as a better (less noisy) volatility proxy. Generally, the better the proxy we use, the better should the model work. Therefore, the natural candidate would be realized volatility. This would lead to models related to Shephard and Sheppard (2010) and Hansen et al. (2011). However, despite the attractiveness of the realized variance we do not use it as a volatility proxy. Realized variance must be calculated from high frequency data and these data are in many cases not available at all or available only over shorter time horizons and costly to obtain and work with. Moreover, due to market microstructure effects the estimation of volatility from high frequency data is a rather complex issue (see Dacorogna et al. (2001)). Contrary to high frequency data, high (H) and low (L) prices, which are usually widely available, can be used to estimate volatility (Parkinson (1980)):

$$\widehat{\sigma_P^2} = \frac{[\ln(H/L)]^2}{4\ln 2}. \tag{8}$$

This estimator is derived under the assumption that, during the day, the logarithm of the price follows a Brownian motion with a zero drift. Even though this is not always true, Parkinson's volatility estimator performs very well with the real world data (Chou et al. (2010)).

An alternative volatility proxy we could use is the Garman and Klass (1980) volatility estimator, which utilizes additional open (O) and close (C) data:

$$\widehat{\sigma_{GK}^2} = 0.5\left[\ln(H/L)\right]^2 - (2\ln 2 - 1)\left[\ln(C/O)\right]^2. \tag{9}$$

Under ideal conditions (Brownian motion with zero drift) this estimator is less noisy than the Parkinson volatility estimator[4], because it utilizes open and close prices too. However, in this paper we use Parkinson's volatility estimator ($\sigma_{proxy}^2 = \sigma_P^2$). We have done all the calculations for the Garman-Klass volatility estimator too and found out that for this particular purpose the Garman-Klass estimator does not improve the results more than Parkinson estimator, the results are practically the same. Moreover, for the same data sets where high and low prices are available, open price is sometimes not available.

In this paper we therefore study the following model

$$\sigma_t^2 = \omega + \alpha\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2, \tag{10}$$

which we denote as RGARCH(1,1) (range GARCH) model. This model can obviously be extended to the RGARCH(p,q) model

$$\sigma_t^2 = \omega + \sum_{i=1}^{p}\alpha_i\widehat{\sigma_{P,t-i}^2} + \sum_{j=1}^{q}\beta_i\sigma_{t-j}^2. \tag{11}$$

Since it is generally known that GARCH(p,q) of order higher than (1,1) is seldom useful (see e.g. Hansen and Lunde (2005)), we study the RGARCH model only in its simplest version (10), i.e. the RGARCH(1,1) model. Most of the paper is devoted to the comparison of

[4]For comparison of range-based volatility estimators see Molnár (2011).

the standard GARCH(1,1) model (5) and the RGARCH(1,1) model (10). Since we do not study GARCH and RGARCH models of higher orders, we sometimes refer to GARCH(1,1) and RGARCH(1,1) models simply as GARCH and RGARCH models.

Our hypotheses are the following:

Hypothesis 1 *An RGARCH(1,1) outperforms the standard GARCH(1,1) model, both in sense of the in sample fit and out of sample forecasting performance.*

Additionally, as previously explained, we expect that the estimated coefficients of the GARCH models will be changed in such a way that more weight will be put on the recent observation(s) of the volatility proxy. This leads us to the second hypothesis.

Hypothesis 2 *If we modify the GARCH(1,1) to the RGARCH(1,1) model, we expect α to increase and β to decrease.*

Since the RGARCH(1,1) model puts more weight on the most recent observation of the volatility, this model will provide largest improvement in those situations when the recent observation tells us much more about the future volatility then the past observations. This leads us to the following hypothesis.

Hypothesis 3 *The superiority of the RGARCH(1,1) model over the GARCH(1,1) model is the strongest when day-to-day changes in volatility are large.*

However, this does not mean that GARCH should be better model in situations when changes in volatility are small. We expect RGARCH model to be superior in both situations, but its superiority should be largest in situations when volatility changes a lot.

Even though we formulated 3 hypotheses, the central one is Hypothesis 1. The purpose of Hypothesis 2 and Hypothesis 3 is mostly to provide some additional insights why and when RGARCH model works better than standard GARCH model.

To evaluate the usefulness of the RGARCH model, we briefly compare it not only with the basic GARCH(1,1) model, but with the other commonly used GARCH models too. GARCH models we compare the RGARCH to are the following:

The GJR-GARCH of Glosten et al. (1993):

$$\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 + \gamma r_{t-1}^2 I_{t-1}. \tag{12}$$

where $I_t = 1$ if $r_t < 0$ and zero otherwise.

The Exponential GARCH (EGARCH) of Nelson (1991):

$$\log\left(\sigma_t^2\right) = \omega + \alpha \left| \frac{r_{t-1}}{\sigma_{t-1}} \right| + \beta \log\left(\sigma_{t-1}^2\right) + \gamma \frac{r_{t-1}}{\sigma_{t-1}}. \tag{13}$$

The standard deviation GARCH of Taylor (1986), denoted in this paper as stdGARCH, both in its symmetric version:

$$\sigma_t = \omega + \alpha r_{t-1} + \beta \sigma_{t-1} \tag{14}$$

and in the asymmetric version, similar to (12), taking into account the leverage effect (astdGARCH):

$$\sigma_t = \omega + \alpha r_{t-1} + \beta \sigma_{t-1} + \gamma r_{t-1} I_{t-1}. \tag{15}$$

The last model we use is the component GARCH (cGARCH):

$$\sigma_t^2 - m_t = \overline{\omega} + \alpha\left(r_{t-1}^2 - m_t\right) + \beta\left(\sigma_{t-1}^2 - m_t\right) \tag{16}$$

$$m_t = \omega + \rho\left(m_t - \omega\right) + \phi\left(r_{t-1}^2 - \sigma_{t-1}^2\right). \tag{17}$$

The intuition for the component GARCH is the following. The standard GARCH(1,1) model, which can be rewritten as

$$\sigma_t^2 = \overline{\omega} + \alpha\left(r_{t-1}^2 - \overline{\omega}\right) + \beta\left(\sigma_{t-1}^2 - \overline{\omega}\right), \tag{18}$$

exhibits mean reversion around $\overline{\omega}$, which is constant. The component GARCH allows mean reversion around the time varying level m_t.

2.2 Estimation

All the GARCH models, including the models (5), (12) - (15) in our paper are estimated via Maximum Likelihood. Since the RGARCH model changes only the specification of the variance equation (equation (10) instead of (5)), we do not need to derive a new likelihood function for estimation of this model. This in turns mean that our model can be estimated without any programming in widely available econometric packages which allow to include exogenous variables in the variance equation, e.g. EViews, R or OxMetrics. We simply specify that we want to estimate a GARCH(0,1) model with an exogenous variable $\widehat{\sigma^2_{P,t-1}}$.

As mentioned earlier, we assume returns to be normally distributed with zero mean (equation (2)) and variance evolving according to a given GARCH model. However, there are alternative distributions for residuals to consider (e.g. Student's t-distribution or GED distribution). We did the calculations for alternative distributions too, but found that comparison of the. RGARCH model with the standard GARCH model is unaffected by the assumption of the residuals' distribution as long as the return distribution is the same for both models. For the sake of brevity, we report only the results for normally distributed residuals.

Two most closely related models are The Conditional Autoregressive Range model (CARR) of Chou (2005) and Range-Based EGARCH model (REGARCH) of Brandt and Jones (2006). Common feature of these models with the standard GARCH models is the variance equation. The variance equation for RGARCH model is created by a modification of the GARCH(1,1) (5), the variance equation of the CARR model is a modification of the GJR-GARCH (12) and the variance equation of the REGARCH is a modification of EGARCH (13).

However, CARR and REGARCH are otherwise significantly different from RGARCH and other GARCH models. Standard GARCH models as well as our RGARCH model are estimated by fitting the conditional distribution of returns. On contrary, estimation of the CARR and the REGARCH models is based on the distribution of the range. Denote range as

$$D_t = \ln\left(H_t/L_t\right). \tag{19}$$

The REGARCH model is estimated by fitting the conditional distribution of log-range:

$$\ln\left(D_t\right) \sim N(0.43 + \ln\left(\sigma_t\right), 0.29^2), \tag{20}$$

and the CARR model is estimated by fitting the conditional distribution of range

$$D_t = \lambda_t \varepsilon_t, \tag{21}$$

where λ_t is the conditional mean of the range (varying according to equation similar to (12)) and ε_t is distributed according to either the exponential or the Weibull distribution.

In other words, these models are not estimated to capture the conditional distribution of the returns, but the conditional distribution of range instead. Since these estimations are not implemented in standard econometric software, CARR and RGARCH models must be programmed first.

On the contrary, RGARCH model combines the ease of estimation of the standard GARCH models with the precision of the range-based models.

Now we evaluate the performance of the RGARCH model (10). To do so, we mainly compare it with the standard GARCH(1,1) model (5), because these two models are very closely related and their direct comparison is very intuitive. We do this comparison for both in-sample fit and out-of-sample forecasting performance. The analysis of the in-sample fit will give us some insights about how these models work. The forecasting ability is typically the most important feature of a volatility model. Therefore when evaluating the overall usefulness of the RGARCH model, we focus mostly on its forecasting ability.

2.3 In-sample comparison

We start the in-sample comparison between RGARCH(1,1) and standard GARCH(1,1) models by an estimation of equations (5) and (10). This allows us to see whether the coefficients change according to our Hypothesis 2. To evaluate which model is a better fit for the data, we use Akaike Information Criterion (AIC). However, as we are comparing models with an equal number of parameters, any information criterion would necessary produce the same raking of these models. We believe that in our particular case, when we are comparing two very closely related models (the conditional distribution of returns is the same, models differ in specification of variance equation only), AIC is a sensible criterion.

Moreover, we estimate the combined GARCH(1,1) model

$$\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 + \alpha_2 \widehat{\sigma_{P,t-1}^2} + \beta \sigma_{t-1}^2 \qquad (22)$$

too. This allows us to better understand which volatility proxy: squared returns r_{t-1}^2 or the Parkinson volatility proxy $\widehat{\sigma_{P,t-1}^2}$, is a more relevant variable in the variance equation.

2.4 Out-of-sample forecasting evaluation

To evaluate forecasting performance of two competing models, we first create forecasts from these models and afterwards evaluate which of these forecasts is on average closer to the true volatility.

To do this, we must first decide how to create the forecasts, particularly how much data to use for the forecasting. If we use too little data, the model will be estimated imprecisely and the forecasting will not be very good. On the other hand, if we use too much data, we can estimate the model precisely, but when the dynamics of the true volatility changes, our model will adapt to this change too slowly. To avoid this problem, we use rolling window forecasting[5] with four different window sizes: 300, 400, 500 and 600 trading days. These numbers are obviously somewhat arbitrary, but we are focused on the comparison of different volatility models, not on the search for the optimal forecasting window. Due to space limitations, we restrict our attention to one-day-ahead forecasts.

Next we must first decide on what to use as a benchmark (as the "true" volatility). The most common benchmark is squared returns. Squared returns are so widely used due to the data availability. Squared returns are a natural candidate too, since the main reason for the existence of volatility models is to capture the volatility of returns.

However, squared returns are a very noisy volatility proxy. Therefore we use the Parkinson volatility estimator and the realized variance too. Due to space limitations, we do not report results when the Parkinson volatility estimator is used as a benchmark, though the results are even more convincing than for squared returns. Conversely, whenever the data on the realized variance is available, we use it as a benchmark.

To evaluate which forecast is closer to the true value, we must next decide on the loss function. We use the Mean Squared Error (MSE) as a loss function. For the sake of exposition, we report Root Mean Squared Error (RMSE) instead of MSE in all the

[5]By rolling window forecasting with window size 100 we mean that we use the first 100 observations to forecast volatility on the 101, then we use observations 2 to 101 to forecast volatility for day 102 and so on.

tables. MSE is not only the most common loss function, but it has many other convenient properties, particularly the robustness. Since we are using imperfect volatility proxies, the choice of an arbitrary loss function (e.g. Mean Absolute Error or Mean Percentage Error) could lead to problems, particularly to the inconsistent ranking of different models (see Hansen and Lunde (2006) and Patton Patton (2011)).

Next we want to know whether the MSE for two different models are statistically different. We adopt the Diebold and Mariano (1995) test for this purpose. The Diebold-Mariano test statistic (DM) is computed in the following way: denote two competing forecasts as $\widehat{\sigma^2_{1,t}}$ and $\widehat{\sigma^2_{2,t}}$ and the true volatility as $\sigma^2_{true,t}$. In our case $\widehat{\sigma^2_{1,t}} = \widehat{\sigma^2_{RGARCH,t}}$ and $\widehat{\sigma^2_{2,t}}$ is the competing model; in the majority of this paper it is the GARCH(1,1) model. First we construct the vector of differences in squared errors

$$d_t = \left(\widehat{\sigma^2_{1,t}} - \sigma^2_{true,t}\right)^2 - \left(\widehat{\sigma^2_{2,t}} - \sigma^2_{true,t}\right)^2. \tag{23}$$

Next we construct the Diebold-Mariano test statistic

$$DM = \frac{\overline{d}}{\sqrt{\widehat{V}\left(\overline{d}\right)}}, \tag{24}$$

where $\overline{d}$ denotes the sample mean of d_t and $\widehat{V}\left(\overline{d}\right)$ is variance of the sample mean. DM is assumed to have a standard normal distribution. Later in the results we denote by asterisk * (**) cases when the DM test statistics lies below 5-percentile (1-percentile), i.e. the cases where we can reject at 5% (1%) confidence level the hypothesis that the competing model has smaller MSE than the RGARCH(1,1) model.[6]

[6] In our data the DM test statistic never lies above 95-percentile.

2.5 Opening jump

In the previous discussion we assumed that all the models are estimated on the close-to-close returns defined by equation (1). This is typically the case for the standard GARCH models. On the other hand, a common approach in the literature dealing with high frequency data is to model open-to-close returns

$$r_t = \log(P_t) - \log(O_t). \tag{25}$$

The reason for this is that volatility for the trading period (from open to close of the market) can be estimated quite precisely, whereas this precision is not available for estimation of the period over the night, which is summarized in opening jump. As Parkinson volatility estimator (8) estimates open-to-close volatility only, we must deal with the same problem. There are basically three ways how to solve this problem.

First, we could add opening jump component to the Parkinson volatility estimator. We do not do this for the same reason this is seldom done in the realized variance literature: this would decrease the precision of the estimated volatility.

Second, we could ignore the fact that Parkinson volatility estimator estimates the volatility only for the open-to-close period and still estimate our model on close-to-close returns. In this case we must be careful with interpretation of the α coefficient in the RGARCH model. As long as opening jumps are present, the Parkinson volatility estimator underestimates volatility of daily returns,

$$E\left(\widehat{\sigma_P^2}\right) < E\left(r^2\right) = \sigma^2. \tag{26}$$

As a result, the estimated coefficient α will be larger to balance this bias in $\widehat{\sigma_P^2}$. This intuition can explain one seemingly surprising result which is documented later in the appendix. The RGARCH model estimated on the close-to-close data typically yield coefficients α and β such that $\alpha + \beta > 1$, even though estimation of the standard GARCH(1,1) model yields coefficients α and β such that $\alpha+\beta < 1$. However, as we just explained, these α coefficients are not directly comparable in presence of opening jumps. We illustrate this on a simple example. If we specify GARCH(1,1) in the following form

$$\sigma_t^2 = \omega + \alpha\frac{r_{t-1}^2}{2} + \beta\sigma_{t-1}^2,$$

then the estimated coefficient α will be exactly twice as large as when we estimate equation (5). Therefore, if the RGARCH model is estimated on the close-to-close returns, the coefficient α does not have the same interpretation as in standard GARCH models. Even though we expect α to increase and β to decrease, if we use close-to-close returns, we must focus on the coefficient β only. The coefficient β will change only because a less noisy volatility proxy is used, whereas change in coefficient α is caused by both high precision and bias of the Parkinson volatility estimator.

Our final choice is to estimate the RGARCH model on the open-to-close returns. In this case the interpretation of the coefficient α remains the same as in the standard GARCH models. Moreover, the dynamics of the opening jumps is arguably different from the volatility of the trading part of the day. Results from the estimation on the close-to-close returns are in the appendix.

3 Data and results

To show the generality of our idea we study a wide class of assets, particularly 30 individual stocks, 6 stock indices and simulated data. Due to space limitations, our analysis cannot be as detailed as it would be if we studied a single asset. We believe that the analysis of the main features of the problem on the broad data set is more convincing than very detailed analysis based on a small data set. We use daily data, particularly the highest, the lowest, the opening and the closing price of the day.

3.1 Stocks

We study the components[7] of the Dow Jow Industrial Average, namely the stocks with tickers AA, AXP, BA, BAC, C, CAT, CVX, DD, DIS, GE, GM, HD, HPQ, IBM, INTC, JNJ, JPM, CAG[8], KO, MCD, MMM, MRK, SFT, PFE, PG, T, UTX, VZ and WMT. Data were obtained from the CRSP database and consist of 4423 daily observations of high, low and close prices from June 15, 1992 to December 31, 2010.

3.1.1 In-sample analysis

Table 1 presents estimated coefficients for the equations (5) and its modified version (7) together with values of Akaike Information Criterion (AIC).

For every single stock, the coefficients in the modified GARCH(1,1) have changed in exactly the same way we expected. Additionally, according to AIC, modified GARCH(1,1) is superior to its standard counterpart for every single stock in our sample.

Next we estimate equation (22). Results of this estimation (reported in Table 2 together with respective p-values) show that whereas coefficients α_2 is always significant both statistically and economically, the coefficient α_1 is insignificant in most of the cases. Even when it is statistically significant, it is rather small. This confirms that σ_P^2 is a better volatility proxy than r^2 and when we have the first one available, the inclusion of

[7]Components of stock indices change over time. These stocks were DJI components on January 1, 2009.

[8]Since historical data for KFT (component of DJI) are not available for the complete period, we use its competitor CAG instead.

Table 1: Estimated coefficients of the GARCH(1,1) model $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2$ and the RGARCH(1,1) model $\sigma_t^2 = \omega + \alpha\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$, reported together with the values of Akaike Information Criterion (AIC) of the respective equations.

Ticker	GARCH(1,1)				RGARCH(1,1)			
	ω	α	β	AIC	ω	α	β	AIC
AA	1.61E-06	0.036	0.960	-5.121	4.21E-06	0.066	0.926	-5.131
AXP	1.61E-06	0.071	0.927	-5.320	2.26E-06	0.160	0.842	-5.348
BA	2.67E-06	0.057	0.934	-5.497	5.20E-06	0.148	0.830	-5.520
BAC	1.69E-06	0.080	0.917	-5.508	1.77E-06	0.197	0.816	-5.529
CAT	2.78E-06	0.045	0.947	-5.303	1.11E-05	0.145	0.826	-5.325
CSCO	2.98E-06	0.078	0.921	-4.756	4.04E-06	0.184	0.814	-4.787
CVX	3.29E-06	0.066	0.917	-5.838	5.20E-06	0.134	0.840	-5.854
DD	1.04E-06	0.038	0.959	-5.551	2.53E-06	0.088	0.901	-5.573
DIS	2.57E-06	0.053	0.939	-5.460	5.51E-06	0.107	0.867	-5.494
GE	8.38E-07	0.062	0.937	-5.742	2.54E-06	0.180	0.811	-5.765
HD	2.82E-06	0.053	0.939	-5.313	7.22E-06	0.121	0.852	-5.334
HPQ	2.15E-06	0.035	0.961	-4.997	3.06E-06	0.054	0.941	-5.008
IBM	8.21E-07	0.054	0.946	-5.552	6.67E-07	0.153	0.860	-5.574
INTC	2.60E-06	0.054	0.942	-4.943	4.52E-06	0.142	0.855	-4.966
JNJ	1.28E-06	0.069	0.926	-6.021	1.47E-06	0.170	0.824	-6.044
JPM	1.82E-06	0.080	0.919	-5.273	1.86E-06	0.158	0.841	-5.307
CAG	1.80E-06	0.057	0.936	-5.815	5.68E-06	0.238	0.740	-5.843
KO	5.68E-07	0.044	0.954	-5.965	6.22E-07	0.114	0.883	-5.980
MCD	1.84E-06	0.046	0.947	-5.654	2.28E-06	0.091	0.898	-5.673
MMM	1.57E-06	0.033	0.959	-5.890	8.19E-06	0.136	0.814	-5.911
MRK	6.02E-06	0.058	0.920	-5.513	1.17E-05	0.124	0.826	-5.533
MSFT	1.05E-06	0.062	0.937	-5.392	6.69E-07	0.195	0.809	-5.408
PFE	1.80E-06	0.046	0.948	-5.509	6.52E-06	0.177	0.805	-5.520
PG	1.69E-06	0.057	0.934	-5.953	4.79E-06	0.213	0.764	-5.989
T	1.27E-06	0.057	0.940	-5.621	2.36E-06	0.109	0.881	-5.629
TRV	3.95E-06	0.074	0.913	-5.544	9.41E-06	0.198	0.782	-5.586
UTX	2.44E-06	0.074	0.918	-5.700	5.05E-06	0.198	0.788	-5.723
VZ	1.46E-06	0.052	0.943	-5.695	4.34E-06	0.159	0.826	-5.704
WMT	1.39E-06	0.058	0.939	-5.617	1.91E-06	0.127	0.861	-5.638
XOM	2.70E-06	0.074	0.912	-5.922	5.32E-06	0.164	0.807	-5.949

the second one can improve the model only marginally. Note that the coefficient α_1 is negative in most cases. This is expected, since an optimal volatility estimate (9) combines the Parkinson volatility estimator with squared returns in such a way that squared returns have negative weight. We discuss this more in the subsection with simulated data.

Table 2: Estimated coefficients and p-values for the combined GARCH(1,1) model $\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 + \alpha_2 \widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$.

Ticker	combined GARCH(1,1)							
	ω	p-value	α_1	p-value	β	p-value	α_2	p-value
AA	4.37E-06	0.000	-0.002	0.811	0.925	0.000	0.069	0.000
AXP	2.33E-06	0.004	-0.041	0.003	0.827	0.000	0.218	0.000
BA	6.07E-06	0.000	-0.028	0.013	0.810	0.000	0.191	0.000
BAC	1.76E-06	0.002	0.007	0.546	0.819	0.000	0.187	0.000
CAT	1.47E-05	0.000	-0.052	0.000	0.783	0.000	0.231	0.000
CSCO	3.82E-06	0.015	-0.025	0.058	0.812	0.000	0.211	0.000
CVX	5.67E-06	0.000	-0.018	0.135	0.829	0.000	0.161	0.000
DD	2.78E-06	0.000	-0.025	0.002	0.896	0.000	0.117	0.000
DIS	5.88E-06	0.000	-0.034	0.001	0.864	0.000	0.140	0.000
GE	2.56E-06	0.000	-0.005	0.704	0.809	0.000	0.186	0.000
HD	8.19E-06	0.000	-0.018	0.095	0.837	0.000	0.150	0.000
HPQ	3.01E-06	0.000	0.001	0.849	0.941	0.000	0.053	0.000
IBM	6.69E-07	0.353	-0.010	0.178	0.853	0.000	0.171	0.000
INTC	4.90E-06	0.012	-0.032	0.006	0.842	0.000	0.187	0.000
JNJ	1.47E-06	0.000	0.005	0.598	0.826	0.000	0.162	0.000
JPM	1.90E-06	0.017	-0.030	0.013	0.829	0.000	0.200	0.000
CAG	6.83E-06	0.000	-0.042	0.002	0.699	0.000	0.315	0.000
KO	6.15E-07	0.046	-0.002	0.773	0.882	0.000	0.117	0.000
MCD	4.61E-06	0.000	-0.041	0.000	0.841	0.000	0.178	0.000
MMM	9.43E-06	0.000	-0.092	0.000	0.790	0.000	0.242	0.000
MRK	1.41E-05	0.000	-0.029	0.009	0.796	0.000	0.173	0.000
MSFT	5.69E-07	0.534	-0.018	0.240	0.798	0.000	0.224	0.000
PFE	6.28E-06	0.000	0.007	0.496	0.813	0.000	0.163	0.000
PG	5.18E-06	0.000	-0.061	0.000	0.733	0.000	0.303	0.000
T	1.95E-06	0.001	0.026	0.000	0.894	0.000	0.072	0.000
TRV	1.03E-05	0.000	-0.041	0.000	0.768	0.000	0.252	0.000
UTX	5.49E-06	0.000	-0.020	0.107	0.773	0.000	0.232	0.000
VZ	3.96E-06	0.000	0.018	0.009	0.840	0.000	0.129	0.000
WMT	1.97E-06	0.003	-0.010	0.338	0.855	0.000	0.142	0.000
XOM	5.75E-06	0.000	-0.030	0.021	0.794	0.000	0.204	0.000

3.1.2 Out-of-sample forecasting performance

As seen in the previous subsection, the RGARCH model outperforms the standard GARCH model in the in-sample fit of the data. The next obvious question is the comparison of the predictive ability of these models. To answer this question, we compare one-day ahead forecasts of the models (5) and (7) with squared returns as a benchmark. Results are presented in the Table 3.

As we can see from Table 3, the RGARCH(1,1) model outperforms GARCH(1,1). All the cases (stock-estimation window pairs) when the difference is statistically significant favour the RGARCH model. The reason the difference is often insignificant is a very noisy volatility benchmark (squared returns). Therefore we should wait with evaluation of size of the improvement of RGARCH(1,1) model over GARCH(1,1) model until next subsections, where we use less noisy volatility proxies in addition to squared returns.

The next obvious question is how our RGARCH performs relative to other more complicated GARCH models. Even though a detailed answer to this question is beyond the scope of this paper, we provide some basic comparison. We now compare the RGARCH model (10) not only with the basic GARCH model (5), but with its other versions (12)-(16) as well. We chose an estimation window equal to 400. A shorter estimation window would favour the RGARCH model even more. A too long estimation window is not desirable, because, as Table 3 documents, volatility forecasting becomes less precise when we use a too long estimation window.

As we can see from Table 4, the comparison of the RGARCH model with other GARCH models is very similar to the previous comparison, the RGARCH model outperforms other GARCH models. When we consider the cases where the difference is statistically signifi-

Table 3: Comparison of the forecasting performance of the GARCH(1,1) model $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2$ and the RGARCH(1,1) model $\sigma_t^2 = \omega + \alpha\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$. Numbers in this table are 1000×RMSE of the one-day-ahead rolling window forecast reported for different window sizes w. An asterisk * (**) indicates when the difference is significant at the 5% (1%) level.

Ticker	GARCH(1,1)				RGARCH(1,1)			
	w=300	w=400	w=500	w=600	w=300	w=400	w=500	w=600
AA	1.277	1.296	1.309	1.322	1.268	1.281	1.291	1.305
AXP	1.167	1.177	1.189	1.202	1.179	1.199	1.203	1.215
BA	0.656	0.657	0.657	0.662	0.649	0.650	0.651	0.657
BAC	2.594	2.621	2.646	2.673	2.791	2.824	2.701	2.761
CAT	0.710	0.717	0.722	0.731	0.694*	0.701	0.710	0.719
CSCO	1.749	1.761	1.781	1.806	1.700	1.708*	1.736*	1.747*
CVX	0.643	0.648	0.657	0.662	0.634	0.635	0.642	0.647
DD	0.675	0.679	0.686	0.692	0.660*	0.665**	0.671**	0.677**
DIS	0.684	0.688	0.696	0.703	0.665*	0.669*	0.678*	0.682*
GE	0.869	0.870	0.879	0.888	0.882	0.865	0.862	0.871
HD	0.794	0.801	0.809	0.815	0.789	0.800	0.800	0.844
HPQ	1.050	1.058	1.070	1.083	1.043	1.057	1.063	1.077
IBM	0.631	0.635	0.641	0.648	0.624*	0.629*	0.637	0.643
INTC	1.194	1.195	1.205	1.218	1.161*	1.169*	1.180*	1.193*
JNJ	0.359	0.358	0.356	0.357	0.350*	0.349*	0.350	0.351
JPM	1.757	1.787	1.805	1.817	1.711	1.724*	1.736**	1.758**
CAG	0.534	0.537	0.538	0.543	0.514	0.531	0.536	0.542
KO	0.496	0.495	0.497	0.500	0.488	0.488	0.491	0.496
MCD	0.670	0.670	0.676	0.678	0.665	0.667	0.682	0.694
MMM	0.446	0.446	0.451	0.455	0.444	0.445	0.449	0.452
MRK	0.642	0.649	0.653	0.660	0.632*	0.636**	0.639*	0.649**
MSFT	0.676	0.683	0.688	0.696	0.676	0.673*	0.675**	0.684**
PFE	0.540	0.546	0.545	0.553	0.546	0.547	0.552	0.555
PG	0.505	0.508	0.509	0.510	0.493*	0.493**	0.498	0.498
T	0.612	0.614	0.619	0.626	0.597	0.601*	0.608*	0.613*
TRV	1.161	1.169	1.177	1.190	1.180	1.178	1.185	1.188
UTX	0.689	0.698	0.701	0.710	0.681*	0.686**	0.695*	0.702**
VZ	0.570	0.573	0.577	0.583	0.561**	0.563**	0.569*	0.575**
WMT	0.625	0.628	0.633	0.640	0.612	0.618	0.619	0.628
XOM	0.610	0.612	0.614	0.621	0.588**	0.590**	0.597*	0.604*

Table 4: Comparison of the forecasting performance of the RGARCH(1,1) model $\sigma_t^2 = \omega + \alpha\widehat{\sigma^2_{P,t-1}} + \beta\sigma^2_{t-1}$and several different GARCH models. Numbers in this table are 1000×RMSE of the one-day-ahead rolling window forecast with forecasting window equal to 400.

ticker	RGARCH	GARCH	GJR	EGARCH	stdGARCH	astdGARCH	cGARCH
AA	1.281	1.296	1.286	1.277	1.294	1.270	1.309
AXP	1.199	1.177	1.189	1.174	1.173	1.178	1.177
BA	0.648	0.655	0.647	0.659	0.655	0.650	0.650
BAC	2.825	2.623	2.654	2.549	2.631	2.595	2.550
CAT	0.705	0.720	0.716	0.718	0.722*	0.716	0.723*
CSCO	1.881	1.928**	1.963	1.895	1.909*	1.888	1.937*
CVX	0.633	0.646*	0.628	0.630	0.653	0.632	0.662**
DD	0.663	0.678**	0.676*	0.683**	0.678**	0.680**	0.678**
DIS	0.668	0.688*	0.685	0.688	0.690	0.689	0.690*
GE	0.863	0.869	0.862	0.855	0.866	0.863	0.887
HD	0.803	0.804	0.799	0.799	0.807	0.799	0.803
HPQ	1.057	1.058	1.056	1.059	1.058	1.056	1.071*
IBM	0.639	0.645	0.633	0.635	0.642	0.633*	0.650*
INTC	1.170	1.196*	1.160	1.158	1.175	1.156	1.207*
JNJ	0.347	0.355*	0.351	0.351	0.353	0.351	0.355*
JPM	1.724	1.786*	1.711	1.715	1.782*	1.730	1.761
CAG	0.531	0.537	0.536	0.533	0.530	0.532	0.529
KO	0.485	0.492	0.505	0.492	0.487	0.488	0.491
MCD	0.669	0.672	0.695	0.824	0.663	0.663	0.668
MMM	0.442	0.443	0.444	0.441	0.442	0.442	0.447
MRK	0.635	0.648**	0.652**	0.648*	0.647*	0.647*	0.653**
MSFT	0.674	0.684*	0.675	0.676	0.686*	0.677	0.686*
PFE	0.562	0.561	0.567	0.555	0.556	0.554	0.560
PG	0.492	0.507**	0.507**	0.503*	0.503*	0.502*	0.508**
T	0.601	0.613*	0.607	0.611	0.613	0.609	0.613
TRV	1.176	1.167	1.174	1.173	1.176	1.175	1.171
UTX	0.685	0.697**	0.697	0.695	0.697**	0.691	0.703
VZ	0.562	0.571**	0.569	0.569	0.570**	0.566	0.574*
WMT	0.621	0.632	0.625	0.629	0.626	0.624	0.633
XOM	0.588	0.609**	0.595	0.594	0.613	0.600	0.618**

cant, the RGARCH model always outperforms all other studied GARCH models. In rest of the cases, when the difference is not statistically significant, the RGARCH model outperforms other studied GARCH models most of the time. Moreover, the comparison of the RGARCH model with other GARCH models shows that the RGARCH model typically either performs better than any of the competing GARCH models or worse than all of them. Therefore comparison of the RGARCH model with the GARCH(1,1) model can to some extent serve as an evaluation of the overall performance of the RGARCH model. However, remember that we do not argue that RGARCH model is the best volatility model. It is clearly not, as it does not take into account e.g. leverage effect. Therefore, the comparison of the RGARCH model with other GARCH models serves mostly the illustrative purposes, particularly to show that even such a simple model (but based on more precise data) can outperform more complicated models.

The results summarized in Tables 3 and 4 show the superior performance of the RGARCH model. The improvement in the RGARCH model in comparison to the basic GARCH(1,1) model seems to be rather small at the first glance. Even though the RGARCH model outperforms the basic GARCH(1,1) model in most cases, the average improvement of the RMSE reported in Table 3 is about 1.2%. This could give us a first impression that the improvement of the RGARCH(1,1) model over the GARCH(1,1) model is rather small.

However, there is a potential problem with this standard evaluation procedure, where we compare the forecasted volatility with the squared returns. Even though the squared returns are unbiased estimates of the volatility, they are very noisy. The most natural solution to this problem is to use the true volatility as a benchmark, or, if unavailable, some other less noisy volatility proxy. Following subsections use less noisy volatility proxies (realized variance for the stock indices and true volatility for simulated data). However, due to stock data limitations, we suggest an alternative measure for the comparison of the basic GARCH(1,1) model and the RGARCH(1,1) model.

Comparison of the volatility forecasts from two different models, forecast 1 ($\sigma^2_{1,1}$, $\sigma^2_{2,1}$, $\sigma^2_{3,1}$,..., $\sigma^2_{n,1}$) and forecast 2 ($\sigma^2_{1,2}$, $\sigma^2_{2,2}$, $\sigma^2_{3,2}$,..., $\sigma^2_{n,2}$) when we observe only returns r_1, r_2, r_2,...,r_n is problematic for two reasons. First, the comparison of the forecasted volatility with squared returns will always penalize the volatility forecast when the squared return is different from the forecasted volatility, even if the volatility was perfectly forecasted. Second, when we have two models and one of them forecasts volatility to be $\sigma^2 = 0.1^2$ on the day when the stock return is $r = 1$ and the second model forecasts volatility to be $\sigma^2 = 3^2$ on the day when stock return is $r = \sqrt{10}$, then MSE (RMSE) will slightly favour the first model ($\left(0.1^2 - 1^2\right)^2 < \left(10 - 3^2\right)^2$, even though the probability of the return $r = 1$ being drawn from the distribution $N\left(0, 0.1^2\right)$ is more than 10^{40}-times smaller than probability of the return $r = \sqrt{10}$ being drawn from the distribution $N\left(0, 3^2\right)$.

An alternative way to compare different volatility forecasts is to not compare squared returns with volatility directly, but to compare the likelihood that the return was drawn from the distribution parametrized by the given volatility. This approach is not perfect either, because the calculated probability depends on the specification of the distribution of the stock returns. However, in our case, when we are comparing two models with the same specification of the conditional distribution of returns, $N\left(0, \sigma^2_{t,1}\right)$ and $N\left(0, \sigma^2_{t,2}\right)$, which differ only in the specification of the variance equation, this is not a problem. Therefore we now compare the basic GARCH(1,1) model with the RGARCH model in terms of the value of the log-likelihood function. The log-likelihood is calculated simply according to the following formula:

$$LLF = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\ln\left(\widehat{\sigma_t^2}\right) - \frac{1}{2}\sum_{t=1}^{n}\frac{r_t^2}{\widehat{\sigma_t^2}}, \tag{27}$$

where σ_t^2 is the volatility forecasted from the studied volatility model (using past information only).

Table 5 confirms our previous comparison between the RGARCH model and the standard GARCH model. The RGARCH model outperforms the standard GARCH(1,1) model for basically every stock and every estimation window.

3.2 Stock indices

In addition to the individual stocks of the Dow Jones Industrial Average stock index we decided to compare the performance of the RGARCH model to the standard GARCH model on the major world indices (French CAC 40, German DAX, Japanese Nikkei 225, Britain's FTSE 100 and American DJI and NASDAQ 100). There are two reasons for this. First, volatility dynamics is generally different for individual stocks and for the whole stock markets. Second, estimates of realized variance, which is a proxy for the true variance, are publicly available for these indices[9]. Open, high, low and close prices are downloaded from finance.yahoo.com. Data covers the period January 3, 1993 - April 27, 2009 for open, high and low prices and the period January 3, 1996 - April 27, 2009 for the realized variance. Due to small differences in trading days in different markets, the number of observations varies accordingly.

For the in-sample analysis we use the data ranging from January 3, 1993 to April 27, 2009. For the out of sample comparison we use the volatilities forecasted for the period

[9]Heber, Gerd, Asger Lunde, Neil Shephard and Kevin K. Sheppard (2009) "Oxford-Man Institute's Realized Library", Oxford-Man Institute, University of Oxford

Table 5: Comparison of forecasting performance GARCH(1,1) model $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2$ and the RGARCH(1,1) model $\sigma_t^2 = \omega + \alpha\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$. Numbers in this table are the LLF of the returns r_t beind drawn from the distributions $N\left(0, \widehat{\sigma_t^2}\right)$, where $\widehat{\sigma_t^2}$ is a one-day-ahead rolling window volatility forecast reported for different window sizes w.

Ticker	GARCH(1,1)				RGARCH(1,1)			
	w=300	w=400	w=500	w=600	w=300	w=400	w=500	w=600
AA	9803	9580	9267	9020	9873	9597	9320	9032
AXP	10377	10166	9881	9595	10502	10242	9969	9688
BA	10434	10225	9809	9660	10500	10258	9993	9708
BAC	10687	10451	10154	9875	10783	10527	10236	9949
CAT	10105	9916	9631	9342	10202	9950	9675	9385
CSCO	9309	9080	8825	8528	9478	9237	8955	8646
CVX	11371	11017	10853	10576	11440	11145	10882	10599
DD	10853	10593	10321	10050	10916	10641	10377	10095
DIS	10535	10298	10024	9747	10681	10411	10142	9859
GE	11086	10860	10567	10258	11176	10902	10617	10325
HD	10266	10024	9729	9479	10372	10084	9809	9542
HPQ	9587	9255	9076	8792	9715	9415	9174	8869
IBM	10813	10575	10247	9972	10986	10716	10378	10130
INTC	9420	9208	8937	8665	9484	9278	9001	8735
JNJ	12013	11776	11492	11230	12063	11126	11522	11264
JPM	10158	10014	9730	9464	10345	10113	9830	9554
CAG	11421	11192	10939	10681	11563	11301	10994	10722
KO	11682	11454	11155	10924	11782	11517	11250	10980
MCD	11058	10846	10564	10288	11129	10871	10596	10320
MMM	11238	11105	10819	10542	11377	11153	10878	10599
MRK	10131	9775	9632	9294	10348	10120	9813	9570
MSFT	10234	10038	9724	9478	10396	10171	9867	9611
PFE	10741	10499	10222	9959	10827	10564	10269	10004
PG	11512	11236	10962	10709	11571	11369	11081	10777
T	10948	10704	10459	10172	11002	10744	10473	10206
TRV	10801	10614	10312	10069	10899	10678	10395	10111
UTX	11013	10790	10477	10179	11054	10840	10556	10269
VZ	11132	10892	10605	10329	11198	10930	10645	10361
WMT	11004	10778	10510	10109	11130	10860	10558	10276
XOM	11464	11223	10947	10657	11567	11294	11014	10729

January 3, 1996 - April 27, 2009. However, estimates of realized variance are not available for some trading days. These days are included in the volatility forecast comparison when squared returns are used as a benchmark, but excluded when the benchmark is realized variance.

3.2.1 In-sample analysis

Table 6 presents estimated coefficients for the equations (5) with its modified version (10) together with the values of Akaike Information Criterion (AIC). The results are again in line with those in Table 1. RGARCH model performs better than the standard GARCH model for every index. Coefficients in the RGARCH are changed as expected - coefficient α is increased and coefficient β is decreased for all the indices.

Table 6: Estimated coefficients of the GARCH(1,1) model $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2$ and it modified version RGARCH(1,1) $\sigma_t^2 = \omega + \alpha\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$, reported together with the values of Akaike Information Criterion (AIC) of the respective equations for the simulated data.

Index	GARCH(1,1)				RGARCH(1,1)			
	ω	α	β	AIC	ω	α	β	AIC
CAC40	1.03E-06	0.075	0.920	-6.327	1.80E-06	0.182	0.821	-6.352
DAX	6.16E-07	0.088	0.911	-6.417	1.28E-06	0.174	0.842	-6.446
DJI	9.39E-07	0.083	0.910	-6.674	-1.77E-06	0.128	0.717	-6.645
FTSE	7.64E-07	0.085	0.910	-6.581	1.47E-06	0.188	0.837	-6.598
NASDAQ	9.43E-07	0.056	0.942	-5.534	4.30E-07	0.135	0.893	-5.561
NIKKEI	3.20E-06	0.093	0.890	-6.084	1.64E-06	0.179	0.854	-6.113

Now we estimate the combined GARCH model (22). The results (presented in Table 7) are consistent with those in Table 2.

3.2.2 Out–of-sample forecasting performance

Now we compare the forecasting performance of the RGARCH model and the standard GARCH model against both squared returns (r^2) and realized variance (RV) used as a

Table 7: Estimated coefficients and p-values for the combined GARCH(1,1) model $\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 + \alpha_2 \widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$.

Index	combined GARCH(1,1)							
	ω	p-value	α_1	p-value	β	p-value	α_2	p-value
CAC40	1.93E-06	0.000294	-0.064	7.02E-05	0.789	0	0.286	0
DAX	1.61E-06	4.00E-15	-0.064	2.74E-05	0.815	0	0.276	0
DJI	5.69E-07	0.003	0.080	0	0.896	0	0.008	1.45E-04
FTSE	1.51E-06	1.33E-05	-0.005	0.723	0.834	0	0.198	5.06E-11
NASDAQ	-3.07E-07	0.553	-0.050	2.97E-05	0.891	0	0.204	0
NIKKEI	1.11E-06	0.031043	-0.088	1.72E-10	0.837	0	0.319	0

benchmark. Results are in Table 8.

Table 8: Comparison of the forecasting performance of the GARCH(1,1) model $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2$ and the RGARCH(1,1) model $\sigma_t^2 = \omega + \alpha\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$. Numbers in this table are 1000×RMSE of the one-day-ahead rolling window forecasts reported for different window sizes w and different benchmarks (squared returns r^2 and the realized variance RV) for the stock indices.

Index	Bench	GARCH(1,1)				RGARCH(1,1)			
		w=300	w=400	w=500	w=600	w=300	w=400	w=500	w=600
CAC40	r^2	0.335	0.339	0.342	0.346	0.331	0.335	0.338	0.342
	RV	0.185	0.181	0.179	0.180	0.172**	0.169**	0.167**	0.167**
DAX	r^2	0.474	0.477	0.481	0.488	0.446**	0.454**	0.461*	0.469*
	RV	0.252	0.242	0.236	0.235	0.212**	0.208**	0.207**	0.207**
DJI	r^2	0.353	0.355	0.362	0.367	0.336	0.341	0.347	0.350
	RV	0.174	0.172	0.176	0.179	0.142**	0.142**	0.141**	0.139**
FTSE	r^2	0.376	0.382	0.385	0.390	0.364*	0.368**	0.372**	0.377**
	RV	0.201	0.226	0.212	0.209	0.196	0.202*	0.189*	0.186*
NASDAQ	r^2	0.931	0.939	0.949	0.963	0.908**	0.917**	0.929**	0.942**
	RV	0.464	0.452	0.440	0.446	0.431*	0.423*	0.426	0.432
NIKKEI	r^2	0.467	0.475	0.478	0.478	0.456*	0.461	0.467	0.470
	RV	0.237	0.283	0.269	0.249	0.196**	0.188**	0.177**	0.173**

This table is the strongest evidence for the superiority of the RGARCH model over the standard GARCH model. For every single index and for every single estimation window size, the RGARCH model outperforms the standard GARCH model. The difference in the forecasting performance of these two models is much more obvious when we use realized variance as a benchmark (since it is much less noisy than squared returns).

3.3 Simulated data

In reality, we can never know for sure what the true volatility was. However, if we simulate the data, we know the true volatility exactly. Simulation therefore provides a convenient tool to study different volatility models. We can compare not only the overall performance of different models, but we can study under which conditions these models perform particularly good or bad. On the other hand, it is always questionable how close the simulated data are to the real world. In order to convince the reader that the simulated data are close to reality (and we did not construct them deliberately to show superiority of our model), we borrow the credibility of Alizadeh et al. (2002). They simulate the data in the following way. First we simulate the volatility process

$$\ln \sigma_t = \ln \overline{\sigma} + \rho_H \left(\ln \sigma_{t-1} - \ln \overline{\sigma} \right) + \mu_1 \varepsilon_{t-1} \tag{28}$$

with parameters $\ln(\overline{\sigma}) = -2.5$, $\rho_H = 0.985$ and $\mu_1 = 0.75/\sqrt{257} = 0.048$. Afterwards we simulate for every day $t = 1, 2, ..., 100000$ a Brownian motion[10] with zero drift term and diffusion term equal to σ_t. Save the highest, the lowest and the final value of this Brownian motion. According to Alizadeh et al. (2002), volatility dynamics (28) together with mentioned parameters is broadly consistent with literature on stochastic volatility.

The volatility process (28) does not favour directly either of the competing models GARCH (5) and RGARCH (10). Volatility gradually evolves over the time, and neither past returns nor past high or low prices influence the future volatility in any way. Note that there are no opening jumps in this these simulated data.

In addition to data simulated according to (28) with parameter $\mu_1 = 0.75/\sqrt{257}$, we simulate the data for two other parameter values too, $\mu_{0.5} = 0.5\mu_1$ and $\mu_2 = 2\mu_1$.

[10] We use 100000 discrete steps for the approximation of the continuous Brownian motion.

Parameter μ_1 represents a case with medium daily changes in volatility and parameters $\mu_{0.5}$ and μ_2 represent cases with small and large changes in daily volatility.

3.3.1 In-sample analysis

Table 9 presents estimated coefficients for the standard GARCH model (5) and the RGARCH model (10) together with the values of Akaike Information Criterion (AIC). As expected, the RGARCH model performs better than the standard GARCH model.

Table 9: Estimated coefficients of the GARCH(1,1) model $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2$ and the RGARCH(1,1) model $\sigma_t^2 = \omega + \alpha \widehat{\sigma_{P,t-1}^2} + \beta \sigma_{t-1}^2$, reported together with the values of Akaike Information Criterion (AIC) of the respective equations for the simulated data.

	GARCH(1,1)				RGARCH(1,1)			
	ω	α	β	AIC	ω	α	β	AIC
$\mu_{0.5}$	1.77E-04	0.016	0.958	-2.143	1.76E-04	0.053	0.922	-2.149
μ_1	1.73E-04	0.044	0.933	-2.112	1.61E-04	0.122	0.857	-2.133
μ_2	1.50E-04	0.114	0.875	-2.037	1.20E-04	0.274	0.723	-2.101

Coefficients in the RGARCH are changed in exactly the same way as in the previous section - coefficient α is increased and coefficient β is decreased. Note that $\alpha+\beta$ is smaller than one for both GARCH and RGARCH model (implying stationarity) and $\alpha+\beta$ is the same (0.98) for both models. This means that both GARCH and RGARCH models imply the same (high) volatility persistence. This is very natural, since we simulated volatility as a highly persistent process. Note that when volatility changes more rapidly (μ increases), more weight is put on the recent (noisy) observation of volatility (α increases) and less weight is put on the past observation of volatility (β decreases).

Now we estimate the combined GARCH model (22). As we can see (Table 10), the results are generally consistent with those in Table 2.

The main difference is that the negative coefficient α_1 is now clearly significant. As Garman and Klass (1980) showed, the optimal volatility forecast based on open, high, low

Table 10: Estimated coefficients and p-values for the combined GARCH(1,1) model $\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 + \alpha_2 \widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$ for the simulated data.

	combined GARCH(1,1)			
	ω	α	β	γ
$\mu_{0.5}$	1.71E-04	-0.027	0.908	0.094
μ_1	1.53E-04	-0.057	0.834	0.204
μ_2	1.13E-04	-0.119	0.686	0.431

and close price is (9). It is a weighted average of the Parkinson volatility estimator (8) and squared open-to-close returns, where squared returns have negative weight. This is the reason why coefficient α_1 is negative. Note that the ration between the coefficients α_1 and α_2 is very close to the ratio predicted from the Garman-Klass formula.

As previously mentioned, we use the Parkinson volatility estimator (8) instead of Garman and Klass (9) volatility estimator because of the data concerns (open prices are sometimes not available). Another reason is that for the purpose of volatility modelling, the Garman and Klass volatility estimator brings only a small improvement over the Parkinson estimator even in the ideal case. This can be seen from the coefficient β, which decreases from 0.933 (for the standard GARCH) to 0.857 (for RGARCH), but afterwards only a little bit to 0.834 (for the combined GARCH, which is basically the same as GARCH based on the Garman and Klass volatility estimator).

3.3.2 Out–of-sample forecasting performance

Now we compare the forecasting performance of the RGARCH model and the standard GARCH model on the simulated data. Results are shown in Table 11.

These results illustrate the benefit of using simulated data. Now we know exactly what the true volatility is and we can use it as a benchmark. Additionally, simulation allows us to have much larger data sample (100000 observations of the simulated data instead

Table 11: Comparison of the forecasting performance of the GARCH(1,1) model $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2$ and the RGARCH(1,1) model $\sigma_t^2 = \omega + \alpha\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$. Numbers in this table are 1000×RMSE of the one-day-ahead rolling window forecasts reported for different window sizes w and different benchmarks squared returns (r^2) and the realized variance (RV) for the simulated data. The differences in MSE are significant at any significance level (due to very large number of observations).

	GARCH(1,1)				RGARCH(1,1)				σ_{true}^2
	w=300	w=400	w=500	w=600	w=300	w=400	w=500	w=600	
r^2 as a benchmark									
$\mu_{0.5}$	10.16	10.14	10.12	10.11	10.15	10.12	10.10	10.09	9.99
μ_1	11.90	11.86	11.84	11.83	11.78	11.74	11.72	11.71	11.49
μ_2	20.31	20.22	20.11	20.07	19.78	19.71	19.63	19.60	18.98
σ_{true}^2 as a benchmark									
$\mu_{0.5}$	1.81	1.71	1.63	1.57	1.71	1.59	1.49	1.43	0
μ_1	3.00	2.88	2.80	2.75	2.52	2.32	2.21	2.15	0
μ_2	7.15	6.97	6.84	6.72	5.52	5.30	5.22	5.15	0

of 4423 observations of the real data), which in turns mean that all the results are highly statistically significant.

First note that the results obtained from the simulated data (Table 11) are consistent with results in Table 3 and Table 8. Table 3 and Table 8 show that the RGARCH model outperforms the standard GARCH model most of the time. Since the simulated data are much larger, we basically got rid of the noise and now we can see (in Table 11) exactly how much better the RGARCH performs. Let us focus for now primarily on the data simulated with the parameter μ_1, which is arguably closest to the real world. The improvement seems to be small, just around 1% decrease in RMSE, when we use squared returns as a benchmark. However, use of the true volatility as a benchmark shows that the real improvement of the RGARCH in comparison to the standard GARCH model is much larger, around 20%.

In fact, the mean squared error (MSE) between the forecasted volatility ($\widehat{\sigma^2}$) and a noisy volatility proxy (σ^2_{noisy}) can be rewritten in the following way:

$$MSE\left(\widehat{\sigma^2}, r^2\right) = MSE\left(\widehat{\sigma^2}, \sigma^2_{true}\right) + MSE\left(\sigma^2_{true}, \sigma^2_{noisy}\right) \tag{29}$$

where σ^2_{true} is the true volatility. This means that part of the MSE is due to the model imperfection (first term) and second part is due to the noisiness of the volatility proxy. When squared returns are used as a benchmark, then the second term typically dominates and it is therefore difficult to choose between competing volatility models based on the MSE (RMSE).

To understand when the RGARCH model provides the largest improvement over GARCH model (Hypothesis 3), let us look at Table 11. As we cam see, the larger the day-to-day changes in volatility, the larger the improvement of the RGARCH model (relatively to the GARCH model). The decrease in RMSE (with the true volatility as a benchmark) when we use RGARCH instead of GARCH is 6%-9% in case of small day-to-day changes in volatility, 16%-22% for moderate changes in volatility and 23%-24% for large changes in volatility. This confirms our Hypothesis 3.

4 Summary

The goal of this paper was to show a simple, effective and general way to incorporate range (the difference between the highest and the lowest price of the day) into the standard GARCH volatility models. We illustrated our idea on the GARCH(1,1) model, which we modify and create a Range GARCH(1,1) model. Empirical tests performed on 30 stocks, 6 stock indices and simulated data show that the RGARCH model outperforms the standard GARCH model, both in the in-sample fit and in the out-of-sample forecasting. The main intuition behind this result is that replacing squared returns by less noisy volatility proxy has two advantages. First, using more precise volatility proxy in a given model obviously helps. Second, when the model is estimated, more weight than before will be attributed to the most recent volatility estimate, because this estimate is now less noisy. As a consequence, this model adjusts more quickly to the changes of volatility and therefore performs particularly well when volatility changes quickly. This is very desirable feature, because volatility forecasting is most important in situations when volatility changes the most.

Another advantage of our model is that it provides both high precision of range as a volatility proxy with simplicity and ease of estimation of the standard GARCH model. This model offers increased precision in volatility modelling at almost no costs: additional required data (high and low prices) are typically widely available and the model itself can be easily estimated using standard econometric software, e.g. EViews, R or OxMetrics. Therefore we encourage both academics and practitioners to use the RGARCH model instead of the standard GARCH model whenever high and low data are available.

References

Alizadeh, S., M. W. Brandt, and F. X. Diebold (2002): "Range-Based Estimation of Stochastic Volatility Models," *The Journal of Finance*, 57, 1047–1091.

Andersen, T. G., T. Bollerslev, P. F. Christoffersen, and F. X. Diebold (2006): "Volatility and Correlation Forecasting," in *Handbook of Economic Forecasting*, ed. by G. Elliott, C. W. J. Granger, and A. Timmermann, Elsevier, vol. 1, chap. 15, 777–878.

Bollerslev, T. (1986): "Generalized autoregressive conditional heteroskedasticity," *Journal of Econometrics*, 31, 307–327.

Bollerslev, T. and H. O. Mikkelsen (1996): "Modeling and pricing long memory in stock market volatility," *Journal of Econometrics*, 73, 151–184.

Brandt, M. W. and C. S. Jones (2006): "Volatility Forecasting With Range-Based EGARCH Models," *Journal of Business and Economic Statistics*, 24, 470–486.

Broto, C. and E. Ruiz (2004): "Estimation methods for stochastic volatility models: a survey," *Journal of Economic Surveys*, 18, 613–649.

Chou, R. Y. (2005): "Forecasting Financial Volatilities with Extreme Values: The Conditional Autoregressive Range (CARR) Model," *Journal of Money, Credit, and Banking,*, 37, 561–582.

Chou, R. Y., H. Chou, and N. Liu (2010): "Range Volatility Models and Their Applications in Finance," in *Handbook of Quantitative Finance and Risk Management*, ed. by C.-F. Lee and J. Lee, Springer, chap. 83.

Dacorogna, M. M., R. Gencay, U. Muller, R. B. Olsen, and O. V. Olsen (2001): *An introduction to high frequency finance*, Academic Press, New York.

Diebold, F. X. and R. S. Mariano (1995): "Comparing Predictive Accuracy," *Journal of Business and Economic Statistics*, 13, 253–265.

Engle, R. F. (1982): "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation," *Econometrica*, 50, 987–1007.

Engle, R. F. and A. J. Patton (2001): "What good is a volatility model?" *Quantitative Finance*, 1, 237–245.

Garman, M. B. and M. J. Klass (1980): "On the Estimation of Security Price Volatilities from Historical Data," *The Journal of Business*, 53, 67–78.

Ghysels, E., A. C. Harvey, and E. Renault (1996): "Stochastic volatility," in *Statistical Methods in Finance*, ed. by C. R. Rao and G. S. Maddala, Amsterdam: North-Holland, 119–191.

Glosten, L. R., R. Jagannathan, and D. E. Runkle (1993): "On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks," *The Journal of Finance*, 48, 1779–1801.

Hansen, P. R., Z. Huang, and H. H. Shek (2011): "Realized GARCH: a joint model for returns and realized measures of volatility," *Journal of Applied Econometrics*, forthcoming.

Hansen, P. R. and A. Lunde (2005): "A forecast comparison of volatility models: does anything beat a GARCH(1,1)?" *Journal of Applied Econometrics*, 20, 873–889.

——— (2006): "Consistent ranking of volatility models," *Journal of Econometrics*, 131, 97–121.

Kim, S., N. Shephard, and S. Chib (1998): "Stochastic Volatility: Likelihood Inference and Comparison with ARCH Models," *The Review of Economic Studies*, 65, 361–393.

Molnár, P. (2011): "Properties of range-based volatility estimators," *International Review of Financial Analysis*, forthcoming.

Nelson, D. B. (1990): "ARCH models as diffusion approximations," *Journal of Econometrics*, 45, 7–38.

——— (1991): "Conditional Heteroskedasticity in Asset Returns: A New Approach," *Econometrica*, 59, 347–370.

——— (1992): "Filtering and forecasting with misspecified ARCH models I: Getting the right variance with the wrong model," *Journal of Econometrics*, 52, 61–90.

Nelson, D. B. and D. P. Foster (1994): "Asymptotic Filtering Theory for Univariate Arch Models," *Econometrica*, 62, 1–41.

Parkinson, M. (1980): "The Extreme Value Method for Estimating the Variance of the Rate of Return," *The Journal of Business*, 53, 61–65.

Patton, A. J. (2011): "Volatility forecast comparison using imperfect volatility proxies," *Journal of Econometrics*, 160, 246–256.

Poon, S. H. and C. W. J. Granger (2003): "Forecasting Volatility in Financial Markets: A Review," *Journal of Economic Literature*, 41, 478–539.

Shephard, N. (1996): "Statistical Aspects of ARCH and Stochastic Volatility," in *Time Series Models in Econometrics, Finance and Other Fields*, ed. by D. R. Cox, D. V. Hinkley, and B. O. E. Nielsen, London: Chapman & Hall, 1–67.

Shephard, N. and K. Sheppard (2010): "Realising the future: forecasting with high-frequency-based volatility (HEAVY) models," *Journal of Applied Econometrics*, 25, 197–231.

Taylor, S. J. (1986): *Modelling Financial Time Series*, John Wiley and Sons, 2nd ed.

5 Appendix

In previous parts of this paper, open-to-close returns were used for the estimation and squared open-to-close returns were used as a volatility benchmark. In some cases, a variable of interest are close-to-close returns. This appendix (Table 12 – Table 19) documents that all the conclusions remain the same when we use close-to-close returns for the estimation and squared close-to-close returns as a volatility benchmark.

Table 12: Estimated coefficients of the GARCH(1,1) model $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2$ and the RGARCH(1,1) model $\sigma_t^2 = \omega + \alpha\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$ (both estimated on the close-to-close returns), reported together with the values of Akaike Information Criterion (AIC) of the respective equations.

Ticker	GARCH(1,1)				RGARCH(1,1)			
	ω	α	β	AIC	ω	α	β	AIC
AA	2.81E-06	0.039	0.956	-4.864	2.29E-06	0.075	0.941	-4.880
AXP	2.41E-06	0.075	0.923	-5.080	1.38E-06	0.227	0.829	-5.119
BA	3.03E-06	0.052	0.942	-5.166	4.90E-06	0.212	0.835	-5.184
BAC	2.21E-06	0.062	0.933	-5.244	3.21E-06	0.245	0.818	-5.274
CAT	3.08E-06	0.026	0.967	-4.990	2.08E-05	0.190	0.813	-5.021
CSCO	6.27E-06	0.061	0.933	-4.465	8.99E-06	0.227	0.817	-4.515
CVX	3.97E-06	0.061	0.922	-5.616	5.10E-06	0.141	0.866	-5.632
DD	1.30E-06	0.037	0.960	-5.351	8.71E-07	0.091	0.923	-5.373
DIS	3.56E-06	0.056	0.938	-5.140	3.61E-06	0.189	0.848	-5.202
GE	7.74E-07	0.046	0.952	-5.521	7.70E-07	0.156	0.874	-5.549
HD	1.51E-06	0.044	0.955	-5.022	5.25E-06	0.230	0.819	-5.043
HPQ	2.19E-06	0.020	0.976	-4.669	1.25E-06	0.058	0.957	-4.703
IBM	2.31E-06	0.061	0.937	-5.225	3.98E-06	0.380	0.742	-5.272
INTC	5.30E-06	0.046	0.947	-4.588	8.54E-06	0.207	0.848	-4.614
JNJ	1.47E-06	0.081	0.916	-5.829	9.42E-07	0.161	0.866	-5.840
JPM	1.37E-06	0.061	0.939	-5.015	-8.62E-08	0.126	0.905	-5.052
CAG	5.98E-07	0.031	0.968	-5.614	1.73E-05	0.452	0.571	-5.647
KO	1.07E-06	0.050	0.946	-5.750	-4.18E-07	0.157	0.878	-5.771
MCD	2.27E-06	0.046	0.947	-5.468	1.99E-06	0.086	0.920	-5.487
MMM	2.94E-06	0.029	0.958	-5.613	1.71E-05	0.243	0.735	-5.650
MRK	2.86E-05	0.047	0.876	-5.118	4.20E-05	0.271	0.690	-5.164
MSFT	6.44E-06	0.067	0.921	-5.011	1.01E-05	0.362	0.724	-5.066
PFE	4.65E-06	0.055	0.932	-5.257	1.18E-05	0.242	0.782	-5.271
PG	8.65E-07	0.041	0.957	-5.715	-5.96E-07	0.080	0.941	-5.750
T	1.64E-06	0.059	0.937	-5.411	1.91E-06	0.126	0.892	-5.421
TRV	4.97E-06	0.070	0.916	-5.385	9.58E-06	0.198	0.811	-5.433
UTX	4.72E-06	0.101	0.894	-5.407	3.75E-06	0.350	0.743	-5.454
VZ	2.09E-06	0.059	0.935	-5.494	4.07E-06	0.180	0.843	-5.500
WMT	1.28E-06	0.043	0.954	-5.389	1.92E-06	0.121	0.892	-5.410
XOM	2.38E-06	0.058	0.932	-5.706	4.19E-06	0.175	0.841	-5.737

Table 13: Estimated coefficients and p-values for the combined GARCH(1,1) model $\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 + \alpha_2 \widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$ estimated on the close-to-close returns.

Ticker	combined GARCH(1,1)							
	ω	p-value	α_1	p-value	β	p-value	α_2	p-value
AA	2.29E-06	0.001	0.000	0.936	0.941	0	0.075	1.6E-13
AXP	1.38E-06	0.083	0.000	0.949	0.830	0	0.227	0
BA	4.74E-06	0.000	0.002	0.788	0.838	0	0.205	0
BAC	3.17E-06	0.000	-0.003	0.657	0.817	0	0.251	0
CAT	2.06E-05	0.000	-0.010	0.015	0.815	0	0.202	0
CSCO	8.18E-06	0.000	-0.014	0.002	0.828	0	0.232	0
CVX	5.11E-06	0.000	-0.001	0.956	0.866	0	0.142	1.5E-13
DD	7.39E-07	0.160	-0.010	0.075	0.923	0	0.104	0
DIS	3.19E-06	0.003	-0.013	0.000	0.852	0	0.203	0
GE	6.16E-07	0.165	-0.011	0.009	0.876	0	0.168	0
HD	5.47E-06	0.000	0.008	0.321	0.821	0	0.216	0
HPQ	1.03E-06	0.004	-0.005	0.000	0.958	0	0.064	0
IBM	3.97E-06	0.000	0.000	0.963	0.742	0	0.379	0
INTC	8.54E-06	0.000	0.000	0.957	0.848	0	0.206	0
JNJ	1.07E-06	0.001	0.038	0.000	0.873	0	0.105	0
JPM	-2.95E-07	0.584	-0.010	0.070	0.907	0	0.137	0
CAG	1.77E-05	0.000	-0.016	0.003	0.563	0	0.478	0
KO	-1.55E-07	0.662	0.018	0.001	0.873	0	0.141	0
MCD	1.95E-06	0.001	0.004	0.496	0.921	0	0.080	2.2E-16
MMM	1.54E-05	0.000	-0.019	0.000	0.755	0	0.249	0
MRK	4.21E-05	0.000	-0.001	0.687	0.689	0	0.274	0
MSFT	1.00E-05	0.000	-0.010	0.006	0.725	0	0.374	0
PFE	1.04E-05	0.000	0.026	0.001	0.807	0	0.182	0
PG	-5.93E-07	0.000	0.000	0.989	0.941	0	0.080	0
T	1.83E-06	0.003	0.030	0.000	0.895	0	0.086	3.1E-12
TRV	9.60E-06	0.000	-0.001	0.814	0.811	0	0.200	0
UTX	3.67E-06	0.008	-0.011	0.221	0.741	0	0.367	0
VZ	3.16E-06	0.000	0.037	0.000	0.874	0	0.098	0
WMT	1.92E-06	0.001	0.000	0.957	0.892	0	0.122	0
XOM	4.41E-06	0.000	-0.040	0.000	0.828	0	0.240	0

Table 14: Comparison of the forecasting performance of the GARCH(1,1) model $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2$ and the RGARCH(1,1) model $\sigma_t^2 = \omega + \alpha\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$(both estimated on the close-to-close returns with squared close-to-close returns as a benchmark). Numbers in this table are 1000×RMSE of the one-day-ahead rolling window forecast reported for different window sizes w. An asterisk * (**) indicates when the difference is significant at the 5% (1%) level.

Ticker	GARCH(1,1)				RGARCH(1,1)			
	w=300	w=400	w=500	w=600	w=300	w=400	w=500	w=600
AA	1.892	1.915	1.937	1.968	1.837	1.874	1.893	1.924
AXP	1.728	1.733	1.763	1.786	1.711	1.736	1.747	1.768
BA	1.247	1.254	1.262	1.280	1.235	1.240	1.248*	1.263*
BAC	4.561	4.572	4.615	4.652	4.435	4.454	4.351	4.432
CAT	1.135	1.142	1.152	1.170	1.098*	1.111*	1.123*	1.147*
CSCO	2.003	2.027	2.047	2.077	1.988	2.003	2.008	2.037
CVX	0.883	0.888	0.899	0.909	0.850**	0.859*	0.876*	0.885*
DD	0.855	0.864	0.874	0.883	0.836*	0.845*	0.857*	0.871*
DIS	1.311	1.328	1.341	1.358	1.307	1.310	1.328	1.342
GE	1.137	1.153	1.168	1.183	1.276	1.227	1.138	1.181
HD	2.709	2.158	2.204	2.228	2.938	2.698	2.519	2.500
HPQ	1.800	1.814	1.825	1.853	1.775*	1.792**	1.812*	1.840**
IBM	1.099	1.109	1.122	1.134	1.095	1.107	1.120	1.133
INTC	1.998	2.007	2.026	2.050	1.951*	1.969*	1.992*	2.016*
JNJ	0.690	0.691	0.696	0.699	0.665**	0.669**	0.678*	0.681*
JPM	2.443	2.471	2.508	2.510	2.284**	2.317**	2.352**	2.386**
CAG	0.967	0.977	0.989	0.999	0.970	0.979	0.991	1.016
KO	0.655	0.661	0.664	0.671	0.647	0.649*	0.655*	0.655**
MCD	0.735	0.737	0.744	0.751	0.730	0.735	0.741	0.746
MMM	0.621	0.624	0.629	0.637	0.613	0.612	0.615	0.624*
MRK	1.811	1.830	1.839	1.863	1.792	1.808*	1.833	1.845*
MSFT	1.338	1.347	1.363	1.375	1.308*	1.310**	1.332**	1.345*
PFE	0.795	0.800	0.811	0.818	0.800	0.800	0.812	0.816
PG	2.337	2.447	2.444	2.466	2.297	2.317**	2.343**	2.371**
T	0.851	0.857	0.865	0.874	0.830	0.838	0.848	0.854
TRV	1.479	1.493	1.512	1.526	1.428*	1.440**	1.453**	1.468**
UTX	1.871	1.880	1.905	1.929	1.862	1.875	1.900	1.922
VZ	0.788	0.794	0.801	0.809	0.772**	0.779**	0.789*	0.794*
WMT	0.735	0.742	0.745	0.754	0.723*	0.730*	0.736	0.744
XOM	0.801	0.804	0.815	0.824	0.753*	0.761*	0.781*	0.791*

Table 15: Comparison of the forecasting performance of the RGARCH(1,1) model $\sigma_t^2 = \omega + \alpha\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$and several different GARCH models, all of them estimated on the close-to-close returns with squared close-to-close returns as a benchmark. Numbers in this table are 1000×RMSE of the one-day-ahead rolling window forecast with forecasting window equal to 400.

ticker	RGARCH	GARCH	GJR	EGARCH	stdGARCH	astdGARCH	cGARCH
AA	1.874	1.915	1.871	1.882	1.916	1.862	1.926
AXP	1.738	1.735	1.732	1.731	1.733	1.729	1.743
BA	1.240	1.254	1.265	1.258	1.250	1.245	1.260
BAC	4.454	4.572	4.556	4.459	4.545	4.458	4.571*
CAT	1.112	1.144*	1.140*	1.141*	1.143*	1.134	1.143*
CSCO	2.254	2.288	2.247	2.236	2.267	2.234	2.309*
CVX	0.857	0.887*	0.854	0.863	0.907**	0.864	0.900**
DD	0.845	0.865*	0.863	0.863	0.865**	0.858	0.871**
DIS	1.310	1.328	1.337*	1.317	1.322	1.317	1.330*
GE	1.226	1.152	1.169	1.143	1.154	1.135	1.172
HD	2.699	2.159	2.441	19.615	2.111	2.117	2.420
HPQ	1.800	1.823**	1.812*	1.813*	1.814**	1.800	1.830**
IBM	1.143	1.145	1.150	1.135	1.137	1.127	1.156
INTC	1.972	2.010**	2.068**	1.987	1.999**	1.988	2.034**
JNJ	0.668	0.690**	0.693*	0.678	0.681*	0.673	0.693*
JPM	2.317	2.471**	2.395	2.367	2.445*	2.365	2.451*
CAG	0.980	0.978	0.985	0.980	0.981	0.981	0.981
KO	0.647	0.658*	0.662	0.651	0.656*	0.652	0.663**
MCD	0.738	0.740	0.752	0.744	0.736	0.733	0.744
MMM	0.609	0.621	0.621	0.626	0.618	0.618	0.629
MRK	0.778	0.793**	0.790*		0.790*	0.785	0.799**
MSFT	1.312	1.349**	1.346**	1.342**	1.342**	1.341*	1.365**
PFE	0.804	0.804	0.806	0.803	0.802	0.799	0.807
PG	2.317	2.447**	2.519**	2.318	8.394	12.721?	2.353**
T	0.838	0.857	0.853	0.855	0.860	0.853	0.857
TRV	1.439	1.492**	1.476*	1.483*	1.496*	1.485*	1.487
UTX	1.876	1.881	2.470*	1.920	2.031*	1.905*	2.326
VZ	0.778	0.793**	0.790*		0.791*	0.785	0.799**
WMT	0.735	0.745*	0.745		0.745*	0.740	0.749*
XOM	0.759	0.803*	0.779		0.821*	0.798	0.827**

Table 16: Comparison of forecasting performance GARCH(1,1) model $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2$ and the RGARCH(1,1) model $\sigma_t^2 = \omega + \alpha\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$. Numbers in this table are the LLF of the returns r_t beind drawn from the distributions $N\left(0, \widehat{\sigma_t^2}\right)$, where $\widehat{\sigma_t^2}$ is a one-day-ahead rolling window volatility forecast reported for different window sizes w.

Ticker	GARCH(1,1)				RGARCH(1,1)			
	w=300	w=400	w=500	w=600	w=300	w=400	w=500	w=600
AA	9803	9580	9267	9020	9873	9597	9320	9032
AXP	10377	10166	9881	9595	10502	10242	9969	9688
BA	10434	10225	9809	9660	10500	10258	9993	9708
BAC	10687	10451	10154	9875	10783	10527	10236	9949
CAT	10105	9916	9631	9342	10202	9950	9675	9385
CSCO	9309	9080	8825	8528	9478	9237	8955	8646
CVX	11371	11017	10853	10576	11440	11145	10882	10599
DD	10853	10593	10321	10050	10916	10641	10377	10095
DIS	10535	10298	10024	9747	10681	10411	10142	9859
GE	11086	10860	10567	10258	11176	10902	10617	10325
HD	10266	10024	9729	9479	10372	10084	9809	9542
HPQ	9587	9255	9076	8792	9715	9415	9174	8869
IBM	10813	10575	10247	9972	10986	10716	10378	10130
INTC	9420	9208	8937	8665	9484	9278	9001	8735
JNJ	12013	11776	11492	11230	12063	11126	11522	11264
JPM	10158	10014	9730	9464	10345	10113	9830	9554
CAG	11421	11192	10939	10681	11563	11301	10994	10722
KO	11682	11454	11155	10924	11782	11517	11250	10980
MCD	11058	10846	10564	10288	11129	10871	10596	10320
MMM	11238	11105	10819	10542	11377	11153	10878	10599
MRK	10131	9775	9632	9294	10348	10120	9813	9570
MSFT	10234	10038	9724	9478	10396	10171	9867	9611
PFE	10741	10499	10222	9959	10827	10564	10269	10004
PG	11512	11236	10962	10709	11571	11369	11081	10777
T	10948	10704	10459	10172	11002	10744	10473	10206
TRV	10801	10614	10312	10069	10899	10678	10395	10111
UTX	11013	10790	10477	10179	11054	10840	10556	10269
VZ	11132	10892	10605	10329	11198	10930	10645	10361
WMT	11004	10778	10510	10109	11130	10860	10558	10276
XOM	11464	11223	10947	10657	11567	11294	11014	10729

Table 17: Estimated coefficients of the GARCH(1,1) model $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2$ and it modified version RGARCH(1,1) $\sigma_t^2 = \omega + \alpha\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$, both of them estimated on the close-to-close returns, reported together with the values of Akaike Information Criterion (AIC) of the respective equations for various stock indices.

Index	GARCH(1,1)				RGARCH(1,1)			
	ω	α	β	AIC	ω	α	β	AIC
CAC40	1.49E-06	0.072	0.922	-5.965	2.98E-06	0.251	0.823	-5.995
DAX	2.19E-06	0.088	0.902	-5.946	9.88E-06	0.207	0.808	-5.954
DJI	1.34E-06	0.084	0.909	-6.323	-1.51E-06	0.140	0.761	-6.341
FTSE	8.90E-07	0.080	0.914	-6.495	1.93E-06	0.183	0.846	-6.515
NASDAQ	4.23E-05	0.044	0.880	-4.796	1.25E-05	0.032	0.959	-4.827
NIKKEI	4.40E-06	0.089	0.893	-5.758	3.64E-06	0.267	0.837	-5.780

Table 18: Estimated coefficients and p-values for the combined GARCH(1,1) model $\sigma_t^2 = \omega + \alpha_1 r_{t-1}^2 + \alpha_2\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$ estimated on the close-to-close returns for various stock indices.

Index	combined GARCH(1,1)							
	ω	p-value	α_1	p-value	β	p-value	α_2	p-value
CAC40	3.26E-06	5.22E-06	-0.017	9.01E-02	0.813	0	0.290	0.00E+00
DAX	5.00E-06	3.40E-13	0.051	1.09E-08	0.854	0	0.104	4.34E-13
DJI	-3.09E-07	0.438	0.041	8.01E-05	0.823	0	0.077	1.90E-11
FTSE	1.95E-06	1.14E-06	-0.001	0.931365	0.846	0	0.184	5.57E-12
NASDAQ	4.63E-04	0.014	-0.002	0.363481	0.575	0.001	-0.006	0.561
NIKKEI	3.63E-06	2.40E-06	0.002	0.838884	0.837	0	0.262	0

Table 19: Comparison of forecasting performance of the GARCH(1,1) model $\sigma_t^2 = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1}^2$ and its modified version RGARCH(1,1) $\sigma_t^2 = \omega + \alpha\widehat{\sigma_{P,t-1}^2} + \beta\sigma_{t-1}^2$ (both them estimated on the close-to-close returns). As a benchmark is used both squared close-to-close returns and realized variance. Numbers in this table are 1000×RMSE of the one-day-ahead rolling window forecasts reported for different window sizes w and different benchmarks (squared returns r^2 and the true volatility σ_{true^2}) for various stock indices.

Index	Bench	GARCH(1,1)				RGARCH(1,1)			
		w=300	w=400	w=500	w=600	w=300	w=400	w=500	w=600
CAC	r^2	0.609	0.607	0.603	0.602	0.583**	0.582**	0.581**	0.581**
	RV	0.257	0.257	0.242	0.241	0.233*	0.227**	0.217**	0.218**
DAX	r^2	0.642	0.636	0.633	0.633	0.622	0.623	0.621	0.620
	RV	0.272	0.259	0.249	0.250	0.263	0.263	0.260	0.258
DJI	r^2	0.501	0.507	0.504	0.513	0.482*	0.486	0.483	0.493
	RV	0.199	0.207	0.202	0.209	0.171**	0.167**	0.164**	0.165**
FTSE	r2	0.480	0.482	0.479	0.478	0.466*	0.465**	0.463**	0.463**
	RV	0.205	0.230	0.216	0.213	0.202	0.207**	0.195**	0.192**
NASDAQ	r2	1.390	1.290	1.232	1.206	1.253	1.202**	1.165**	1.145**
	RV	0.549	0.674	0.678	0.657	0.517	0.578*	0.567**	0.550**
NIKKEI	r^2	0.699	0.702	0.695	0.689	0.683*	0.678**	0.675*	0.669*
	RV	0.356	0.394	0.366	0.351	0.363	0.365*	0.345*	0.326*