An alternative view on cross hedging*

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Abstract

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JEL classification: D81; G11

Keywords: risk management, cross hedging, multiplicative dependence, basis risk, futures contracts

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1 Introduction

Due to the incompleteness of financial markets, a risk-averse decision maker will typically not have the opportunity to trade hedging instruments that perfectly replicate his initial exposure to some price risk. Hence, he has to bear basis risk when managing price risk with a cross hedge. For example, the producer or processor of a particular grade of crude oil may have to rely on commodity derivatives written on another grade of crude oil such that there is basis risk. Another example is derivatives trading by investment banks: Selling customized products to corporate clients and cross hedging with standardized exchange-traded financial derivatives also creates basis risk.

This paper focuses on optimal cross hedging with futures contracts under various assumptions on the nature of basis risk. Most hedging models assume that the spot price is equal to the futures price plus basis risk. Since Benninga, Eldor and Zilcha

(1983), this additive type of dependence (also known as the regression approach) is particularly popular. For any risk-averse utility function, the optimal cross hedge in an unbiased futures market is a full hedge, adjusted by the regression coefficient. This position also minimizes the variance of wealth. Based on the reverse relation where the futures price a linear function of the spot price and basis risk, Paroush and Wolf (1986) derive the optimality of an underhedge in an unbiased futures market. The extent of underhedging usually depends on the utility function.

In contrast to the literature, this paper proposes an alternative view where basis risk is *multiplicatively* related to either the spot price or the futures price. The amount of basis risk the decision maker has to bear is no longer independent of the level of the spot price or the futures price but is proportional to them. This multiplicative relation is consistent with the log-log model widely used in empirical hedging studies.

This paper analyzes two types of multiplicative dependence: In the first, the spot price is the product of the futures price and a conditionally independent basis risk. The second type is based on the reverse assumption: The futures price is the product of the spot price and an independent basis risk. For an unbiased futures market, the results are as follows: Given the first type of dependence, the optimal hedge ratio is determined by the decision maker's absolute prudence as a measure for the sensitivity to particularly low realizations of final wealth. Positive prudence is a necessary and sufficient condition for underhedging (a hedge ratio below one), negative prudence is equivalent to overhedging (a hedge ratio above one). A full hedge is optimal under quadratic utility. For the second type of dependence, non-negative absolute prudence is only a sufficient condition for the optimality of underhedging. Numerical examples illustrate the optimal futures position for the second type of dependence. These examples indicate that the optimal futures position can be substantially lower (up to 35 %) than the variance-minimizing position even under moderate assumptions on the decision maker's prudence. Hence, it is of crucial importance for practical applications whether basis risk is independent of the underlying's spot or futures price or proportional to it. This question has to be answered empirically.

In a recent paper, Mahul (2002) analyzes the first type of multiplicative dependence and contrasts it with the corresponding additive relation. He derives characteristics of first-best financial instruments to be used for hedging purposes and considers the role of options and futures under multiplicative dependence. This paper goes one step further because it also analyzes the reverse type of multiplicative dependence and explicitly characterizes the optimal futures position.

The paper is organized as follows: Section 2 delineates the framework of analysis. Section 3 shortly summarizes the results for an additive basis risk. Section 4 presents the two types of multiplicative combination in detail. Optimal futures hedging for each type is analyzed in Sections 5.1 and 5.2. Numerical examples are given in Section 6. Section 7 concludes. All proofs are given in the Appendix.

2 The model

The analysis is based on a two-date expected utility model: At date 0, the risk-averse decision maker has a given exposure Q to a price risk $\tilde{P}_1$. At date 0, his aim is to maximize expected utility from wealth at date 1 by choosing a hedging position in order to manage this price risk. Final wealth is denoted $\tilde{W}_1$.[1] The utility function $U(W_1)$ is at least three times continuously differentiable and exhibits risk aversion, $U'(W_1) > 0$, $U''(W_1) < 0$.[2] In addition, $\lim_{W_1 \to 0} U'(W_1) \to +\infty$ and $\lim_{W_1 \to +\infty} U'(W_1) \to 0$. At date 1, the decision maker sells the given quantity Q at the random future spot price $\tilde{P}_1$. At date 0, the decision maker can sell an amount X in a futures contract at the given futures price F_0 but has to repurchase the futures contracts at date 1 at the random futures price $\tilde{F}_1$.[3] Then, final wealth is given by

$$\tilde{W}_1 = \tilde{P}_1\, Q + \left(F_0 - \tilde{F}_1\right) X + \bar{W}_1\,, \tag{1}$$

where $\bar{W}_1$ denotes some deterministic amount. The decision maker's problem is

$$\max_X \mathrm{E}\left[U(\tilde{W}_1)\right]$$

[1]Throughout the paper, random variables have a tilde ($\tilde{}$) but their realizations do not.

[2]It is obvious that privately held, owner-managed firms behave in a risk-averse manner. But even firms with many shareholders and separation of ownership and control tend to behave as if they were risk-averse. This can be attributed to agency considerations (Stulz, 1984), corporate taxes and costs of financial distress (Smith and Stulz, 1985) and other capital market imperfections (Stulz, 1990, and Froot, Scharfstein and Stein, 1993). Hence, the results derived here are applicable to corporate risk management as well.

[3]$\tilde{P}_1$ and $\tilde{F}_1$ are assumed to have positive realizations only, $P_1 > 0$, $F_1 > 0$.

where $\tilde{W}_1$ is defined in (1). Since the utility function is concave and $U'(W_1) \in (0, \infty)$, the first-order condition for the optimal futures position X^*,[4]

$$\mathrm{E}\Big[U'(\tilde{W}_1^*)\Big(F_0 - \tilde{F}_1\Big)\Big] = 0\,, \tag{2}$$

is necessary and sufficient for a unique and interior optimum.

In order to focus on the hedging role of futures contracts, it is assumed that the futures market is unbiased, that is, the expected futures price is equal to the current futures price, $\mathrm{E}[\tilde{F}_1] = F_0$. Thus, the first-order condition reduces to $-\operatorname{cov}(U'(\tilde{W}_1^*), \tilde{F}_1) = 0$. In the case of backwardation or contango, the decision maker will also enter into a speculative position which can be easily incorporated into this model.

3 Cross hedging under additive basis risk

This section shortly reviews the results derived under additive basis risk. In a futures market, the basis is defined as the difference between the futures price $\tilde{F}_1$ and the spot price $\tilde{P}_1$. In the standard hedging model, tracing back to Holthausen (1979), the basis is deterministic such that *perfect hedging* or *direct hedging* is possible. In particular, Holthausen (1979) considers the simplest case where the basis is always zero, $\tilde{F}_1 = \tilde{P}_1$. He shows that, given an unbiased futures market, it is optimal to fully hedge the given exposure, $X^* = Q$, such that final wealth does no longer depend on price risk but is deterministic.

If the basis is stochastic, the decision maker faces basis risk in addition to price risk. In this case, risk management has to rely on *cross hedging* or *indirect hedging* as analyzed by Anderson and Danthine (1981), Broll, Wahl and Zilcha (1995), Chang and Wong (2002) and others. In order to formalize basis risk, an additional random variable $\tilde{\gamma}$ has to be incorporated into the model. This can be done either by assuming that price risk is a function of the futures price and basis risk, $\tilde{P}_1 = p(\tilde{F}_1, \tilde{\gamma})$, or by assuming that the futures price is a function of price risk and basis risk, $\tilde{F}_1 = f(\tilde{P}_1, \tilde{\gamma})$.[5]

[4] Optimal values are given a star (*).

[5] Both cases have to be considered separately since $f(\cdot)$ and $p(\cdot)$ are not necessarily invertible.

So far, the literature focused on the case of an *additive* combination of basis risk and either $\tilde{F}_1$ or $\tilde{P}_1$. Lence (1995) generalized the contributions of Benninga, Eldor and Zilcha (1983) and others by considering $\tilde{P}_1 = p(\tilde{F}_1, \tilde{\gamma}) = c\,\tilde{F}_1 + g(\tilde{\gamma})$ for some arbitrary constant c and some arbitrary function $g(\cdot)$. Hence, $\tilde{\gamma}$ and $\tilde{F}_1$ are additively combined to generate $\tilde{P}_1$. Lence (1995) has shown that conditional independence of $\tilde{F}_1$ from $\tilde{\gamma}$ is necessary and sufficient for the optimality of a full hedge in an unbiased futures market, $X^* = c\,Q$.[6] This full hedge does not require any additional assumptions on the utility function beyond risk aversion.

The reverse relation where F_1 is an additive combination of $\tilde{P}_1$ and $\tilde{\gamma}$ has first been proposed by Paroush and Wolf (1986). A version slightly more general than theirs is given by $\tilde{F}_1 = f(\tilde{P}_1, \tilde{\gamma}) = k\,\tilde{P}_1 + h(\tilde{\gamma})$ where $\tilde{P}_1$ and $\tilde{\gamma}$ are stochastically independent and $h(\cdot)$ is a linear function.[7] Briys, Crouhy and Schlesinger (1993) show that an underhedging position is optimal in an unbiased futures market under this type of additive basis risk, $X^* < Q/k$ for $k > 0$. It follows from Adam-Müller (2000) that this result holds for any risk averter, in particular, it is independent of the sign of $U'''(\cdot)$ as an indicator of the decision maker's prudence.

Following Kimball (1990), absolute prudence is defined as $-\,U'''(\cdot)/U''(\cdot)$. Positive prudence, being a necessary condition for decreasing absolute risk aversion, is a commonly accepted property of utility functions since it leads to plausible behavior in various decision problems under risk.[8] It also implies that the decision maker is particularly sensitive to low realizations of final wealth (more than under quadratic utility) such that there is a precautionary motive to avoid such realizations.

[6] A random variable $\tilde{x}$ is said to be conditionally independent of another random variable $\tilde{y}$ if $\mathrm{E}[\tilde{x}|y] = E[\tilde{x}]\,\forall\,y$. Under mild regularity conditions, conditional independence of $\tilde{x}$ of $\tilde{y}$ is equivalent to $\mathrm{cov}(\tilde{x}, l(\tilde{y})) = 0$ for all functions $l(\tilde{y})$. See Ingersoll (1987, p. 15).

[7] This is not exactly the reverse assumption because independence is stronger than conditional independence. In addition, the linearity assumption imposed on $h(\cdot)$ is not needed for $g(\cdot)$.

[8] See Gollier (2001, Ch. 16). For example, the hyperbolic absolute risk aversion (HARA) class of utility function, given by $U^{HARA}(W_1) = (1-\gamma)/\gamma \times ((A + W_1)/(1-\gamma))^{\gamma}$, exhibits positive prudence if $\gamma < 1$. The HARA class contains the most commonly used types of utility functions such as those with constant absolute and constant relative risk aversion.

4 Two multiplicative types of basis risk

As argued in the previous section, the theoretical literature on risk management has a focus on an *additive* incorporation of basis risk into the decision maker's problem. In this case, the amount of basis risk is independent of the level of F_1 or P_1 which is a strong assumption that does not necessarily hold in any particular application. Therefore, this paper proposes the alternative of a *multiplicative* relation between basis risk and either the futures price $\tilde{F}_1$ or the spot price $\tilde{P}_1$. Hence, the amount of basis risk is assumed to be proportional to the level of the futures price or the spot price. In particular, this paper analyzes the following cases, labelled A.1 and A.2:

Assumption A.1:

$$\tilde{P}_1 = \alpha + \beta\,\tilde{F}_1\,(1 + \tilde{\varepsilon})$$

where $\tilde{\varepsilon}$ is conditionally independent of $\tilde{F}_1$. The support of $\tilde{\varepsilon}$ is contained in the interval $[\underline{\varepsilon}, \overline{\varepsilon}]$ with $-1 < \underline{\varepsilon} < 0 < \overline{\varepsilon} < \infty$ and $\mathrm{E}[\tilde{\varepsilon}] = 0$. The support of $\tilde{F}_1$ is a subset of $[\underline{F_1}, \overline{F_1}]$ with $0 < \underline{F_1} < \overline{F_1} < \infty$. In addition, $\beta > 0$ and $\alpha > -\,\beta\,\underline{F_1}(1 + \underline{\varepsilon})$.

These assumptions imply $P_1 > 0$ and $\mathrm{E}[\tilde{P}_1] = \alpha + \beta\mathrm{E}[\tilde{F}_1]$. Under A.1, the spot price $\tilde{P}_1$ is a multiplicative combination of basis risk $\tilde{\varepsilon}$ and the futures price $\tilde{F}_1$. Basis risk from $\tilde{\varepsilon}$ is proportional to F_1.

A.1 is closely related to the log-log model which is widely used in empirical hedging studies on the relationship between the spot price $\tilde{P}_1$ and the futures price $\tilde{F}_1$.[9] In this model, it is assumed that $\log(\tilde{P}_1) = c + \log(\tilde{F}_1) + \tilde{e}$ where $\tilde{F}_1$ and $\tilde{e}$ are stochastically independent. Setting $\alpha = 0$ and redefining $\log(\beta) = c$ and $\log(1+\tilde{\varepsilon}) = \tilde{e}$, taking the log of A.1 yields $\log(\tilde{P}_1) = \log(\beta) + \log(\tilde{F}_1) + \log(1+\tilde{\varepsilon}) = c + \log(\tilde{F}_1) + \tilde{e}$. Hence, A.1. is consistent with the log-log model.

Assumption A.2:

$$\tilde{F}_1 = a + b\,\tilde{P}_1\,(1 + \tilde{\epsilon})\,,$$

where $\tilde{\epsilon}$ and $\tilde{P}_1$ are stochastically independent. The support of $\tilde{\epsilon}$ is a subset of $[\underline{\epsilon}, \overline{\epsilon}]$ with $-1 < \underline{\epsilon} < 0 < \overline{\epsilon} < \infty$ and $\mathrm{E}[\tilde{\epsilon}] = 0$. The support of $\tilde{P}_1$ is contained in $[\underline{P_1}, \overline{P_1}]$ with $0 < \underline{P_1} < \overline{P_1} < \infty$. In addition, $b > 0$ and $a > -\,b\,\underline{P_1}(1 + \underline{\epsilon})$.[10]

[9] See Baillie and Myers (1991) and the review article by Tomek and Peterson (2001).

[10] All results can also be derived for $\beta < 0$ and $b < 0$, with some obvious modifications.

These assumptions imply $F_1 > 0$ and $\mathrm{E}[\tilde{F}_1] = a + b\,\mathrm{E}[\tilde{P}_1]$. Under A.2, the futures price $\tilde{F}_1$ is a multiplicative combination of basis risk $\tilde{\epsilon}$ and the spot price $\tilde{P}_1$. Basis risk from $\tilde{\epsilon}$ is proportional to P_1.

Whether A.1 or A.2 or an additive specification is appropriate in a particular hedging problem, has to be answered empirically.

5 Cross hedging under multiplicative basis risk

5.1 Basis risk of type A.1

Under A.1, price risk $\tilde{P}_1$ can be interpreted as a bundle of tradable futures price risk $\tilde{F}_1$ and untradable basis risk $\tilde{\varepsilon}$. From (1) and A.1, final wealth is given by

$$\tilde{W}_1 = \tilde{\varepsilon}\,\tilde{F}_1\,\beta Q + w(\tilde{F}_1, X)\,, \tag{3}$$

where $w(\tilde{F}_1, X) = \tilde{F}_1(\beta Q - X) + (\alpha Q + F_0\,X + \bar{W}_1)$. $w(\tilde{F}_1, X)$ does not depend $\tilde{\varepsilon}$. (3) shows that the decision maker's exposure to untradable basis risk $\tilde{\varepsilon}$ is independent of the futures position X. However, basis risk cannot be interpreted as an additive background risk since it appears in multiplicative combination with $\tilde{F}_1$.[11] Since $\tilde{F}_1$ is tradable, basis risk $\tilde{\varepsilon}$ affects the optimal futures position even though it is not directly related to the decision variable X.

In the absence of basis risk as well as in the presence of an additive basis risk as in Lence (1995), full hedging in an unbiased futures market is optimal for all risk-averse utility functions. In contrast to these results, Proposition 1 shows that this is not the case under A.1. Here, the optimal futures position depends on the decision maker's absolute prudence:

Proposition 1 *Suppose that the futures market is unbiased and that A.1 holds. Suppose further that the sign of $U'''(\cdot)$ is constant. Then, $X^* <\, [=][>]\, \beta Q$ if and only if $U'''(\cdot) >\, [=][<]\, 0$.*

[11] For the case of an additive independent background risk, Briys, Crouhy and Schlesinger (1993) show that full hedging in an unbiased futures market is optimal for any risk-averse utility function.

A full hedge is optimal only under quadratic utility; it minimizes the variance of $\tilde{W}_1$. Hence, the variance-minimizing futures position under A.1 is given by $X^{vm,1} = \beta Q$.[12]

Given A.1, the first-order condition for X^* can be rewritten as $\mathrm{E}\{\mathrm{E}[U'(w(F_1, X^*) + \tilde{\varepsilon}\, F_1 \beta Q)|F_1](F_0 - \tilde{F}_1)\} = 0$, applying the law of iterated expectations. Using a Taylor expansion of $U'(\cdot)$ around $w(F_1, X)$, expected marginal utility for a given F_1 and a small $\tilde{\varepsilon}$-risk can be written as

$$\begin{aligned} \mathrm{E}\Big[U'(w(F_1, X) + \tilde{\varepsilon}\, F_1\, \beta Q)\Big|F_1\Big] &= U'(w(F_1, X)) \\ &\quad + \mathrm{var}(\tilde{\varepsilon})\, \frac{(F_1\, \beta Q)^2}{2}\, U'''(w(F_1, X))\,. \end{aligned} \tag{4}$$

Since marginal utility is decisive for hedging, the second term on the RHS of (4) indicates that the impact of basis risk $\tilde{\varepsilon}$ on the optimal futures position X^* depends on F_1 and on $U'''(\cdot)$. Under quadratic utility, basis risk is ignored such that full hedging is optimal. However, basis risk affects the optimal futures position whenever marginal utility is not linear. Proposition 1 states that an underhedging position is optimal if and only if preferences exhibit positive absolute prudence, $U'''(\cdot) > 0$. Given positive prudence, the decision maker has a precautionary incentive to avoid particularly low realizations of final wealth. Hence, he will increase final wealth in states with very small W_1 as compared to quadratic utility. At $X^{vm,1}$, these states are characterized by a highly negative realization of $\tilde{\varepsilon}$ together with a highly positive realization of $\tilde{F}_1$ as can be seen from

$$\begin{aligned} \tilde{W}_1\Big(X = X^{vm,1}\Big) &= \tilde{\varepsilon}\, \tilde{F}_1\, \beta Q + \alpha Q + F_0\, X^{vm,1} + \bar{W}_1 \\ &= X^{vm,1}\Big(\tilde{\varepsilon}\, \tilde{F}_1 + F_0\Big) + \alpha Q + \bar{W}_1\,. \end{aligned} \tag{5}$$

As (3) indicates, generating additional final wealth in states with high F_1 requires selling less futures contracts than under quadratic utility since this leads to $\partial w(F_1, X)/\partial F_1 > 0$ for all F_1. Thus, the optimal futures position of a prudent decision maker is below the variance-minimizing position that is optimal for quadratic utility. This is equivalent to an underhegde, $X^* < \beta Q = X^{vm,1}$.[13]

[12]Conditional independence implies $\mathrm{cov}(\tilde{F}_1, \tilde{F}_1\tilde{\varepsilon}) = 0$ such that $\mathrm{var}(\tilde{W}_1) = (\beta Q - X)^2\, \mathrm{var}(\tilde{F}_1) + (\beta Q)^2\, \mathrm{var}(\tilde{\varepsilon}\tilde{F}_1)$. $X^{vm,1} = \beta Q$ directly follows.

[13]The case where $U'''(\cdot) < 0$ can be interpreted along the same lines.

Kimball (1990) has shown that prudence plays an important role in the presence of an additive background risk. Proposition 1 shows that the decision maker's prudence is also crucial in the presence of a multiplicative basis risk.[14] As already noticed by Lence (1995), the optimal hedge ratio in the log-log model is not independent of the utility function as claimed by Baillie and Myers (1991). Proposition 1 goes one step further and shows *how* the optimal futures position depends on the decision maker's preferences.

5.2 Basis risk of type A.2

Under A.2 as given by $\tilde{F}_1 = a + b\,\tilde{P}_1\,(1+\tilde{\epsilon})$, tradable futures price risk $\tilde{F}_1$ can be regarded as a package of price risk $\tilde{P}_1$ and basis risk $\tilde{\epsilon}$. The exposure to $\tilde{P}_1$ is exogenously given by Q while the exposure to the package that forms $\tilde{F}_1$ is endogenously determined via the futures position X. In other words, if the decision maker wants to protect himself against fluctuations in $\tilde{P}_1$ by trading $\tilde{F}_1$, he thereby exposes himself to basis risk $\tilde{\epsilon}$. Under A.2, final wealth can be written as

$$\tilde{W}_1 = \tilde{P}_1\Big(Q - bX\Big) - \tilde{\epsilon}\,\tilde{P}_1\Big(bX\Big) + X\,\Big(F_0 - a\Big) + \bar{W}_1\,. \tag{6}$$

(6) indicates that the exposure to basis risk $\tilde{\epsilon}$ is endogenously determined by the futures position X. (This is not the case under A.1 as follows from (3).) The second term on the RHS of (6) shows that basis risk $\tilde{\epsilon}$ enters final wealth only in multiplicative combination with $\tilde{P}_1$ and X. At $X = 0$, there is no basis risk since the second term in (6) is zero. At full hedging, $X = Q/b > 0$, the isolated effect of price risk $\tilde{P}_1$ as represented by the first term is eliminated. But this comes at the cost of exposing final wealth to basis risk $\tilde{\epsilon}$. Obviously, there is a conflict between reducing price risk $\tilde{P}_1$ and avoiding basis risk $\tilde{\epsilon}$. The first term favors a full hedging position, $X = Q/b$, the second a futures position of zero, $X = 0$. The next statement characterizes the optimal futures position.

[14]Wong (2002) provides another example where prudence drives the result in the case of multiplicative combined risks.

Proposition 2 *Suppose that the futures market is unbiased and that A.2 holds. Suppose further that the sign of $U'''(\cdot)$ is constant.*

a) The optimal futures position is a short hedge, $X^* > 0$.

b) If $U'''(\cdot) \geq 0$, the optimal futures position is an underhegde, $X^* < Q/b$.

Part a) of Proposition 2 shows that every decision maker will optimally hedge at least part of the price risk $\tilde{P}_1$ by choosing a positive futures position, $X^* > 0$. This causes final wealth to dependent on basis risk $\tilde{\epsilon}$ as well. The decision maker enjoys a gain from diversification: $X^* > 0$ is the same as selling part of one risk ($\tilde{P}_1$) and acquiring another risk ($\tilde{\epsilon}\tilde{P}_1$) which is not perfectly correlated with the first.

Consider part b) of Proposition 2. For quadratic utility, $U'''(\cdot) = 0$, the optimal futures position under A.2 can be explicitly derived since it coincides with the variance-minimizing position. It is given by[15]

$$X^{vm,2} = \frac{Q}{b}\left(\frac{\operatorname{var}(\tilde{P}_1)}{\operatorname{E}[\tilde{P}_1^2]\operatorname{var}(\tilde{\epsilon}) + \operatorname{var}(\tilde{P}_1)}\right) = \frac{Q}{b}K \tag{7}$$

where K is a positive constant that only depends on the distributions of $\tilde{P}_1$ and $\tilde{\epsilon}$. K is below one such that the variance-minimizing futures position is always an underhedging position as claimed in part b) of Proposition 2. K captures the relative size of price risk and independent basis risk. Since K decreases in $\operatorname{var}(\tilde{\epsilon})$ and increases in $\operatorname{var}(\tilde{P}_1)$, $X^{vm,2}$ exhibits intuitively plausible comparative statics: For example, an increase in the variance of $\tilde{\epsilon}$ implies that the amount of undesirable basis risk taken per unit of price risk $\tilde{P}_1$ hedged via futures contracts increases such that variance-minimizing futures position decreases.[16]

More importantly, part b) of Proposition 2 states that positive prudence is a sufficient condition for underhedging under A.2. To see why, consider a prudent decision maker who starts at full hedging, $X = Q/b$. His final wealth, expressed in

[15]Since $\operatorname{var}(\tilde{P}_1\,\tilde{\epsilon}) = \operatorname{E}[\tilde{P}_1^2]\operatorname{var}(\tilde{\epsilon})$ and $\operatorname{cov}(\tilde{P}_1, \tilde{P}_1\tilde{\epsilon}) = 0$, it is straightforward to show that $\operatorname{var}(\tilde{W}_1) = (Q - bX)^2\operatorname{var}(\tilde{P}_1) + b^2X^2\operatorname{E}[\tilde{P}_1^2]\operatorname{var}(\tilde{\epsilon})$. (7) follows directly.

[16]Without basis risk, K equals one such that full hedging is optimal which, in this case, makes final wealth deterministic.

terms of tradable futures price risk $\tilde{F}_1$ and untradable basis risk $\tilde{\epsilon}$, is given by[17]

$$\tilde{W}_1\Big(X = Q/b\Big) \;=\; -Q\left(\frac{\tilde{\epsilon}}{1+\tilde{\epsilon}}\right)\left(\frac{\tilde{F}_1 - a}{b}\right) + \frac{Q}{b}\Big(F_0 - a\Big) + \bar{W}_1\,. \tag{8}$$

Since b and Q are positive and $(F_1 - a) > 0$ in all states, final wealth is very low if ϵ and F_1 are both very high. Positive prudence creates a strong motive to generate additional wealth in this unfavorable state. Generating additional final wealth in states with high F_1 requires a reduction of the futures position to a level below $X = Q/b$. Hence, underhedging is optimal.

Finally, we compare the variance-minimizing futures position and the futures position optimal for a prudent decision maker. Under A.1, a prudent decision maker's optimal futures position is always less than the variance-minimizing position $X^{vm,1} = \beta\, Q$ which coincides with the full hedge. Under A.2, a similar statement cannot be derived. The variance-minimizing position is itself an underhegde, $X^{vm,2} < Q/b$, see (7). Consider final wealth at $X^{vm,2}$, expressed in terms of tradable futures price risk $\tilde{F}_1$ and untradable basis risk $\tilde{\epsilon}$:

$$\tilde{W}_1\Big(X = X^{vm,2}\Big) \;=\; \tilde{F}_1\,\frac{Q}{b}\left\{\frac{1}{(1+\tilde{\epsilon})} - K\right\} + \frac{Q}{b}\left(F_0\,K - \frac{a}{(1+\tilde{\epsilon})}\right) + \bar{W}_1\,. \tag{9}$$

Consider the curly bracketed term. Given $\epsilon < 0$, the fraction exceeds one such that the curly bracketed term is positive because of $K < 1$. Then, final wealth at $X^{vm,2}$ increases in F_1 since b and Q are positive. Hence, the lowest realization of final wealth occurs at smallest realization of $\tilde{F}_1$, given $\epsilon < 0$. Increasing final wealth in these states requires selling less futures contracts which is in favor of a futures position below $X^{vm,2}$. However, in states where $\epsilon > 0$, the curly bracketed term may also be negative.[18] Then, final wealth decreases in F_1 which is in favor of a futures position above the variance-minimizing position. Since the exact size of these effects depends on the distribution of $\tilde{\epsilon}$, a general statement cannot be made for A.2.[19]

[17] Since $\tilde{F}_1$ is a function of $\tilde{P}_1$ and $\tilde{\epsilon}$, (8) should be used for interpretation only. The same applies to (9) below.

[18] There is at least one such state if the highest possible realization of $\tilde{\epsilon}$ exceeds $(1-K)/K$.

[19] In addition, the realization of $\tilde{\epsilon}$ may also give rise to a wealth effect which is attributable to the second bracketed term in (9). Unless absolute prudence is a constant, this wealth effect will have an additional impact on relative weight of the two effects just described.

6 Numerical examples

This section provides two numerical examples that are based on A.2. The aim is to gain additional insights into the nature of the optimal futures position and its relation to the variance-minimizing futures position.

The first example is based on the following assumptions: The marginal probability distribution of $\tilde{P}_1$ has five-point support: With 2% probability, P_1 equals 70. With 32% probability each, P_1 is either 75, 80 or 85. Finally, there is a outlier at $P_1 = 140$ which also has 2% probability.[20] It follows that $\mathrm{E}[\tilde{P}_1] = 81$; the standard deviation amounts to $\mathrm{std}(\tilde{P}_1) = 9.434$ and the skewness to 4.802 indicating that $\tilde{P}_1$ is skewed to the right.

The marginal distribution of $\tilde{\epsilon}$ has a three-point support: $\epsilon = +0.2$ and $\epsilon = -0.2$ with probability π each and $\epsilon = 0$ with the remaining probability $(1 - 2\,\pi)$. This implies a symmetric distribution with $\mathrm{E}[\tilde{\epsilon}] = 0$ and $\mathrm{std}(\tilde{\epsilon}) = 0.283\,\sqrt{\pi}$. π is a volatility parameter for basis risk $\tilde{\epsilon}$ that affects the probabilities of the non-zero realizations while leaving the support unchanged. In the scenarios considered below, π will be varied between 1% and 10% in steps of 1% and from 10% to 50% in steps of 5%.

Table 1: The distributions of $\tilde{P}_1$, $\tilde{\epsilon}$ and $\tilde{F}_1$ for $\pi = 10\%$

state	probability	P_1	ϵ	F_1
1	0.2%	70	-0.2	56
2	3.2%	75	-0.2	60
3	3.2%	80	-0.2	64
4	3.2%	85	-0.2	68
5	0.2%	140	-0.2	112
6	1.6%	70	0	70
7	25.6%	75	0	75
8	25.6%	80	0	80
9	25.6%	85	0	85
10	1.6%	140	0	140
11	0.2%	70	+0.2	84
12	3.2%	75	+0.2	90
13	3.2%	80	+0.2	96
14	3.2%	85	+0.2	102
15	0.2%	140	+0.2	168

[20] All numerical values except the assumptions are rounded.

Table 2: Optimal futures positions for different levels of π

parameters of the joint distribution			optimal futures position, CRRA = 3			variance minimizing futures position			relative difference
π	$\text{std}(\tilde{F}_1)$	$\text{corr}(\tilde{P}_1, \tilde{F}_1)$	X^*	$\text{std}(\tilde{W}_1)$	$\min(\tilde{W}_1)$	$X^{vm,2}$	$\text{std}(\tilde{W}_1)$	$\min(\tilde{W}_1)$	$\left(\frac{X^* - X^{vm,2}}{X^{vm,2}}\right)$
0.01	9.71	0.971	0.929	2.25	59.18	0.944	2.24	57.91	-1.55%
0.02	9.98	0.945	0.865	3.10	64.73	0.893	3.08	62.29	-3.13%
0.03	10.25	0.921	0.807	3.70	67.58	0.848	3.68	66.23	-4.78%
0.04	10.50	0.898	0.755	4.18	67.74	0.807	4.14	67.58	-6.47%
0.05	10.75	0.877	0.707	4.58	67.88	0.770	4.53	67.69	-8.17%
0.06	11.00	0.858	0.664	4.91	68.01	0.736	4.85	67.79	-9.83%
0.07	11.24	0.840	0.624	5.20	68.13	0.705	5.12	67.89	-11.45%
0.08	11.47	0.822	0.589	5.46	68.23	0.676	5.37	67.97	-12.99%
0.09	11.70	0.806	0.556	5.69	68.33	0.650	5.58	68.05	-14.45%
0.10	11.92	0.791	0.527	5.89	68.42	0.626	5.77	68.12	-15.83%
0.15	12.99	0.726	0.414	6.65	68.76	0.527	6.49	68.42	-21.56%
0.20	13.98	0.675	0.338	7.15	68.99	0.455	6.96	68.63	-25.71%
0.25	14.90	0.633	0.286	7.50	69.14	0.401	7.30	68.80	-28.77%
0.30	15.77	0.598	0.247	7.76	69.26	0.358	7.60	68.93	-31.11%
0.35	16.59	0.569	0.217	7.96	69.35	0.323	7.76	69.03	-32.94%
0.40	17.37	0.543	0.193	8.12	69.42	0.295	7.92	69.12	-34.41%
0.45	18.12	0.521	0.175	8.24	69.48	0.271	8.06	69.19	-35.61%
0.50	18.84	0.501	0.159	8.35	69.52	0.251	8.17	69.25	-36.61%

Combining these marginal distributions leads to a joint distribution of $\tilde{P}_1$ and $\tilde{\epsilon}$ and, hence, a distribution of $\tilde{F}_1$ with fifteen possible states. The distribution of $\tilde{F}_1$ is skewed to the right as well.[21] For simplicity, $a = 0$ and $b = 1$ such that $\text{E}[\tilde{F}_1] = \text{E}[\tilde{P}_1] = F_0 = 81$. Table 1 summarizes these assumption for $\pi = 10\%$.

The utility function is the power function with $\check{U}(W_1) = W_1^\gamma/\gamma$ and $\gamma = -2$ such that relative risk aversion is constant (CRRA) at a moderate level of $(1-\gamma) = 3$. Prudence is positive as well. Relative prudence as defined by $-W_1\,\check{U}'''(W_1)/\check{U}(W_1)''$ is constant at $(2-\gamma) = 4$. In addition, $\bar{W}_1 = 0$. The initial exposure Q is normalized to one such that, at $X = 0$, $\text{std}(\tilde{W}_1) = \text{std}(\tilde{P}_1) = 9.434$ and the smallest possible realization of $\tilde{W}_1$, $\min(\tilde{W}_1)$, is 70 with probability 2%. Since the futures market is unbiased, $\text{E}[\tilde{W}_1] = 81$ for any futures position.

Table 2 presents the result for the optimal futures positions. The first column shows the probability π for the non-zero realizations of $\tilde{\epsilon}$. Columns 2 and 3 show the standard deviation of $\tilde{F}_1$ and the correlation coefficient for $\tilde{P}_1$ and $\tilde{F}_1$.

Columns 4 to 6 characterize the optimal futures position for CRRA = 3 as well

[21]Even if the distributions of $\tilde{P}_1$ and $\tilde{\epsilon}$ are symmetric, the distribution of $\tilde{F}_1$ is skewed. This is due to the multiplicative interaction of $\tilde{P}_1$ and $\tilde{\epsilon}$. For an illustration, see the second example.

as the resulting standard deviation of final wealth $\tilde{W}_1$ and the smallest possible realization of $\tilde{W}_1$. Similarly, the next three columns show the variance-minimizing futures position which is optimal under quadratic utility as well as the resulting standard deviation and smallest possible realization of $\tilde{W}_1$. Since $Q = 1$, the values for X^* and $X^{vm,2}$ can directly be interpreted as hedge ratios. The last column shows the percentage difference between the optimal and the variance-minimizing futures position relative to the latter.

The higher π, the more weight is attached to the non-zero realizations of $\tilde{\epsilon}$. In other words, the package of price risk $\tilde{P}_1$ and basis risk $\tilde{\epsilon}$ that can be traded in the futures market contains more and more basis risk as π increases. Consequently, the standard deviation of $\tilde{F}_1$ increases as column 2 shows. For the same reason, the correlation coefficient between $\tilde{P}_1$ and $\tilde{F}_1$ decreases. Hence, it is not surprising that both the optimal as well as the variance-minimizing futures position, X^* and $X^{vm,2}$, decrease in π (columns 4 and 7). The resulting distributions of final wealth are shortly characterized in columns 5 and 6 for X^* and columns 8 and 9 for $X^{vm,2}$. They indicate that both the standard deviation and the smallest possible realization of final wealth tend to the values of the unhedged position as π increases. This is not surprising since the futures position decreases in π to values as low as $X^* = 0.159$ and $X^{vm,2} = 0.251$ for $\pi = 0.50$.

The two most important results emanating from this example are: First, the optimal futures position in an unbiased futures market can be significantly less than a full hedge, even under a modest CRRA of 3. At $\pi = 0.10$, the optimal futures position is less than 53% of the full hedging position despite the correlation coefficient being still high at 0.791. Second, the difference between the variance-minimizing futures position and the optimal position is quite substantial. As the last column shows, the relative difference increases strongly as basis risk grows. For small basis risk, the change is only 1.55% whereas at $\pi = 0.10$, $X^* = 0.527$ is nearly 16% below $X^{vm,2} = 0.626$. For higher basis risk, the optimal futures position is more than 36% below the variance-minimizing position. Hence, the mistake made by following variance-minimizing hedging routines can be substantial.

Of course, the standard deviation of final wealth is larger for the optimal futures position (column 6) than for the variance-minimizing position reported in column 9.

Taking $\pi = 0.10$ as an example, the standard deviations are 5.77 and 5.89, respectively. In addition, the smallest possible realization of final wealth is consistently higher under CRRA. This is clearly attributable to the decision maker's prudence. For $\pi = 0.10$, one has to compare $W_1 = 68.12$ with $W_1 = 68.42$.

The fact that the smallest possible realization of final wealth increases in the volatility of the basis risk may seem counterintuitive at the first sight. However, the probability for this extreme realization is very small for low values of π. For example, at $\pi = 0.01$, the worst case has a probability as low as 0.02%. At $\pi = 0.50$, this probability is much higher at 1.00%. (These probabilities are not reported in Table 2.) More interestingly, $\min(\tilde{W}_1)$ always occurs at $\epsilon = \overline{\epsilon} = +0.2$. For $\pi \leq 0.02$, it occurs when $P_1 = \overline{P_1} = 140$.[22] For $\pi \geq 0.03$, it occurs when $P_1 = \underline{P_1} = 70$.

In the first example, presented in Tables 1 and 2, the optimal futures position is consistently smaller than the variance-minimizing position. Next, consider the second example where the optimal futures position is above the variance-minimizing position in some scenarios but below in others.

The second example is based on the following assumptions: The marginal probability distribution of $\tilde{P}_1$ is symmetric with a three-point support: P_1 equals either 0.9 or 1.1 with 20% probability, $P_1 = 1.0$ with 60% probability. Therefore, $\mathrm{E}[\tilde{P}_1] = 1$ and $\mathrm{std}(\tilde{P}_1) = 0.004$. The marginal distribution of $\tilde{\epsilon}$ is symmetric as well with three-point support such that $\epsilon = +\delta$ and $\epsilon = -\delta$ with probability 25% each and $\epsilon = 0$ with the probability 50%. Hence, $\mathrm{std}(\tilde{\epsilon}) = \delta/\sqrt{2}$. δ is a volatility parameter for basis risk $\tilde{\epsilon}$ that affects the support of $\tilde{\epsilon}$ but does not change the probabilities. δ will be varied between 1% and 10% in steps of 1%.

The resulting distribution of $\tilde{F}_1$ is skewed to the right and has nine possible states with $\mathrm{E}[\tilde{F}_1] = \mathrm{E}[\tilde{P}_1] = 1$ and $\mathrm{var}(\tilde{F}_1) = 0.004 + 0.502\,\delta^2$. All other assumptions are the same as in the first example, in particular $a = \bar{W}_1 = 0$, $b = Q = 1$ and CRRA $= 3$. The futures market is unbiased, $F_0 = 1$.

Table 3 presents the optimal futures positions for the second example, depending on the spread parameter δ in the first column. All other columns of Table 3 present the same variables as those in Table 2 except the standard deviations of final wealth.

[22] In fact, the curly bracketed term in (9) is negative for $\pi \leq 0.02$ and $\overline{\epsilon}$.

Table 3: Optimal futures positions for different levels of δ

parameters of the joint distribution			optimal futures position, CRRA = 3		variance minimizing futures position		relative difference
δ	$\text{std}(\tilde{F}_1)$	$\text{corr}(\tilde{P}_1, \tilde{F}_1)$	X^*	$\min(\tilde{W}_1)$	$X^{vm,2}$	$\min(\tilde{W}_1)$	$\left(\dfrac{X^* - X^{vm,2}}{X^{vm,2}}\right)$
0.01	0.0636	0.994	0.987	0.990	0.998	0.990	-0.019%
0.02	0.0648	0.976	0.952	0.978	0.952	0.978	-0.062%
0.03	0.0667	0.948	0.898	0.966	0.899	0.966	-0.101%
0.04	0.0693	0.913	0.832	0.953	0.833	0.953	-0.105%
0.05	0.0725	0.873	0.761	0.942	0.761	0.942	-0.060%
0.06	0.0762	0.830	0.689	0.932	0.689	0.932	0.033%
0.07	0.0804	0.787	0.620	0.923	0.619	0.923	0.165%
0.08	0.0849	0.745	0.556	0.916	0.555	0.916	0.321%
0.09	0.0898	0.704	0.498	0.909	0.496	0.909	0.488%
0.10	0.0950	0.666	0.446	0.904	0.443	0.904	0.656%

For $\delta \leq 0.05$, the optimal futures position X^* is larger than the variance-minimizing position $X^{vm,2}$. For $\delta \geq 0.06$, the reverse is true. However, the differences between these two positions are relatively small as indicated in the last column.

7 Conclusions

This paper analyzes two types of multiplicative dependence between basis risk on the one hand and spot price risk or futures price risk on the other. These specifications imply that basis risk is proportional to the level of the spot price or the futures price. The paper shows that the optimal futures position under these multiplicative specifications crucially depends on the decision maker's absolute prudence. This is in sharp contrast to previous contributions (except Mahul, 2002) which had a strong focus on models where basis risk enters the model in an additive way and prudence does not play a role.

If basis risk is multiplicatively combined with futures price risk as under assumption A.1, there is a direct relation between the decision maker's absolute prudence and the optimal futures position relative to the initial exposure: Underhedging in an unbiased futures market is optimal if and only if absolute prudence is positive; quadratic utility is a necessary and sufficient condition for full hedging. Under assumption A.2 where basis risk is multiplicatively combined with spot price risk, non-negative absolute prudence is only a sufficient condition for the optimality of an underhedging position in an unbiased futures market.

Numerical examples show that the difference between the futures position optimal under assumption A.2 and the variance-minimizing position can be significant, even under moderate levels of the decision maker's prudence. Hence, variance-minimizing hedging routines can lead to considerable mistakes if basis risk is proportional to the realizations of the spot price.

Appendix

Proof of Proposition 1

Unbiasedness and the law of iterated expectations imply that the first-order condition in (2) is equivalent to $\operatorname{cov}(\mathrm{E}[U'(\tilde{W}_1^*)|F_1], \tilde{F}_1) = 0$. Hence, $\mathrm{E}[U'(\tilde{W}_1^*)|F_1]$ is either a constant or it is decreasing in some interval of F_1 while increasing in some other interval of F_1. Hence, $\partial\,\mathrm{E}[U'(W_1)|F_1]/\partial\,F_1$ equals zero everywhere or has varying sign. Conditional independence, $\mathrm{E}[\tilde{\varepsilon}|F_1] = \mathrm{E}[\tilde{\varepsilon}] = 0$ for all F_1, implies

$$
\begin{aligned}
\frac{\partial\,\mathrm{E}\Big[U'(W_1)\Big|F_1\Big]}{\partial\,F_1} &= \mathrm{E}\Big[U''(\tilde{W}_1)\Big(\beta Q(1+\tilde{\varepsilon}) - X\Big)\Big|F_1\Big] \\
&= \Big(\beta Q - X\Big)\,\mathrm{E}\Big[U''(\tilde{W}_1)\Big|F_1\Big] \\
&\quad + \beta Q\,\Big\{\mathrm{E}\Big[U''(\tilde{W}_1)\Big|F_1\Big]\,\mathrm{E}[\tilde{\varepsilon}|F_1] + \operatorname{cov}\Big(U''(\tilde{W}_1), \tilde{\varepsilon}\,\Big|F_1\Big)\Big\} \\
&= \Big(\beta Q - X\Big)\,\mathrm{E}\Big[U''(\tilde{W}_1)\Big|F_1\Big] + \beta Q\operatorname{cov}\Big(U''(\tilde{W}_1), \tilde{\varepsilon}\,\Big|F_1\Big)\,.
\end{aligned}
\tag{10}
$$

$\beta, F_1, Q > 0$ and the fact that $\partial\,U''(W_1)/\partial\,\varepsilon = U'''(W_1)\,\beta\,F_1\,Q$ for all F_1, ε imply $\operatorname{sgn}\operatorname{cov}(U''(\tilde{W}_1), \tilde{\varepsilon}|F_1) = \operatorname{sgn} U'''(\cdot)$. Thus, the covariance in (10) is positive for $U'''(\cdot) > 0$. For $(\beta Q - X) \leq 0$, there is no interval in which $\mathrm{E}[U'(\tilde{W}_1)|F_1]$ does not increase in F_1 as follows from (10). This yields a contradiction. Thus, $(\beta Q - X^*) > 0$.

The proof for $U'''(\cdot) < [=]\,0$ is analogous. $\square$

Proof of Proposition 2

To simplify the notation, define functions $A(X)$, $B(X)$ and $C(X)$ as

$$
\begin{aligned}
A(X) &= \operatorname{cov}(U''(\tilde{W}_1), \tilde{\epsilon}|P_1)\,, && (11)\\
B(X) &= \operatorname{cov}\Big(U'(\tilde{W}_1(X)), \tilde{P}_1\Big) = \operatorname{cov}\Big(\mathrm{E}\Big[U'(\tilde{W}_1(X))\Big|P_1\Big], \tilde{P}_1\Big)\,, && (12)\\
C(X) &= \operatorname{cov}\Big(U'(\tilde{W}_1(X))\,\tilde{P}_1, \tilde{\epsilon}\Big) = \operatorname{cov}\Big(\mathrm{E}\Big[U'(\tilde{W}_1(X))\,\tilde{P}_1\Big|\epsilon\Big], \tilde{\epsilon}\Big)\,, && (13)
\end{aligned}
$$

where the second parts of (12) and (13) are due to the law of iterated expectations.

Using A.2, $\mathrm{E}[\tilde{\epsilon}] = 0$ and $F_0 = \mathrm{E}[\tilde{F}_1] = a + b\mathrm{E}[\tilde{P}_1]$ due to unbiasedness, one can rewrite the LHS of the first-order condition (2) as

$$
\begin{aligned}
&\mathrm{E}\Big[U'(\tilde{W}_1(X))\,\Big(F_0 - (a + b\tilde{P}_1(1+\tilde{\epsilon}))\Big)\Big]\\
=\;& \mathrm{E}\Big[U'(\tilde{W}_1(X))\,\Big(F_0 - (a + b\,\tilde{P}_1)\Big)\Big] - b\,\mathrm{E}\Big[U'(\tilde{W}_1(X))\,\tilde{\epsilon}\tilde{P}_1\Big] && (14)\\
=\;& -b\Big[B(X) + C(X)\Big]\,.
\end{aligned}
$$

(2) and (14) imply $[B(X^*) + C(X^*)] = 0$ since $b > 0$.

The remainder of the proof is based on the signs of $A(X)$, $B(X)$ and $C(X)$. In order to sign $A(X)$, it is useful to derive

$$
\frac{\partial\, U''(W_1)}{\partial\, \epsilon} = -\,U'''(W_1)\,bP_1X \qquad \forall\, P_1, \epsilon\,. \tag{15}
$$

To sign $B(X)$, notice that

$$
\begin{aligned}
\frac{\partial\,\mathrm{E}\Big[U'(\tilde{W}_1)\Big|P_1\Big]}{\partial\, P_1} &= \mathrm{E}\Big[U''(\tilde{W}_1)\,\Big(Q - bX(1+\tilde{\epsilon})\Big)\Big|P_1\Big]\\
&= \Big(Q - bX\Big)\mathrm{E}\Big[U''(\tilde{W}_1)\Big|P_1\Big] - bX\,A(X) \qquad \forall\, P_1
\end{aligned} \tag{16}
$$

since $\mathrm{E}[\tilde{\epsilon}] = 0$. Signing $C(X)$ uses the fact that

$$
\frac{\partial\,\mathrm{E}\Big[U'(\tilde{W}_1)\,\tilde{P}_1\Big|\epsilon\Big]}{\partial\,\epsilon} = -\,bX\,\mathrm{E}\Big[U''(\tilde{W}_1)\,\tilde{P}_1^2\Big|\epsilon\Big] \qquad \forall\,\epsilon\,. \tag{17}
$$

Hence, $U''(W_1) < 0$ and $b > 0$ imply $\operatorname{sgn} C(X) = \operatorname{sgn} X$.

Consider $X = 0$. (15) implies $A(0) = 0$. Together with $U''(\cdot) < 0$, $Q > 0$ and (16), $B(0) < 0$ follows. (17) implies $C(0) = 0$. Hence, (14) is positive if evaluated at $X = 0$. The concavity of the problem implies $X^* > 0$. This proves part a).

Now, consider $X = Q/b$. Suppose that $U'''(W_1) \geq 0 \, \forall W_1$. Then, $Q > 0$ and (15) imply $A(Q/b) \leq 0$. Hence, $B(Q/b) \geq 0$ by (16). Finally, $C(Q/b) > 0$ due to (17). Taken together, (14) is negative at $X = Q/b$. Hence, $X^* < Q/b$ due to the concavity of the problem. This proves part b). □

References

Adam-Müller, A.F.A. (2000). Exports and hedging exchange rate risk: The multi-country case. Journal of Futures Markets, 20, 843-864.

Anderson, R.W., & Danthine, J.-P. (1981). Cross hedging. Journal of Political Economy, 89, 1182-1196.

Baillie, R., & Myers, R. (1991). Bivariate GARCH estimation of the optimal commodity futures hedge. Journal of Applied Econometrics, 6, 109-124.

Benninga, S., Eldor, R., & Zilcha, I. (1983). The optimal hedging in the futures market under price uncertainty. Economics Letters, 13, 141-145

Briys, E., Crouhy, M., & Schlesinger, H. (1993). Optimal hedging in a futures market with background noise and basis risk. European Economic Review, 37, 949-960.

Broll, U., Wahl, J.E., & Zilcha, I. (1995). Indirect hedging of exchange rate risk. Journal of International Money and Finance, 14, 667-678.

Chang, E.C., & Wong, K.P. (2002). Cross hedging with currency options and futures. Journal of Financial and Quantitative Analysis, forthcoming.

Froot, K.A., Scharfstein, S., & Stein, J. (1993). Risk management: Coordinating corporate investment and financing policies. Journal of Finance, 68, 1629-1658.

Gollier, Ch. (2001). The economics of risk and time. Cambridge (Mass.) et al., MIT Press.

Holthausen, D.M. (1979). Hedging and the competitive firm under price uncertainty. American Economic Review, 69, 989-995.

Ingersoll, J.E. (1987). Theory of financial decision making. Totowa (NJ), Rowman & Littlefield.

Kimball, M.S. (1990). Precautionary saving in the small and in the large. Econometrica, 58, 53-73.

Lence, S.H. (1995). On the optimal hedge under unbiased futures prices. Economics Letters, 47, 385-388.

Mahul, O. (2002). Hedging in futures and options markets with basis risk. Journal of Futures Markets, 22, 59-72.

Paroush, J., & Wolf, A. (1986). Production and hedging decisions in futures and forward markets. Economics Letters, 21, 139-143.

Smith, C.W., & Stulz, R.M. (1985). The determinants of firms' hedging policies. Journal of Financial and Quantitative Analysis, 20, 391-405.

Stulz, R.M. (1984). Optimal hedging policies. Journal of Financial and Quantitative Analysis, 19, 127-140.

Stulz, R.M. (1990). Managerial discretion and optimal financing policies. Journal of Financial Economics, 26, 3-27.

Tomek, W.G., & Peterson, H.H. (2001). Risk management in agricultural markets: A review. Journal of Futures Markets, 21, 953-985.

Wong, K.P. (2003). Currency hedging with options and futures. European Economic Review, forthcoming.