The Making of Zero Curves

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Risk-free rates are ubiquitous in finance, even more after the Cox-Ross [6] principle of risk-neutral valuation for derivatives. Simple arbitrage shows that they can be negative only if considered net of the storage cost of money.

The phase-out of Libor, its replacement by overnight reference rates, and the way they are used to determine zero rates make this paper's topic relevant.

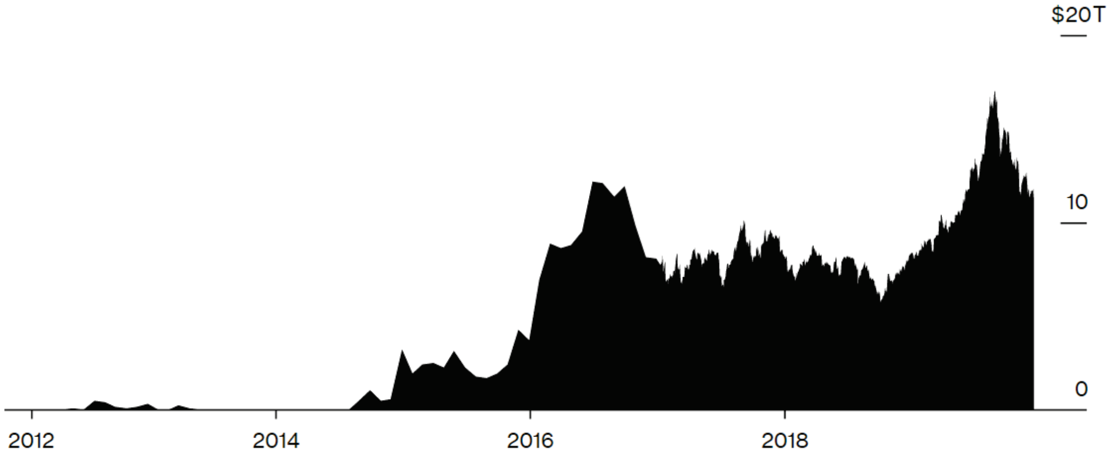
Similarly to the federal funds rate, which is a weighted average of the rates in brokered transactions (with weights being determined by the size of the transaction), we advocate the construction of transaction-weighted zero curves. In particular, we highlight the role of risk-free zero curves defined by the ISDA CDS standard model in order to standardize the quotes in the market for credit default swaps.

Zero curves play a crucial role in finance. They are the tool that instantaneously transforms the contracts' *promised* payments into current values. The sustainability of promises depends on the creditworthiness of the parties, the contract's collaterals, and the risk factors that affect the contract's payoff, including its currency denomination.

> *An interest rate in a particular situation defines the amount of money a borrower promises to pay the lender. For any given currency, many different types of interest rates are regularly quoted. These include mortgage rates, deposit rates, prime borrowing rates, and so on.* {Hull (2022) p. 76 [13]}

An interest rate is the percentage of principal charged by the lender for the use of one of its assets, not only money. Borrowing and lending may involve several assets. Therefore, we can observe interest rates on lending of money, gold, securities, flats, furniture, clothing, and other markets that foster the circular economy (https://wef.ch/2QhuHSI). As stated by Irving Fisher (1907) [10] in his preface: "*... the rate of interest is not a phenomenon restricted to money markets, but is omnipresent in economic relations.*"

The interest rate on money depends on the state of the economy, creditworthiness, liquidity, collateral, storage cost of money, administrative costs, etc. When the default probability is high and the collateral is insufficient, the interest rate can be very high. Fisher Black (1975) [5] showed that, if the risk-free interest rate on money is 6%, the interest rate on a uncollateralized money loan with 33.3% default probability and null recovery rate can be as high as 59% (= 1.06/66.6% − 1). On the contrary, when the risk-free interest rate on money is ultra-low and the storage cost of money is high, the interest rate can be negative.



Source: Ainger (2019) [1]

Figure 1 Market Value of Negative-Yielding Bonds.

Of course, there is not a single interest rate on money as of one moment in time, any more than a *ten-year* lease of a flat at a annual rate would be the same annual rate as a *one-year* lease on the flat, quite apart from how the one-year lease rate on the flat might move from year to year. Those two simultaneous lease rates, if expanded to include other durations, say six months and three years, imply a curve to the lease rates, and raises the question of how to describe that whole curve in a parsimonious manner. Add in the distinction between default / liquidity risk, and the resulting concept is a zero curve. Zero curves might be expected to display positive numbers, but with money and near-monies, the expectation is also that the zero curve should also not be much below zero, if ever (The word "zero" thus involves two concepts).

1. Negative Interest Rates

The market for interest-rate derivatives is huge. At the end of June 2020, the notional capital of interest-rate derivatives was equal to $495 trillion {Bank for International Settlements (2020) [2]}. The bulk of this amount is represented by interest rate swaps ($364 trillion).

In December 2020, the global debt with sub-zero yields rose to $18.04 trillion, the highest level ever recorded, according to data compiled by Bloomberg (Figure 1).[1] Buyers holding the securities to maturity are guaranteed to make a loss. Why do investors lend money at negative rates? Nominal interest rates on money lending can never be negative. This is a simple corollary to the assumption of no arbitrage: "*The assumption of no arbitrage (NA) is compelling because it appeals to the most basic beliefs about human behavior, namely that there is someone who prefers having more wealth to having less.*" {Ross (2005) p. 2 [21]}

Why should one lend money at a negative rate, so receiving back - in the future - an amount lower than the amount currently owned? Nominal interest rates on money *cannot* be negative. However, when the interest rate on money is ultra low and the lender's storage cost

[1] As pointed out by Masera (2019, p. 6, [18]), negative nominal interest rates (NNIRs) "pose serious challenges for risk valuation models – notably those currently used by regulators in risk-weighted capital requirements – and therefore increase uncertainty and risk."

of money is higher then that of the borrower, the lender may find it convenient to enter into a contract with a negative rate in order to *transfer* the storage costs to the borrower.

Gold held in the Bank of England's vaults is a type of money, and there is close to no risk. Gold is held on behalf of other central banks and the bullion dealers in the London Bullion Market Association (LBMA), which is the largest wholesale market for precious metals in the world.[2] The gold bars are approximately 400 troy ounces, worth about $700,000 at prices common in recent months ($1,750 per troy ounce). The Bank of England charges 3.5 pence per bar per night. At 365 nights per year, that would be £12.775 annually. At an exchange rate of $1.3/£1, that would be close to $17 annually. Therefore, the storage cost of gold is about 0.0024% per year (= $17 / $700,000 × 100).

Storage of bulk commodities, such as coffee and wheat, which are near-moneys when in centralized warehouses, costs typically a flat rate (per bag, per bushel, etc.), rather than a percent.[3] Money is unusual in that percentages are common, but this is not a substantive difference. The agreed interest rate on a fully-collateralized contract is equal to the difference between the interest rate on money and the storage cost of money. If the interest rate on money is null, the contract's interest rate is equal to the difference between the borrower's and the lender's storage cost of money. Such a difference can also be seen as the cost of an insurance policy paid to the borrower for taking care of the lender's money. Consider the case of a safe deposit box (Moneyland Magazine [20]):

> *Holding money in bank safe deposit boxes provides an alternative to holding bank balances. They are a popular instrument for avoiding negative interest rates. ... Example: If you hold 50,000 Swiss francs in a bank safe deposit box with a 100-franc annual rental fee, you are effectively paying negative interest at the rate of 0.2% per annum. In this case, holding money in a safe deposit box would only be more profitable than investing it in a bank account if the negative interest charged by the bank were higher than 0.2%. ... In addition to the rent you pay for a bank safe deposit box, you should also account for possible additional costs. These may include extra insurance costs and transportation costs (to and from your safe deposit box) directly related to using a safe deposit box.*

2. Double Loans, Interest Rates, Discount Factors

Sometimes, double loans are negotiated, as in interest-rate swaps and repos. In a fixed-for-floating interest-rate swap (a contract with two rates), a party lends money at a variable rate while the other lends money at a fixed rate. The swap rate is a *par-coupon rate* that makes the contract fair. In a repo (a contract with one rate), a party lends money (and borrows securities), while the other lends securities (and borrows money). The repo rate is the difference between the interest rate on money lending and the interest rate on securities lending {Williams and Barone (1991) [26]}. It is clear that the repo rate is negative when the second rate is higher than the first one.

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[2] There is an active market in borrowing and lending the gold. Those rates have been on the order of .1 to .2% per annum.

[3] The capability to store surplus commodities and the impact of stockpiles on prices and production are analyzed in Williams and Wright (1991) [27].

Table 1 Eurex Clearing: Data fields of the input file “Interest Rate Curves Report”.

Field name	*Value type**	*Value example*	*Remark*
Value DateTime	Datetime	2012-01-21 14:30:00	Value date and time of the curve: YYYY-MM-DD hh:mm:ss
Curve ID	String (25)	EUR.EURIBOR.3M	Curves: “ccy.index.rfq”
Maturity Offset	Number (0)	138	Tenor grid point in offset days
Maturity Date	Datetime	2012-03-28 00:00:00	Date of the tenor grid point (time is 00:00:00 by default)
Value Type	String (1)	Z	S: Discount factor for IRD (mid); Z: Zero rate (mid)
Value	Number (14)	0.9753250047	IRD curve value and inflation index levels

* The number in the bracket indicates how many digits after the decimal point are provided for numbers and the maximum length of the field for String fields.

Table 2 Eurex Clearing: Discount factors and zero rates (14 May 2021).

Value DateTime	*Curve ID*	*Maturity Offset*	*Maturity Date*	*Value Type*	*Value*
13/05/2021 23:59:00	EUR.ESTR.1D	1	14/05/2021 00:00:00	S	1.00001566691211
13/05/2021 23:59:00	EUR.ESTR.1D	2	15/05/2021 00:00:00	S	1.00003134500738
...	...	...	...	...	...
13/05/2021 23:59:00	EUR.ESTR.1D	18267	18/05/2071 00:00:00	S	0.83598584610068
13/05/2021 23:59:00	EUR.ESTR.1D	18268	19/05/2071 00:00:00	S	0.83598115132305
13/05/2021 23:59:00	EUR.ESTR.1D	1	14/05/2021 00:00:00	Z	-0.00564004418046
13/05/2021 23:59:00	EUR.ESTR.1D	2	15/05/2021 00:00:00	Z	-0.00564201290377
...	...	...	...	...	...
13/05/2021 23:59:00	EUR.ESTR.1D	18267	18/05/2071 00:00:00	Z	0.00353050280589
13/05/2021 23:59:00	EUR.ESTR.1D	18268	19/05/2071 00:00:00	Z	0.00353042021402

Interest rates are just a mathematical transformation of discount factors, subject to several conventions (compounding frequencies, day count and quotation conventions). What really matters is the discount factor for a certain date, a concept that dates back to Leonardo da Pisa (1202) [15], alias Fibonacci.

The complex intertwinement among discount factors, risk attitudes (as measured by utility functions), subjective probabilities, risk-neutral probabilities, state prices and the pricing kernel is shown in Appendix A, where we study the valuation of an insurance policy, a case originally considered by Rubinstein (1999) [23].

As a short introduction to the comparison of various computations used by traders, the next sections show some of the info about zero curves that is available on the web. How would we judge which technique is best? Goodness of fit? Simplicity? Speed of computation? Lower fees for users? In other words, how much do these techniques offer in practice?

3. Central Counterparties (CCPs), Eiopa, ISDA and IHS Markit

All major CCPs (ICE, CME, LCH, Eurex, JSCC, TFX, etc.) need to discount the payments of Interest-Rate Derivatives (IRDs) for their daily settlement operations. For instance, Eurex Clearing makes available, on a daily basis, all zero rates and discount factors used to price and settle OTC IRD transactions: https://www.eurex.com/ec-en/clear/eurex-otc-clear/settlement-prices.

The data are provided for a given valuation timestamp and for a given offset from the valuation date (in days). The file “Interest Rate Curves Report” contains the data fields shown in Table 1. A sample of the 1-day to 50-year discount factors / zero rates used by Eurex Clearing on 14 May 2021 is shown in Table 2.

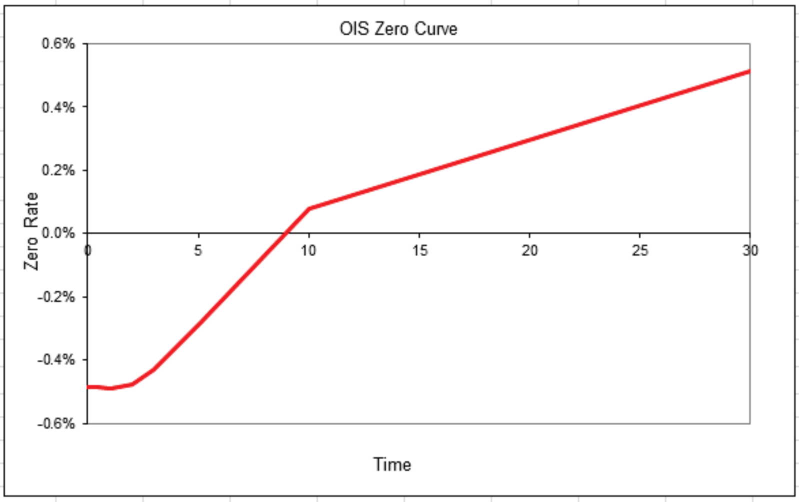
Table 3 SwapClear: Settlement prices for IRSs and OISs (14 May 2021).

	IRSs			OISs
	Anl Bnd / 1M Euribor	Anl Bnd / 3M Euribor	Anl Bnd / 6M Euribor	Anl Mny / Eonia
3 month	-	-	-	-0.48492
6 month	-	-	-	-0.48674
1 year	-	-	-	-0.48719
2 year	-0.52875	-0.50592	-0.46700	-0.47488
3 year	-0.46700	-0.44117	-0.39600	-0.42854
5 year	-0.30650	-0.27850	-0.46700	-0.28476
10 year	0.08600	0.11200	0.16000	0.07809
30 year	0.08600	0.53760	0.54775	0.49485

Note: Settlement prices are yields, although percentage sign (%) is not shown. The following abbreviations are used in column headings of the table: Annual frequency = Anl; Daycount scheme: Money market= Mny /, Bond= Bnd/.

Table 4 DerivaGem: OIS zero rates.

OIS Zeroes	
Time	Zero
0.000	-0.4852%
0.250	-0.4852%
0.500	-0.4873%
1.000	-0.4884%
2.000	-0.4760%
3.000	-0.4297%
5.000	-0.2859%
10.000	0.0790%
30.000	0.5114%



Another example is offered by SwapClear (LCH), which publishes the settlement prices of interest rate swaps (IRSs) and overnight indexed swaps (OISs) on the following web page: https://www.lch.com/services/swapclear/essentials/settlement-prices. The SwapClear settlement prices for EUR-denominated IRSs and OISs, used on 14 May 2021, are reported in Table 3.

A free software that calculates zero rates from OIS rates, using the bootstrap method, is provided by John C. Hull [13]: http://www-2.rotman.utoronto.ca/~hull/software/DG400a.zip. The DerivaGem software is made up by 4 files: DG400a.xls, DG400 Applications.xls (to carry out assignments), DG400 functions.xls (to see the functions used by DG400a.xls), readme.txt (to get start). The “Zero_Curve” worksheet of DG400a.xls allows to calculate OIS and Treasury (continuously compounded) zero rates. Accrual periods are assumed to be exact fractions of a year and day counts are Actual/Actual. Table 4 shows the OIS zero rates determined by DG400a.xls on the basis of SwapClear’s OIS rates reported in Table 3.

Table 5 EIOPA: risk fre rates denominated in EUR (30 April 2021).

1	-0.605%	21	0.492%	140	3.027%
2	-0.575%	22	0.550%	141	3.031%
3	-0.505%	23	0.613%	142	3.035%
4	-0.436%	24	0.680%	143	3.039%
5	-0.356%	25	0.748%	144	3.043%
6	-0.276%	26	0.818%	145	3.047%
7	-0.199%	27	0.887%	146	3.051%
8	-0.118%	28	0.955%	147	3.054%
9	-0.038%	29	1.022%	148	3.058%
10	0.034%	30	1.088%	149	3.062%
...	...	...	...	150	3.065%

Table 6 IHS Markit: Interest rates (14 May 2021)

Snap time	*Spot date*	*Tenor*	*Maturity date*	*Par Rate*
14/05/2021 14:00	19/05/2021	1M	18/06/2021	-0.005671
14/05/2021 14:00	19/05/2021	3M	18/08/2021	-0.005691
14/05/2021 14:00	19/05/2021	6M	18/11/2021	-0.005717
14/05/2021 14:00	19/05/2021	1Y	18/05/2022	-0.005720
14/05/2021 14:00	19/05/2021	2Y	18/05/2023	-0.005570
14/05/2021 14:00	19/05/2021	3Y	20/05/2024	-0.005092
14/05/2021 14:00	19/05/2021	4Y	19/05/2025	-0.004421
14/05/2021 14:00	19/05/2021	5Y	18/05/2026	-0.003691
14/05/2021 14:00	19/05/2021	6Y	18/05/2027	-0.002961
14/05/2021 14:00	19/05/2021	7Y	18/05/2028	-0.002202
14/05/2021 14:00	19/05/2021	8Y	18/05/2029	-0.001452
14/05/2021 14:00	19/05/2021	9Y	20/05/2030	-0.000751
14/05/2021 14:00	19/05/2021	10Y	19/05/2031	-0.000071
14/05/2021 14:00	19/05/2021	12Y	18/05/2033	0.001149
14/05/2021 14:00	19/05/2021	15Y	19/05/2036	0.002569
14/05/2021 14:00	19/05/2021	20Y	20/05/2041	0.003818
14/05/2021 14:00	19/05/2021	30Y	18/05/2051	0.004077

Zero rates for very long maturities are used by the insurance industry. For instance, risk-free rates (RFRs) are published monthly by the European Insurance and Occupational Pensions Authority (EIOPA): eiopa.europa.eu/tools-and-data/risk-free-interest-rate-term-structures_en. Table 5 shows a sample of the EUR risk free rates reported for 30 April 2021 in the file EIOPA_RFR_20210430_Term_Structures.xlsx.

The ISDA CDS Standard Model, whose goal is to enhance transparency and to optimize use of standard technology for CDS pricing, requires the determination of risk-free zero curves for discounting the payments of credit default swaps, with tenors (3M, 6M, 1Y, ..., 30Y) that are currency specific. IHS Markit (formerly Markit [17]), the administrator of this open source project, publishes deposit and swap rates, on every weekday, for ten currencies (USD, CAD, EUR, GBP, JPY, CHF, AUD, NZD, HKD, and SGD). The website is https://www.markit.com/news/. The syntax of the file name is as follows: InterestRates_CCY_yyyymmdd.zip. For example, the EUR file published on 14 May 2021 is named InterestRates_EUR_20210514.zip. The zip file contains InterestRates_EUR_20210514.xml (Table 6) and Disclaimer.txt (with the terms of use).

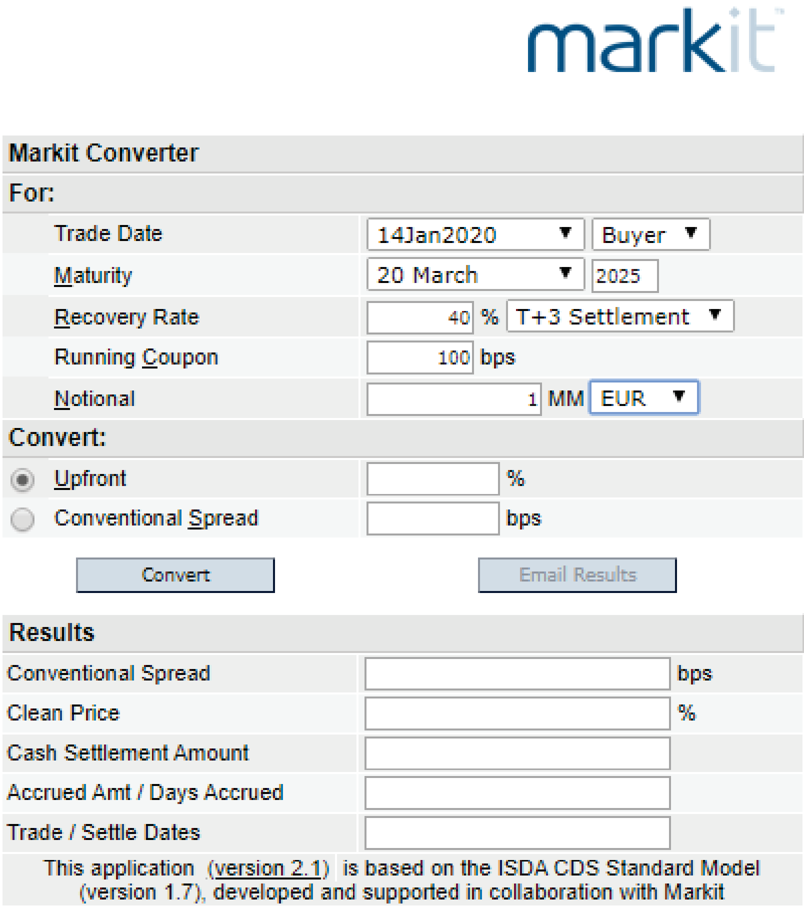


Figure 2 Markit Converter.

The xml file can be imported by using the Data menu of Excel. The usual publication time is 16.00 local time (New York, Toronto, Frankfurt, London, Tokyo, Zurich, Sidney, Wellington, Hong Kong, Singapore, respectively).

ISDA "Big Bang" of 2009

In 2009, the International Swaps and Derivatives Association (ISDA) introduced a number of documentation changes to help standardize the coupon payments made by single-name credit default swaps (CDSs). Currently, CDSs are characterized by a small number of fixed coupons (25, 100, 300, 500, 750, 1000 bps) and require an upfront payment.

In order to transform the upfront into a conventional spread (or vice versa), Markit has developed an application available on the web: www.markit.com/converter.jsp (Figure 2). To ensure reproducibility, this application (version 2.1), which is based on the ISDA CDS Standard Model (version 1.7), requires the above-mentioned "locked" interest rates [Markit (2014)].

The file in xml format (partly reported in Table 6) contains 17 interest rates observed on 14 May 2021 (Friday) and used by the converter on 17 May 2021 (Monday): 1M, 3M, 6M, 12M Euribor rates (with ACT/360 day count convention) and 2Y, 3Y, 4Y, 5Y, 6Y, 7Y, 8Y, 9Y, 10Y, 12Y, 15Y, 20Y, 30Y EUR swap rates (6M variable with ACT/360 day count convention vs. 1Y fixed with 30/360 day count convention). These rates can be used to build the EUR zero curve.

Bootstrap Method

In its 2014 document, Markit describes the USD interest rates used by its CDS Converter and the calculations made to derive yield curves. The database is represented by 6 Libor rates [1M, 2M, 3M, 6M, 9M, 1Y | Source ICE Benchmark administration (IBA)] and 14 swap rates [2Y, 3Y, 4Y, 5Y, 6Y, 7Y, 8Y, 9Y, 10Y, 12Y, 15Y, 20Y, 25Y, 30Y | Source: ICAP]. Since the 14 swaps considered require 6-month payments, and the non-Libor payment dates of the swaps are 58 (= 30 × 2 − 2), the calculations of the USD yield curve involve 64 dates (= 6 + 58).[4]

In a series of articles published on the web from December 6th 2017 to January 24th 2018, Mark Rotchell (rotchvba.wordpress.com [22]) has explored the ISDA CDS Standard Model and has made it accessible to Excel and VBA users. To go through the calculations, we will use his YieldCurveInExcel.xlsm file downloadable from https://drive.google.com/file/d/19lj3HgWQaWSP6maCyRJtEqP7R2Ue1wmf/view.

The *trade date*, T_0, is Monday, September 7th, 2009 and the *spot date*, T_{SPOT}, is Wednesday, September 9th, 2009, i.e. the trade date + 2 weekdays. Consider the 6M and 1Y Libor rates, which are equal to 0.7125% and 1.2881%. The *maturity dates* of the two contracts, T_{6M} (Tuesday 9 March 2010) and T_{1Y} (Thursday 9 September 2010), are 181 and 365 days from the spot date. In our notation, $t_{6M} = T_{6M} - T_{SPOT} = 181$ and $t_{1Y} = T_{1Y} - T_{SPOT} = 365$. The spot prices, $P(t_{6M})$ and $P(t_{1Y})$, are equal to 0.996430 and 0.987108:

$$P(t_{6M}) = \frac{1}{1 + \frac{181}{360} \times 0.7125\%} = 0.996430 \qquad P(t_{1Y}) = \frac{1}{1 + \frac{365}{360} \times 1.2881\%} = 0.987108. \tag{1}$$

Markit uses the bootstrap method to convert the swap rates into spot prices. Consider the 2Y swap rate, s_{2Y}, which is equal to 1.2957%. The value, V_{FIX}, of its fixed leg at the spot date is set to $1 - P(t_{2Y})$, the value of the floating leg at the same date:[5]

$$V_{FIX} = P(t_{6M})\, c_1 + P(t_{1Y})\, c_2 + P(t_{1.5Y})\, c_3 + P(t_{2Y})\, c_4 = 1 - P(t_{2Y}) \tag{2}$$

where c_{6M}, c_{1Y}, $c_{1.5Y}$ and c_{2Y} are the swap's fixed coupons (calculated by applying the 30/360 day count convention)

$$c_{6M} = \frac{t^*_{6M}}{t^*_{1Y}} s_{2Y} \quad c_{1Y} = \frac{t^*_{1Y} - t^*_{6M}}{t^*_{1Y}} s_{2Y} \quad c_{1.5Y} = \frac{t^*_{1.5Y} - t^*_{1Y}}{t^*_{1.5Y}} s_{2Y} \quad c_{2Y} = \frac{t^*_{2Y} - t^*_{1.5Y}}{t^*_{2Y}} s_{2Y}. \tag{3}$$

In this case, $t^*_{6M} = 180$, $t^*_{1Y} = 360$, $t^*_{1.5Y} = 540$, $t^*_{2Y} = 720$, so that $c_{6M} = c_{1Y} = c_{1.5Y} = c_{2Y} = 0.5 \times s_{2Y} = 0.64785\%$.

While the 6M and 1Y spot prices, $P(t_{6M})$ and $P(t_{1Y})$, in (2) have already been calculated as a function of 6M and 1Y Libor rates, the remaining spot prices, $P(t_{1.5Y})$ and $P(t_{2Y})$, have to be *bootstrapped*. Since we have one equation [(2)] and two unknowns [$P(t_{1.5Y})$ and $P(t_{2Y})$], we need an additional constraint.

4 Markit assumes that weekends (Saturday, Sunday) are the only non-business days. If the maturity date for a deposit rate falls on a weekend, the maturity date is the next weekday.

5 The value of the floating leg, V_{FL}, is calculated by discounting the Libor forward rates, as follows:

$$V_{FL} = P(t_{6M})\left[\frac{1}{P(t_{6M})} - 1\right] + P(t_{1Y})\left[\frac{P(t_{6M})}{P(t_{1Y})} - 1\right] + P(t_{1.5Y})\left[\frac{P(t_{1Y})}{P(t_{1.5Y})} - 1\right] + P(t_{2Y})\left[\frac{P(t_{1.5Y})}{P(t_{2Y})} - 1\right] = 1 - P(t_{2Y}).$$

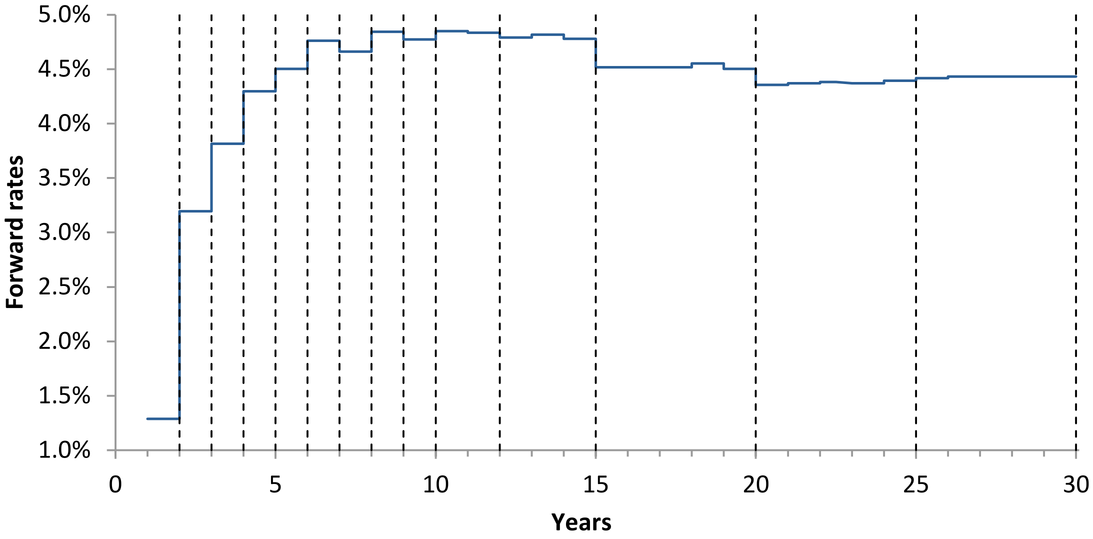


Figure 3 Forward rates.

The assumption made by Markit (2014, pp. 8-9) is the following:

The intermediate discount factor, in this case for 18 months, is interpolated between the 1Y and 2Y discount factors on the basis of a constant forward rate over the period from 1Y to 2Y i.e. the discount factor is log-linearly interpolated.

The exact meaning of *constant forward rate* and *log-linearly interpolated* is as follows:

1. The *quasi-constant* forward rate (Figure 3) is a 6-month continuously compounded forward rate where the 6-month period is made up by a *variable* number of days;
2. The interpolation is linear in the log of the discount factors.

After setting the constancy-constraint for the forward rate, R_F, over the period from 1Y to 2Y

$$R_F(t_{1Y}, t_{1.5Y}) = R_F(t_{1.5Y}, t_2), \tag{4}$$

the Solver function in Excel gives the spot prices $P(t_{1.5Y})$ and $P(t_{2Y})$ as

$$P(t_{1.5Y}) = P(t_{1Y}) \exp[-R_F(t_{1Y}, t_{1.5Y})(t_{1.5Y} - t_{1Y})/(t_{2Y} - t_{1y})] = 0.980827 \tag{5}$$

$$P(t_{2Y}) = P(t_{1.5Y}) \exp[-R_F(t_{1.5Y}, t_2)(t_{2Y} - t_{1.5Y})/(t_{2Y} - t_{1y})] = 0.974482 \tag{6}$$

where

$$R_F(t_{1Y}, t_{1.5Y}) = \frac{\ln[P(t_{1Y})/P(t_{1.5Y})]}{(t_{1.5Y} - t_{1Y})/(t_{2Y} - t_{1Y})} = \frac{\ln[0.987108/0.980827]}{(546-365)/(730-365)} = 1.2874\% \tag{7}$$

$$R_F(t_{1.5Y}, t_{2Y}) = \frac{\ln[P(t_{1.5Y})/P(t_{2Y})]}{(t_{2Y} - t_{1.5Y})/(t_{2Y} - t_{1Y})} = \frac{\ln[0.980827/0.974482]}{(730-546)/(730-365)} = 1.2874\% \tag{8}$$

Tedious algebra shows that the price $P(t_{1.5Y})$ is given by the log-linear interpolation between $P(t_{1Y})$ and $P(t_{2Y})$:

$$\ln[P(t_{1.5Y})] = \ln[P(t_{1Y})] + (t_{1.5Y} - t_{1Y})/(t_{2Y} - t_{1y})[\ln(P(t_{2Y})) - \ln(P(t_{1Y}))]. \quad (9)$$

In other terms

$$P(t_{1.5Y}) = [P(t_{1Y})]^{1-\hat{t}}\,[P(t_{2Y})]^{\hat{t}} = 0.987108^{\,1-0.495890} \times 0.974482^{\,0.495890} = 0.980827 \quad (10)$$

where

$$\hat{t} = (t_{1.5Y} - t_{1Y})/(t_{2Y} - t_{1y}) = (546 - 365)/(730 - 365) = 0.495890. \quad (11)$$

The 6-month continuously compounded forward rates are shown in Figure 3.

CDS Quoted Price

In 2009, the quotation method of CDSs has been changed. Now, instead of the CDS spread, the clean price, P_{CDS}, is quoted (Appendix B):

$$P_{CDS} = 1 - u \quad (12)$$

where

- u is the upfront payment

$$u = [(1 - R) - a_\tau]\, p_b(T) - (V_{pmt} - h_0), \quad (13)$$

- R is the recovery rate;[6]
- a_τ is the expected coupon accrued at the default time τ;
- $p_b(T)$ is the value of a T-maturity first-touch digital option, which pays a unit capital at time τ $(0 < \tau \leq T)$ if the firm defaults at τ;
- V_{pmt} is the present value, at the cash settlement date T_c (three business days after the trade date T_0), of expected coupon payments;
- h_0 is the coupon accrued at the trade date T_0.

To go through the calculations, we can use the Mark Rotchell file cds-valuation.xlsx downloadable from the web (rotchvba.wordpress.com/2018/01/cds-valuation.xlsx).

The zero curve estimates seen in the previous section make it possible to determine the clean price P_{CDS} at the trade date, T_0 (Monday, 7 September 2009). Take the case of a 1Y CDS with cash settlement date, T_c, on Thursday 10 September 2009 (cell C15), maturity, T, on Monday 20 September 2010 (cell C17) and notional amount, N, equal to \$1 billion (cell C21). The coupon rate, c, is 100 bps (cell C22), the recovery rate, R, is 20% (cell C23), the expected coupon accrued at the default time, a_τ, is 0.131786% {cell M14, with changed sign, divided by $[N\, p_b(T)]$}, and the value, $p_b(T)$, of the T-maturity first-touch digital option is 0.130306% {cell M15 divided by $[(1 - R)\, N]$}.

[6] Recovery rates: 40% is used for senior unsecured, 20% is used for subordinate, 25% is used for emerging markets (both senior and subordinate).

Coupons are paid at 5 standard dates (20 September 2009, 20 December 2009, 20 March 2010, 20 June 2010, 20 September 2010), with actual payment dates adjusted according to the following business day convention The first coupon starts to accrue from 20 June 2009 (actual payment date: 21 June 2009). The actual coupons (calculated by applying the Act/360 day count convention) are (cells E31:E35 divided by N):

$$c_{21SEP09} = c_{21DEC09} = c_{22MAR10} = c_{21JUN10} = 0.252778\% \text{ and } c_{20SEP10} = 0.255556\%. \tag{14}$$

At the cash settlement date T_c, the present value, V_{pmt}, of expected coupon payments is 1.258943% (cell M13, with changed sign, divided by N) and the coupon accrued at the trade date, h_0, is 0.216667% (cell M17 divided by N).

Therefore, the upfront payment, u, is

$$u = [(1 - 20\%) - 0.131786\%] \times 0.130306\% - (1.258943\% - 0.216667\%) = -0.938203\% \tag{15}$$

and the quoted price, P_{CDS}, is equal to

$$P_{\text{CDS}} = 1 - u = 1 - (-0.938203\%) = 100.938203\%. \tag{16}$$

4. Principal Components, Autoencoders, Multiple Zero Curves

The bootstrap method followed by ISDA/Markit is the "market standard" to calculate the fair value of interest rate swaps. The method describes each zero curve by a high number of *nodes* or *vertices*, 64 in our USD example: 6 maturities for deposits up to 1 year and 58 (= 29 × 2) for swaps up to 30 years. In other terms, 64 discount factors have been used to perfectly fit the value of 20 contracts (6 deposits plus 14 swaps). However, in order to valuate interest-rate derivatives, the zero curve determined by the bootstrap method is not sufficient. We need to understand how its shape changes as time goes by.

A "movie" showing the daily changes of the EUR zero curve from 29 December 2006 to 14 May 2021 is available here:

http://docenti.luiss.it/barone/files/2021/05/EUR_Zero_Rates_Movie.xlsm.

Historically, the short end of the zero curve has not been smooth, as in Figure 4 dated 14 May 2020. In our time series (2,398 observations) such weird shapes, with a short-end hump (1-year zero rate higher than both the 3-month and 2-year rates) happen 65% of the time. The zero curve calculated for 14 May 2021 is shown in Figure 5.

In our time series, weird shapes of the zero curves happen xx% of the time

To understand the structure of swap data we can use the Principal Components Analysis (PCA) proposed by Karl Pearson as early as 1901. This approach makes it possible to reduce the number of variables that explain the shape of a zero curve.

PCA replaces a set of n variables by n factors so that:

1. Any observation on the original variables is a linear combination of the n factors.
2. The n factors are uncorrelated.
3. The quantity of a particular factor in a particular observation is the factor score.
4. The importance of a particular factor is measured by the standard deviation of its factor score across observations.

The idea is to find a few variables that account for a high percentage of the variance in the observations.

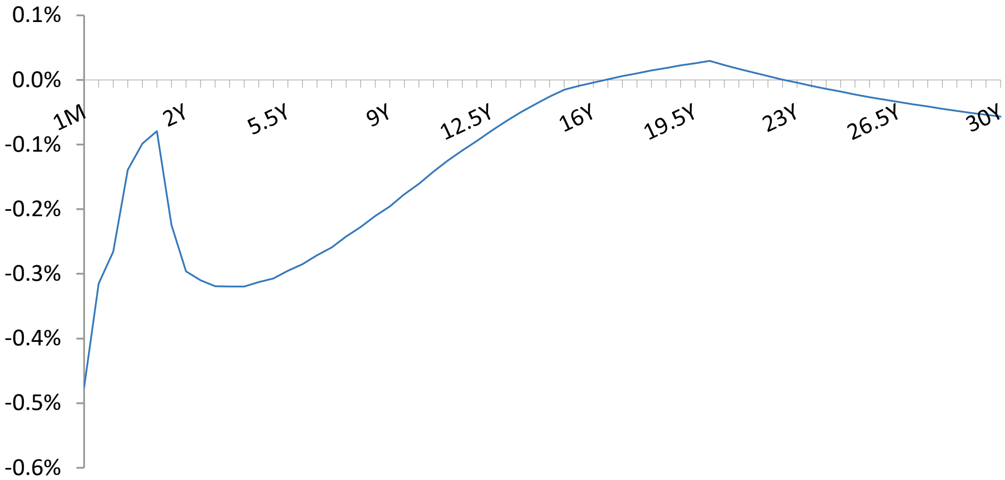


Figure 4 Zero curve (14 May 2020).

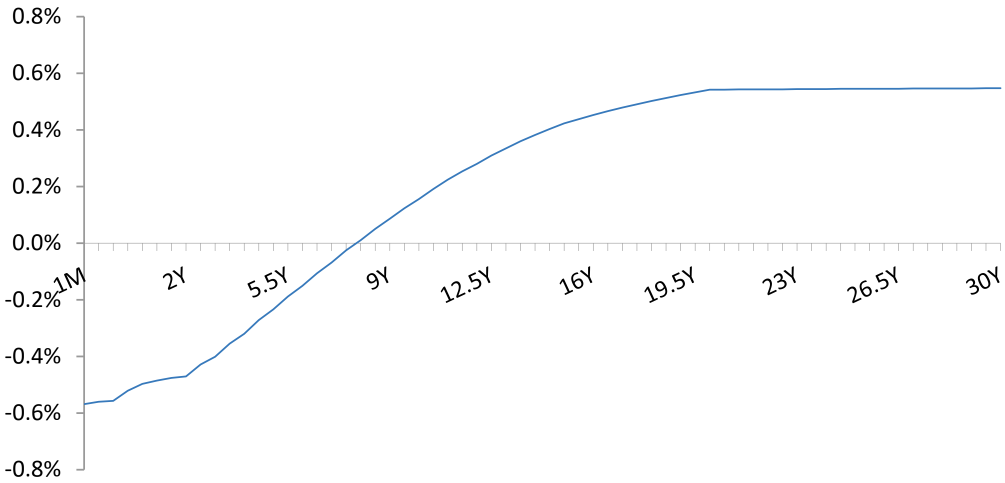


Figure 5 Zero curve (14 May 2021).

We have used data on EUR swap rates with 13 maturities (2Y, 3Y, 4Y, 5Y, 6Y, 7Y, 8Y, 9Y, 10Y, 12Y, 15Y, 20Y, 30Y) to replicate the PCA analysis of U.S. Treasury rates in John C. Hull {[13], pp. 512-4}.[7] Table 7 and Table 8 report the results obtained using 3,686 daily observations from 29 December 2006 to 14 May 2021. Table 7 reports the *factor loadings*, i.e. the interest rate moves determined by 13 factors (PC1 to PC13). Because there are 13 rates and 13 factors, we can solve a system of 13 simultaneous equations to express the swap rate changes observed on any given day as a linear sum of the factors. The quantities of the 13 factors obtained by solving the system on a particular day are the *factor scores* for that day. The importance of factors is measured by their standard deviation: the higher is the standard deviation, the more important is the factor. Table 8 reports the standard deviations of the factor scores, from the highest to the lowest Therefore, PC1, is the most important factor.

[7] An Excel file for carrying out the calculations is available for download on the website of John C. Hull: (www-2.rotman.utoronto.ca/·hull/ofod).

Table 7 Factor Loadings for EUR Swap Rates.

	PC1	PC2	PC3	PC4	PC5	PC6	PC7	PC8	PC9	PC10	PC11	PC12	PC13
2Y	-0.13	-0.29	0.75	-0.57	-0.07	-0.01	0.00	0.00	0.01	0.00	0.00	0.00	0.00
3Y	-0.21	-0.46	0.19	0.46	0.56	-0.14	0.04	0.12	0.16	0.08	0.02	-0.03	0.34
4Y	-0.24	-0.38	0.02	0.27	0.00	0.05	-0.14	-0.28	-0.35	-0.17	-0.04	0.07	-0.68
5Y	-0.26	-0.30	-0.12	0.11	-0.46	0.35	0.52	0.41	0.18	0.02	0.00	0.02	-0.04
6Y	-0.28	-0.21	-0.17	-0.01	-0.35	0.08	-0.20	-0.40	-0.30	0.22	0.36	-0.15	0.48
7Y	-0.28	-0.12	-0.18	-0.10	-0.18	-0.09	-0.29	-0.12	0.21	-0.13	-0.79	0.07	0.20
8Y	-0.29	-0.04	-0.19	-0.16	-0.03	-0.22	-0.35	0.17	0.44	-0.24	0.48	0.40	-0.11
9Y	-0.30	0.03	-0.20	-0.21	0.10	-0.29	-0.04	0.21	0.06	0.13	0.03	-0.78	-0.24
10Y	-0.30	0.08	-0.18	-0.24	0.19	-0.29	0.24	0.14	-0.38	0.50	-0.11	0.45	-0.05
12Y	-0.31	0.16	-0.11	-0.19	0.27	0.05	0.38	-0.11	-0.26	-0.69	0.05	-0.04	0.21
15Y	-0.31	0.25	0.01	-0.07	0.27	0.45	0.14	-0.48	0.44	0.29	0.01	0.00	-0.16
20Y	-0.31	0.35	0.20	0.13	0.08	0.47	-0.45	0.46	-0.27	0.01	-0.04	-0.01	0.06
30Y	-0.32	0.44	0.40	0.42	-0.35	-0.44	0.16	-0.13	0.07	-0.02	0.01	0.00	-0.01

Table 8 Standard Deviation of Factor Scores (basis points).

PC1	PC2	PC3	PC4	PC5	PC6	PC7	PC8	PC9	PC10	PC11	PC12	PC13
11.81	3.55	1.42	1.09	0.51	0.33	0.20	0.16	0.13	0.10	0.09	0.08	0.12

In Table 7, the values of PC1 monotonously decrease from -0.13 to -0.32. Therefore the first factor corresponds to a roughly parallel shift in the yield curve: When we have one unit of PC1, the two-year swap rate decreases by 0.13 basis points, the three-year rate decreases by 0.21 basis points, and so on.

The second factor, PC2, corresponds to a rotation or change of slope of the yield curve: Rates between 2 year and 8 years decrease while rates between 9 years and 30 years increase. The third factor, PC3, corresponds to a "bowing" of the yield curve: Relatively short (2-year to 4-year) and relatively long (9-year to 30-year) rates increase, while the intermediate rates (5-year to 12-year) decrease.

The total variance of the original data, calculated by using the data reported in Table 8, is equal to 155.78:

$$\begin{gathered} 11.81^2 + 3.55^2 + 1.42^2 + 1.09^2 + 0.51^2 + 0.33^2 + 0.20^2 + \\ 0.16^2 + 0.13^2 + 0.10^2 + 0.09^2 + 0.08^2 + 0.12^2 = 155.78 \end{gathered} \tag{17}$$

Therefore, the first factor accounts for 89.6% (= 11.81^2 / 155.78) of the total variance.

As shown by Hull ([12], pp. 155-8), an autoencoder with one hidden layer and linear activation functions produces results that are equivalent to a PCA.[8]

However, the two methods are basically different (Hull [12], p. 158):

The advantage of PCA is that the features produced are uncorrelated and their relative importance is clear from the standard deviation of the feature scores. The advantage of autoencoders is that they can be used with non-linear activation functions ... . To use PCA terminology, they allow data to be explained with non-linear factors.

[8] An autoencoder is a neural network designed to reduce the dimensionality of data, i.e., a form of (unsupervised) machine learning where the objective is to replace a set of features (the variables from which the predictions about the target are made) with a smaller number of manufactured features that provide similar information.

Hull ([12], pp. 41-5) applies the principal components analysis to daily movements in U.S. dollar Treasury interest rates with maturities of 1, 2, 3, 5, 7, 10, 20, and 30 years over the period January 2010 - July 2020. Then, he shows how to get PC1 and PC2 by an autoencoder. See http://www-2.rotman.utoronto.ca/~hull/MLThirdEditionFiles/mlindex1_3rdEd.html.

Traders have to keep track of interest rates in all currencies in which they transact government bonds, corporate bonds and interest-rate derivatives. In order to take into account the liquidity of traded instruments, they also model bid and ask rates.

Risk managers have to take into account the correlations among many zero curves in order to measure the riskiness of their portfolios. The most common approach is to determine the spot zero rates for canonical dates ("vertices") and to estimate the correlations among the vertices. The multiple vertices approach is described in John C. Hull (2018) {[11], pp. 328-31}.

5. Term Structure Models

The results obtained by applying the Principal Components Analysis support the adoption of a *one-factor* term structure model to describe the main movements of zero curves.

Merton

The first one-factor term-structure model was sketched by Robert Merton (1973) [19] in a footnote of his seminal article in option pricing theory, where he assumes that the short rate follows an arithmetic Brownian motion, a process that he considers "not realistic because it implies a positive probability of negative interest rates".[9]

Vasicek

The model proposed by Oldrich Vasicek (1977) [25] has the same stochastic term as Merton but the drift term is mean-reverting (the short interest rate is pulled back to some long-run average level over time):

$$dr = a(b - r)\, dt + \sigma\, dz. \tag{18}$$

Also this process entails a non-null probability of negative interest rates.

Cox, Ingersoll, and Ross

The model proposed by Cox, Ingersoll, and Ross (CIR [7]) has the same mean-reverting drift as Vasicek, but the stochastic term is proportional to $\sqrt{r}$ instead of r:

$$dr = a(b - r)\, dt + \sigma \sqrt{r}\, dz. \tag{19}$$

This change makes it impossible for interest rates to become negative.

[9] Merton (1973) {[19], footnote 43, p. 163}: "... Suppose that bond prices for all maturities are only a function of the current (and future) short-term interest rates. Further, assume that the short-rate, r, follows a Gauss-Wiener process with (possibly) some drift, i.e., $dr = adt + gdz$, where a and g are constants. Although this process is not realistic because it implies a positive probability of negative interest rates, it will still illustrate the point. Suppose that all bonds are priced so as to yield an expected rate of return over the next period equal to r (i.e., a form of the expectations hypothesis):

$$P(\tau;r) = \exp\left[-r\tau - \frac{a}{2}\tau^2 + \frac{g^2\tau^3}{6}\right]$$

and

$$\frac{dP}{P} = rdt - g\tau\, dz.$$

... ."

Perfect fit: Shreve's Suggestion

In the above models, there is a pitfall highlighted by John C. Hull (2022). They do not provide a *perfect fit* to the current term structure of interest rates:

> *Most traders, when they are valuing derivatives, find this unsatisfactory. Not unreasonably, they argue that they can have very little confidence in the price of a bond option when the model used does not price the underlying bond correctly.*
> Hull (2022) {[13], p. 710}

Shreve (1997) {[24], pp. 315-8} suggests to define two timescales: the market time (or process time) and the calendar time (or real time).[10] The deterministic, monotonically increasing function, φ, that links the two timescales is estimated in such a way as to fit the theoretical contract values perfectly to the actual values.

Shreve's approach has been implemented in Barone (1998) {[3], p. 54}. The values of the function φ, estimated for the contracts considered, show that the process time is slower than the real time for maturities up to one year; it becomes faster in the interval between two and seven years and then slows again in the interval between eight and 10 years.

An Extended CIR Model

In order to deal with negative interest rates, we can assume that the CIR process [7] describes the dynamics of an instantaneous interest rate made up by two components: a default-free rate on money and a premium reflecting "market imperfections" such as the storage cost of money. A similar approach has been implemented in Barone, Barone-Adesi, Castagna (1998) [4].

6. Libor Phase-Out and New Reference Rates

In July 2017, the Financial Conduct Authority (FCA) announced the discontinuation of Libor {FCA [9]}. The decision was taken because of the absence of active underlying markets, which requires the use of "expert judgement" by the panel banks to form many of their submissions. The Libor panel banks agreed to continue to submit to Libor until end-2021 (subsequently extended to end-June 2023 for US dollar Libor only).

What is the best proxy for a risk-free rate (RFR)? This is a long-standing issue. The most natural candidate is a rate based on the quotes of Treasuries. However, there are pros and cons.

> *It might be thought that derivatives traders would use the rates on Treasury bills and Treasury bonds as risk-free rates. In fact they do not do this. This is because there are tax and regulatory factors that lead to Treasury rates being artificially low. For example:*
>
> 1. *Banks are not required to keep capital for investments in a Treasury instruments, but they are required to keep capital for other very low risk instruments.*

[10] This hypothesis is consistent with the literature on the weekend effect and with traders' practice of determining options volatility with reference to market time and interest payments with reference to calendar time. See "Trading Days vs. Calendar Days" in Hull (2022) {[13], pp. 323-4}.

2. *In the United States, Treasury instruments are given favorable tax treatment compared with other very low risk instruments because the interest earned by investors is not taxed at the state level.* {Hull [13], p. 79}

Traditionally derivatives dealers have assumed that Libor rates are risk-free. Unfortunately, Libor is based on estimates of the rates at which banks can borrow on unsecured terms in wholesale markets, not actual transactions. As such it is open to manipulation.

Following the financial crisis that started in 2007, many dealers switched to using overnight indexed swap (OIS) rates as risk-free rates, at least for collateralized transactions. Regulators and leading financial institutions are now taking steps to make the transition from Libor as smooth as possible. The markets most affected by Libor phase-out are the markets for USD, GBP, EUR, JPY, CHF, AUD, CAD and HKD.

Libor will be replaced by overnight reference rates such as Ester (*euro short-term rate*) in the Eurozone, Sofr (*secured overnight financing rate*) in the United States, Sonia (*sterling overnight index average*) in the U.K., Saron (*Swiss average rate overnight*) in Switzerland, Tonar (*Tokyo overnight average rate*) in Japan.

A fundamental difference between Libor and reference rates based on averaging overnight rates on periods longer than 1 day is that Libor rates for a period are known at the beginning of the period to which they apply, whereas the result of the averaging process for overnight rates is known only at the end of the period.[11] Besides, Libor rates incorporate some credit risk, whereas rates based on overnight rates are considered risk-free.[12] However, while Sofr is *secured*, being equal to the volume-weighted median average of the rates on overnight repo transactions in the United States, the other above-mentioned overnight rates are not *secured*, because they are not based on repo transactions.

Interest Rate Curve Standardization

The use of risk-free rates (RFRs) will become the market convention. As a consequence, IHS Markit, the administrator and responsible authority for publishing the interest rates underlying the ISDA CDS standard model, will replace "old" interest rates with "new" ones.[13]

On April 2021, IHS Markit [14] published an introduction to the risk free interest rates to be used as input into the ISDA CDS Standard Model.[14] Differently from the past, when they were based on Deposit and Swap Rates, the new interest rate curves are now based on "RFR OIS (Swap) rates", where RFR and OIS are the acronyms of Risk Free Rates and Overnight Indexed Swaps, respectively.

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[11] Lyashenko and Mercurio [16] show that backward-looking in-arrears forward rates and forward-looking forward rates can be expressed by a single rate and that their framework can be used to simultaneously model the evolution of forward-looking (Libor-like) and backward-looking term rates using the same stochastic process.

[12] See Hull {[13], p. 79}: "This has led banks to ask for a way of creating risky reference rates by adding a credit spread to the new reference rates. There have been a number of proposals and the new risk-free reference rates may be augmented by credit spread measures in the future."

[13] On November 30th, 2020, S&P Global and IHS Markit announced they have entered into a definitive merger agreement to combine in an all-stock transaction which values IHS Markit at an enterprise value of $44 billion. S&P Global and IHS Markit continue to expect to close the proposed merger in the fourth quarter of 2021.

[14] The transition of the interest rate inputs to the ISDA CDS Standard Model to SONIA took place on the weekend of May 24, 2021.

To quote Hull ([13], p. 151):

> *An OIS is an agreement to exchange a fixed rate of interest for a reference rate of interest that is calculated from realized overnight rates. A simple example of an OIS is a swap lasting for three months. This leads to a single exchange at the end of the three months. The fixed rate of interest applied to a certain principal is exchanged for the reference rate applied to the same principal. Other similar OISs that last for one month, six months, and one year lead to a single exchange calculated in a similar way. When the life of the OIS is greater than one year, it is typically divided into three-month subperiods, with the fixed rate being exchanged at the end of each three-month period for the three-month reference rate that is calculated for that period from one-day rates.*

In the case of quotes published by IHS Markit, the frequency of both floating and fixed payments of OISs with life greater than one year is annual, instead of quarterly:[15]

> *Swaps up to and including the 1Y node are zero coupon instruments, whereas longer maturity nodes have annual coupons.*

As regards the interpolation method, no change has been made with respect to the previous approach.

To download the RFR interest rate curves from the “new” website (rfr.ihsmarkit.com), this is the URL to be entered:

https://rfr.ihsmarkit.com/InterestRates_CCY_yyyymmdd.zip?email=xxxx@company.com

where xxxx@company.com is the email address provided to IHS Markit in the registration process.

Similarly to downloads of EUR interest rates observed on 14 May 2021 from the ”old” website (www.markit.com/news/), the zip file contains InterestRates_EUR_202100514.xml and ISDA Standard Rate Curves Disclaimer.txt (a file containing the terms of use).[16]

New and Old Risk-Free Zero Curves: A Comparison

Starting from July 9th, 2021, IHS Markit [14] is publishing the EUR-denominated interest rate curves based on both the old and new methodology:

> *Although the use of RFRs will become the market convention, the RFR and IBOR rates will be available in parallel for the shorter period of one year after the relevant RFR goes live or until the IBOR ceases to exist.*

---

15 In the case of the old interest rate curves, the frequencies were as follows ([17], p. 7):

	EUR	USD	GBP	JPY	CHF	AUD
fixedpaymentfrequency	1Y	6M	6M	6M	1Y	6M
floatingpaymentfrequency	6M	3M	6M	6M	6M	3M

In the case of the new interest rate curves, the frequencies are as follows ([14], p. 7):

	EUR	USD	GBP	JPY	CHF	AUD
fixedpaymentfrequency	1Y	1Y	1Y	1Y	1Y	1Y
floatingpaymentfrequency	1Y	1Y	1Y	1Y	1Y	1Y

16 On 14 May 2021, the file InterestRates_EUR_202100514.xml had the same content in both the “new” and the “old” website.

Table 9 Average difference between "old" and "new" risk-free zero rates (basis points).

Maturity	1M	3M	6M	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	12Y	15Y	20Y	30Y
Delta	0.90	2.36	4.93	7.91	8.45	10.42	12.31	13.76	14.70	15.32	15.80	16.16	16.35	16.37	15.87	14.89	13.28

Source: elaborations of IHS Markit data (9 July 2021 to 20 August 2021).

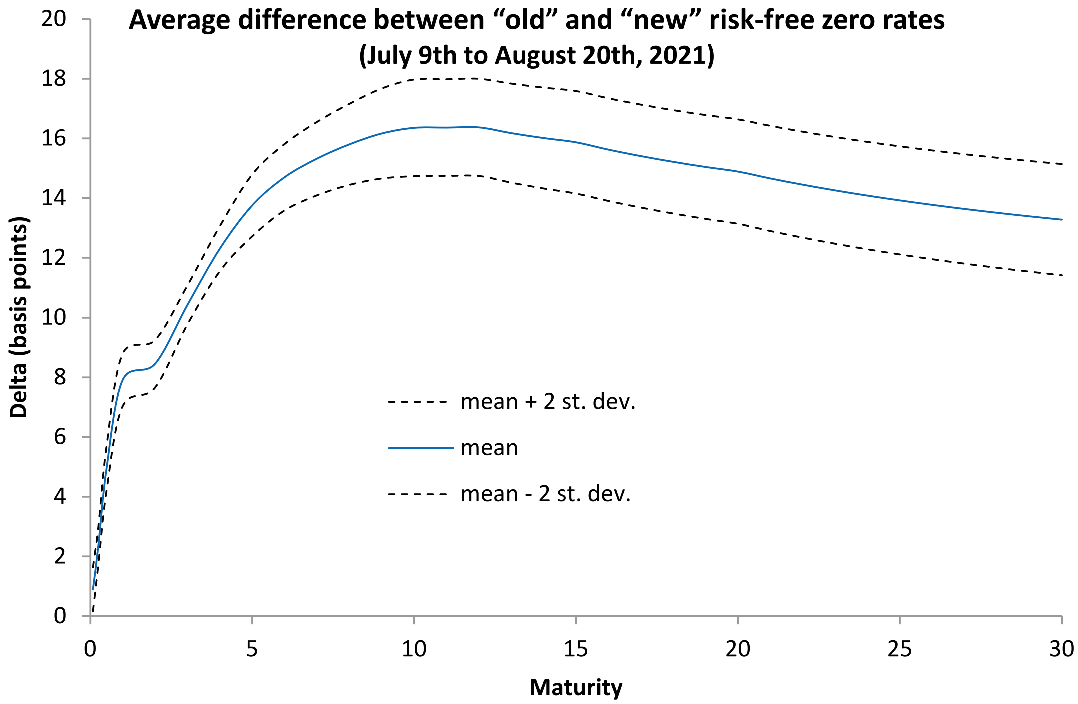

Figure 6 Average difference between "old" and "new" risk-free zero rates.

This makes it possible to compare the new and old risk-free zero curves. In the period from July 9th to August 20th, 2021, the average difference between the zero rates calculated according to the old methodology and the new one ranges from 0.90 basis points at the 1 month maturity to 16.37 basis points at the 12-year maturity (Table 9 and Figure 6).

The new EUR-denominated zero curve of August 20th, 2021, is shown in Figure 7. In the short end there is no evidence of the "asperities" that have been often observed in the past when the zero curve was based also on Libor rates.

7. Conclusions

The key puzzles for this paper were: 1) how to explain negative interest rates on money? 2) how to estimate (describe parsimoniously) a zero curve?

As regards the first puzzle, the answer is that the risk-free zero curves do not measure the *pure* risk-free rates on money. The zero rates that we generally observe are net of the storage cost of money. Therefore, when the *pure* risk-free rates on money are ultra low, as in recent years, the storage costs push the zero curve in the negative region. The storage cost of

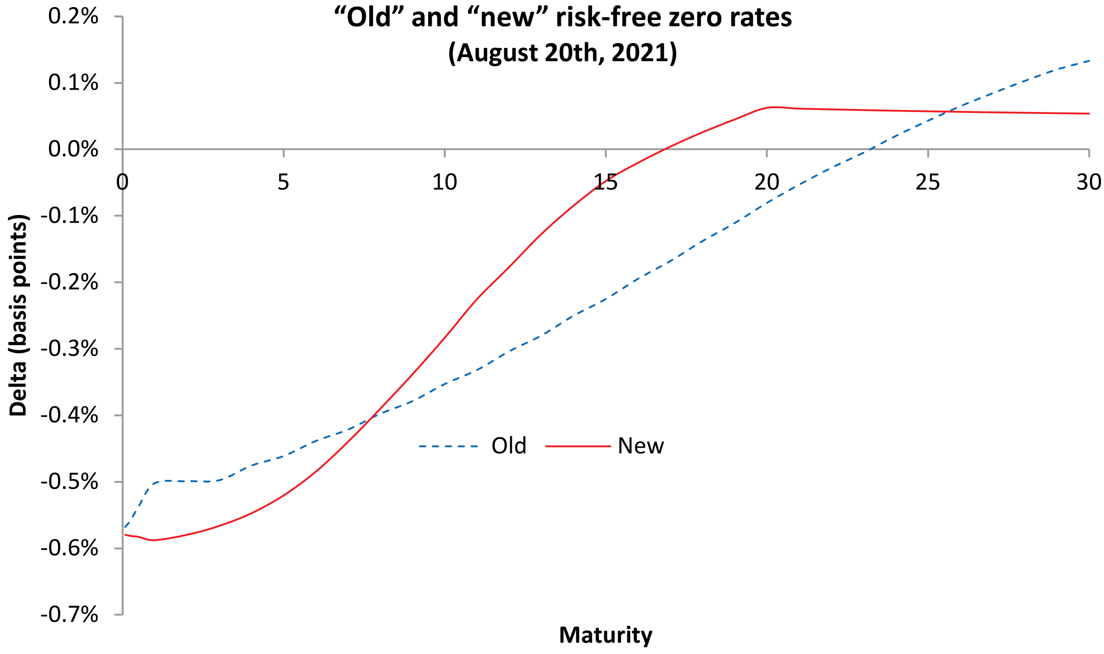

Figure 7 Average difference between "old" and "new" risk-free zero rates.

money can also be seen as the cost of an insurance policy paid to the borrower (e.g. a bank) for taking care of the lender's (e.g. a depositor's) money.

We do not have an obvious answer for the second puzzle. Clearly, the zero curve in May 14, 2020 (Figure 4), which is not at all smooth, cannot be consistent with a simple stochastic process for the short-term rate. Could it have been that some prices at the very short end were "wrong"? Forward curves (forward prices) are also forecasts. Could the forward curve for May 14, 2019 have anticipated even the possibility of the strange shape of the curve the following year, shifted by one year, of course? A lesson from commodity markets, whether in theory or in actual practice, is that zero curves can be highly nonlinear, much like May 14, 2020, that is. Often this has to do with the cycle of the crop year. Even with a small crop for corn, which suggests an extreme backwardation between the last old-crop futures contract and the first new-crop contract, early in that poor crop-year corn will be abundant nevertheless and the interest rate on corn will be close to zero. Often this has to do with capacity constraints, such as for storage of natural gas. Why should money not have any seasonality?

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Appendix A

Mark Rubinstein's earthquake insurance

To show how to evaluate a derivative, Mark Rubinstein (1999, Chapter 1, pp. 2-13) examines the case of a 1-year earthquake insurance policy in which an individual makes a payment in exchange for a potentially much larger payoff from the insurance company should an earthquake destroy his property.

The insurance policy's current value can be calculated in (at least) three ways, by using:

1. the subjective probabilities, π, which take into account only personal opinions;
2. the risk-neutral probabilities, π^*, which take into account personal opinions and risk aversion;
3. the state-prices, q, which take into account personal opinions, risk aversion and time.

Payoffs and Subjective Probabilities

Table a1 contains four columns: the value of the underlying variable (measured on the Richter scale), the corresponding damage's entity, the policy's payoff, the likelihood of an earthquake as measured by the homeowner's subjective probabilities.

Table a1 Earthquake insurance : payoffs and subjective probabilities.

Richter Scale	*Damage's Magnitude*	*Dollar Payoff* (z)	*Subjective Probability* (π)
0 - 4,9	None	0	0.850
5,0 - 5,4	Slight	750	0.100
5,5 - 5,9	Small	10,000	0.030
6,0 - 6,9	Medium	25,000	0.015
7,0 - 8,9	Large	50,000	0.005

Suppose, that the homeowner pays the premium fully in advance at the *beginning* of a year but is paid for any earthquake damage *only at the end* of the year even if the damage occurs during the middle of the year. The expected payoff, $E(z)$, is equal to \$1,000:

$$\begin{aligned} E(z) &= 0.850 \times \$0 + 0.100 \times \$750 + 0.030 \times \$10{,}000 + 0.015 \times \$25{,}000 + 0.005 \times \$50{,}000 \\ &= \$1.000. \end{aligned} \tag{a1}$$

Risk Aversion

To take into account the homeowner's risk aversion, let's assume that his utility function, $U(W)$, is a power function defined by

$$U(W) = W^{\gamma}/\gamma \tag{a2}$$

where W is the homeowner's wealth and γ is the constant elasticity.[1]

[1] Power functions are characterized by *constant* relative risk aversion, $R(W) = 1 - \gamma$. In fact

$$R(W) \equiv -\frac{U_{WW}}{U_W} W = A(W)W = 1 - \gamma$$

where $A(W)$ is the absolute risk aversion defined by de Finetti-Arrow-Pratt

$$A(W) \equiv -\frac{U_{WW}}{U_W} = -\frac{(\gamma - 1)W^{\gamma-2}}{W^{\gamma-1}} = \frac{1-\gamma}{W}.$$

Suppose that the homeowner's entire wealth is US $100,000. A very simple way to adjust for risk-aversion is to weight dollars so that "rich" states are worth less than they actually are and "poor" states are worth more than they actually are. Table a2 does precisely this.

Table a2 Earthquake insurance: marginal utilities and risk-neutral probabilities.

Richter Scale	Payoff (z)	Subjective Probability (π)	Wealth (W)	Marginal Utility ($U_W = W^{\gamma-1}$)	Adjustment Factor ($f = c\, U_W$)	Risk-Neutral Probability ($\pi^* = f\,\pi$)
0 - 4.9	0	0.850	100.000	0.00316	0.9936	0.845
5.0 - 5.4	750	0.100	99.250	0.00317	0.9974	0.100
5.5 - 5.9	10,000	0.030	90.000	0.00333	1.0474	0.031
6.0 - 6.9	25,000	0.015	75.000	0.00365	1.1474	0.017
7.0 - 8.9	50,000	0.005	50.000	0.00447	1.4052	0.007

Note: we suppose that the elasticity, γ, of the utility function is equal to 0.5. The arbitrary constant, c, necessary to set to 1 the sum of risk-neutral probabilities (through the adjustment factors, f) turns out to be 314.2194.

For example, in the future rich state represented by the occurrence of an earthquake of magnitude 0--4.9 on the Richter scale, by multiplying by the risk-aversion adjustment factor of 0.9936 the weight attached to dollars in that state is reduced from 0.850 to 0.845. On the other hand, the weight attached to dollars in the future poor 7.0--8.9 Richter scale state is increased from 0.005 to 0.007 by multiplying by 1.4052.

Using the risk-neutral probabilities, π^*, the expected risk-adjusted payoff, $E^*(z)$, is equal to \$1,170,59:

$$E(z^*) = 0.845 \times \$0 + 0.100 \times \$750 + 0.031 \times \$10{,}000 + 0.017 \times \$25{,}000 + 0.007 \times \$50{,}000 = \$1{,}170.59. \quad \text{(a3)}$$

where E^* is the expectation operator in a risk-neutral world.

If the risk-free (annually compounded) interest rate, r, is 5%, the insurance's present value, V, is equal to \$1,114.85:

$$V = E^*(z) / (1 + r) = \$1{,}170.59 / 1.05 = \$1{,}114.85. \quad \text{(a4)}$$

This amount reflects *subjective probabilities, risk-aversion* and *time*.

State Prices

The *state prices* (or Arrow-Debreu *prices*) express a concept very similar to that of risk-neutral probabilities. The state-prices, q, are equal to the product between risk-neutral probabilities, π^*, and the *risk-free discount factor*, $1/(1 + r)$. In other terms:

$$q = \pi^* / (1 + r). \quad \text{(a5)}$$

Using the state-prices, the insurance's valuation formula becomes:

$$V = \sum_{i=1}^{n} q_i z_i \quad \text{(a6)}$$

where n is the number of states.

On the basis of state prices q given by Table a3, the insurance's present value, V, is equal (as before) to \$1,114.85:

$$V = 0{,}80438 \times \$0 + 0{,}09499 \times \$750 + 0{,}02993 \times \$10{,}000 + 0{,}01639 \times \$25{,}000 + 0{,}00669 \times \$50{,}000 = \$1{,}114.85. \quad \text{(a7)}$$

Table a3 Earthquake insurance: state prices and pricing kernel.

Richter Scale	Payoff (z)	Subjective Probability (π)	Adjustment Factor ($f = c\, U_W$)	Risk-Neutral Probability ($\pi^* = f\, \pi$)	State Price [$q = \pi^*/(1 + r)$]	Pricing Kernel [$\varphi = q/\pi = f/(1 + r)$]
0 - 4.9	0	0.850	0.9936	0.845	0.80438	0.9463
5.0 - 5.4	750	0.100	0.9974	0.100	0.09499	0.9499
5.5 - 5.9	10.000	0.030	1.0474	0.031	0.02993	0.9975
6.0 - 6.9	25.000	0.015	1.1474	0.017	0.01639	1.0927
7.0 - 8.9	50.000	0.005	1.4052	0.007	0.00669	1.3383

By writing the valuation formula in (a6), it is natural to interpret the *qs* as the prices of securities that pay $1 if a certain event (or state of nature) happens and $0 otherwise. This is the reason why the *qs* are called *state-prices* and the corresponding securities are called *state-securities*.

Pricing Kernel

The *pricing kernel*, synonymous of *stochastic discount factors*, is defined as the ratio between the state prices, q, and the subjective probabilities, π:

$$\varphi = q \,/\, \pi. \tag{a8}$$

Substituting $q = \varphi\, \pi$ in (a6) gives the present value of the insurance policy as

$$V = \sum_{i=1}^{n} \varphi_i\, \pi_i\, z_i \;= E(\varphi\, z) \tag{a9}$$

and, in our example,

$$\begin{aligned} V \;&= 0{,}9463 \times 0.850 \times \$0 + 0.9499 \times 0.100 \times \$750 + 0.9975 \times 0.030 \times \$10{,}000 \\ &+ 1.0927 \times 0.015 \times \$25{,}000 + 1.3383 \times 0.005 \times \$50{,}000 = \$1{,}114.85. \end{aligned} \tag{a10}$$

Appendix B

Quotation Method of CDSs

Single-name CDSs trade like bonds.[1] If there is no credit event during the contract's life, the buyer of protection is obliged to pay a given coupon on a quarterly basis until maturity. Coupons are paid quarterly, on standard dates (20 March, 20 June, 20 September and 20 December), and are generally equal to 100 bps per year.

Let P_{CDS} be the quoted price, or clean price, at the trade date T_0 of a single-name CDS with unit notional value, m payment dates T_i (i = 1, 2, ..., m) and maturity $T = T_m$.[2] By definition, we have

$$P_{\text{CDS}} = 1 - u \tag{b1}$$

where

u is the upfront payment made by the protection buyer to the protection seller (or from the seller to the buyer if $u < 0$). If $P_{\text{CDS}} = 1$, then the CDS's coupon rate, c, is exactly equal to the rate that makes the contract fair ($V_{\text{CDS}} = 0$) and no upfront must be paid ($u = 0$).

The contract is fair when the present value of the expected payoff, V^+, is equal to the present value of the expected payments, V^-:

$$V_{\text{CDS}} = V^+ - V^- = 0. \tag{b2}$$

From the standpoint of the protection buyer, the present value of the expected payoff is

$$V^+ = [(1 - R) - a_\tau]\, p_b(T) \tag{b3}$$

where

R is the recovery rate;

a_τ is the expected coupon accrued at the default time τ;

$p_b(T)$ is the value of a T-maturity first-touch digital option, which pays a unit capital at time τ ($0 < \tau \leq T$) if the firm defaults at τ.[3]

The present value of the expected payments made by the buyer is equal to the algebraic sum of three components:

$$V^- = (V_{pmt} - h_0) + u \tag{b4}$$

where

V_{pmt} is the present value of coupon payments at the cash settlement date T_c;[4]

h_0 is the coupon accrued at the trade date T_0;

u is the upfront payment made at the cash settlement date T_c.

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[1] To standardize single-name CDS contracts, the International Swaps and Derivatives Association (ISDA) introduced a number of changes in its "Big Bang" of April 2009. In particular, in order to standardize coupon payments, a small number of coupon rates have been used thereafter: 25, 100, 300, 500, 750, 1000 bps. per year.

[2] Coupons are paid quarterly, on standard dates (20 March, 20 June, 20 September and 20 December). Payment dates are adjusted, as necessary, to good business days using the "following" day count convention and the relevant holiday calendar for the currency.

[3] We make the assumption of an instantaneous recovery at the time of default.

[4] By standard, the cash settlement time T_c is three business days after the trade date T_0.

The present value of expected coupon payments is equal to

$$V_{pmt} = \sum_{i=1}^{m} P(T_c, T_i)\, h_i\, c\,[1 - Q(T_0, T_i)] \tag{b5}$$

where

$P(T_c, T_i)$ is the unit discount factor, at the cash settlement time T_c, for the payment date T_i;

h_i is the i^{th} accrual fraction (i = 1, 2, ..., m) under the Actual/360 convention;[5]

c is the coupon rate (25, 100, 300, 500, 750 or 1000 bps. per year);

$Q(T_0, T_i)$ is the unconditional probability of default by time T_i, estimated at the trade date T_0.

The coupon accrued at the trade date T_0 is equal to

$$h_0 = \frac{T_0 - T_{-1}}{360} c \tag{b6}$$

where T_{-1} is the payment date occurring on or immediately prior to the step-in date (the calendar day after the trade date T_0).

The expected coupon accrued at the default time τ is approximately equal to:[6]

$$a_\tau = ½\,(c/4). \tag{b7}$$

Therefore, by (b2)-(b4), we have

$$V_{CDS} = V^+ - V^- = [(1 - R) - a_\tau]\, p_b(T) - [(V_{pmt} - h_0) + u] = 0 \tag{b8}$$

and the upfront payment u that makes the contract fair is

$$u = [(1 - R) - a_\tau]\, p_b(T) - (V_{pmt} - h_0). \tag{b9}$$

Example no. 1

The CDS price shown by Equation (16), P_{CDS} = 100.938203%, may be obtained by using the Markit interest rates and by setting the following values in the Perpetual Debt Structural model (PDSM):

$$V_0 = 421.351 \qquad Z = 742.480 \qquad q_V = 13.74\% \qquad \sigma_V = 21.02\%. \tag{b10}$$

These values make it possible to match the manual input and static data in Mark Rotchell file cds-valuation.xlsx, in particular the survival probability (cell I14) and the recovery rate (C23).

Example no. 2

If the default probability is null, then $p_b(T)$ = 0 and u = − (V_{pmt} − h_0). Therefore, under the assumption of a null T-maturity zero rate, P_{CDS} = 1 + V_{pmt} − h_0. This is the upper bound of P_{CDS}. It is the price of a risk-free bond with unit face value, paid at maturity T, and coupons periodically paid at the rate c.

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5 Coupons accrue from (and including) one payment date to (but excluding) the next. There is an exception for the final coupon which accrues to (and including) the maturity date T.

6 "For reasonably small default probabilities and inter-coupon periods, the expected difference in time between the credit event and the previous coupon date is just slightly less than one-half, in expectation, of the length of an inter-coupon period, assuming that the default risk is not concentrated at a coupon date." [Duffie-Singleton (2003), pp. 183-4]. Thus, for purposes of pricing in all but extreme cases, we can assume that, at default, the buyer will have to pay one-half of the regular coupon payment, i.e. ½ (c/4).